MTH 3110: Abstract Algebra I
Fall Semester 2018
Problem List 4

Prof: Mandi Schaeffer Fry

Due 10/15/18

You are encouraged to ask questions during office hours. You are also encouraged to work through problems together and bounce ideas off of one another; however, the actual write up should be done on your own. This means your homework should not be identical to another person's.

NOTE: Late homework will NOT be accepted. Solutions should be submitted to me by email or in person/my mailbox **by 7pm on Monday, Oct 15** in clear writing, written neatly, using complete sentences. (This may require re-writing your final draft to turn in!) Multiple pages should be stapled. (I will NOT provide the stapler.) Points will be deducted from assignments not following these guidelines. You also have the option to LaTeX your homework for an extra full redo.

Warning! This assignment may take extra time, which is why I am giving you more time than usual. Start early! Ask questions!

1 Before Class On...

- Wed, 10/3: Read Chapter 6 (except the parts about Cayley's theorem and about Automorphisms - we'll come back to them), and the first 3 pages of Chapter 10.
- Wed, 10/10: Read Chapter 4. (Note that our proofs in class will be slightly different, and in many cases simpler, using the isomorphisms and lemmas we've already discussed in class.)

2 Notation and Definitions to Know

An $*$ denotes Flashquiz - eligible items.

- *homomorphism
- *isomorphism
- *cyclic group
- generator
- know the theorems that every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$ for some $n \in \mathbb{Z}_{>0}$.

3 For Practice...

All exercises listed as practice problems are from Gallian **6th** edition - compare with me or a classmate to make sure you have the right ones.

- Chapter 6, Exercises 1,3-6, 17, 22-26
- Chapter 10, Exercises 1-7
- Let $k \neq 0$ be a fixed integer. Prove that $\mathbb{Z}$ is isomorphic to $k\mathbb{Z} := \{kz | z \in \mathbb{Z}\}$. (You may take for granted that $k\mathbb{Z}$ forms a group under addition.)
- Prove that $(\mathbb{Q} \setminus \{0\}, \cdot)$ is not cyclic.
- Let $\Phi \colon G \to \overline{G}$ be a group homomorphism. Prove that the image of Φ, denoted $\text{Im}\Phi$ or $\Phi(G)$, is a subgroup of $\overline{G}$.

4 To Turn In by 7pm on 10/15/18

1. Prove that the notion of isomorphism of groups is an equivalence relation. (Hint: You may use the properties of functions in Theorem 0.7, which we talked about at the beginning of the semester, but note that as part of your proof you'll have to prove compositions, inverses, etc, of homomorphisms are also homomorphisms!)
2. Show that $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{Q} \setminus \{0\}, \cdot)$. (Hint: Assume BWOC that there is an isomorphism $\Phi\colon \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$, and consider an element $a \in \mathbb{Q}$ such that $\Phi(a) = -1$.)

For the remainder of the assignment, let $\Phi\colon G \to \overline{G}$ be a homomorphism of groups, and let e_G and $e_{\overline{G}}$ be the identity elements of G and $\overline{G}$, respectively. For problems 3-5 you'll need the following definition:

Definition. The *kernel* of the homomorphism Φ is defined to be the set:

$$\ker \Phi := \{a \in G \mid \Phi(a) = e_{\overline{G}}\}.$$

3. Prove that $\ker \Phi$ is a subgroup of G.
4. Prove that Φ is injective if and only if $\ker \Phi = \{e_G\}$. (This result means that in a sense, the kernel tells us how "far" the homomorphism is from being injective!)
5. Prove that $ghg^{-1} \in \ker \Phi$ for every $g \in G$ and every $h \in \ker \Phi$. (We say that a subgroup of *normal* if it satisfies this property.)