MTH 4110: Abstract Algebra 2 - Spring Semester 2019
Problem List 1

Prof: Mandi Schaeffer Fry

Due 2/1/19

You are encouraged to ask questions during office hours. You are also encouraged to work through problems together and bounce ideas off of one another; however, the actual write up should be done on your own. This means your homework should not be identical to another person's.

NOTE: Late homework will NOT be accepted without the use of a "full redo". Solutions should be submitted by email to Dr. Mandi (aschaef6@msudenver.edu) as a **single PDF file** in clear writing, written neatly, using complete sentences. (This may require re-writing your final draft to turn in!) Recall that if you use LaTeX (including Overleaf) to typeset your homework, you'll get an extra "full redo".

1 Before Class On...

- Mon, 1/28: Read Chapter 12
- Wed, 1/30: Read Chapter 13

2 Notation, Definitions, Theorems to Know

An $*$ denotes Flashquiz - eligible items.

- *ring
- *subring
- *subring test
- *commutative ring
- *ring with unity
- unit, unity, inverse, identity, and other terms from group theory in Abstract 1...

3 For Practice...

- Let $n \geq 2$ be an integer. Prove that the set $n\mathbb{Z} := \{nz \colon z \in \mathbb{Z}\}$ is a subring of $\mathbb{Z}$.
- Is $n\mathbb{Z}$ a ring with unity? Prove your answer. (Careful: what did you notice about the unity in Problem 2?!)
- Book Chapter 12, problems 2, 4, 8, 9, 19, 30, 38, 40-43 (these are the same in the 6th- 9th editions)

The problems to turn in are on the other side

4 To Turn In On 2/1/19

Recall that $M_2(\mathbb{R})$ is defined as 2×2 matrices with entries in $\mathbb{R}$:

$$M_2(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Matrix addition is defined component-wise:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} := \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

and matrix multiplication is done using the "row-dot-column" rule:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} := \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

0. (BONUS) Prove that $(M_2(\mathbb{R}), +, \cdot)$ is a ring, where $+$ and $\cdot$ are matrix addition and multiplication as above.
1. (a) Prove that the subset $S := \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subring of $M_2(\mathbb{R})$. (Hint: use the subring test.)
   (b) Is $M_2(\mathbb{R})$ a commutative ring? Prove your answer.
   (c) Is S a commutative ring? Prove your answer.
   (d) Is $M_2(\mathbb{R})$ a ring with unity? Prove your answer. (Just the unity part - it is already a ring by part 0).
   (e) Is S a ring with unity? Prove your answer.
2. Prove that the subset $S' := \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ is a commutative ring with unity. (Note: to show it is a ring, it suffices to show that it is a subring of $M_2(\mathbb{R})$.)
3. Let R be a ring. Prove or disprove: If a and b are elements of R such that $ab = 0$, then $ba = 0$. (Hint: $M_2(\mathbb{R})$ would be a good place to look.)