MTH 4110: Abstract Algebra 2 - Spring Semester 2019
Problem List 2

Prof: Mandi Schaeffer Fry

Due 2/8/19

You are encouraged to ask questions during office hours. You are also encouraged to work through problems together and bounce ideas off of one another; however, the actual write up should be done on your own. This means your homework should not be identical to another person's.

NOTE: Late homework will NOT be accepted without the use of a "full redo". Solutions should be submitted by email to Dr. Mandi (aschaef6@msudenver.edu) as a **single PDF file** in clear writing, written neatly, using complete sentences. (This may require re-writing your final draft to turn in!) Recall that if you use LaTeX (including Overleaf) to typeset your homework, you'll get an extra "full redo".

1 Before Class On...

- Wed, 1/30: Read Chapter 13
- Wed, 2/6: Read Chapter 14

2 Notation, Definitions, Theorems to Know

An $*$ denotes Flashquiz - eligible items.

- *zero divisor
- *integral domain
- *field
- *characteristic
- *Theorem 13.4: What is the characteristic of an Integral Domain?

3 For Practice...

- Ch. 13, 6th Edition: Problems 4-7; 13; 15; 17 (also the elements $a - ab, a + b - ab, a + b - 2ab$ are idempotent if a and b are); 23; 24; 26; 45; 47

OR

- Ch. 13, 8th Edition: Problems 4-7; 15; 17; 19; 29; 30; 32; 53; 55

The problems to turn in are on the other side

4 To Turn In On 2/8/19

1. An *idempotent* is an element $a \in R$ of a ring R such that $a^2 = a$.
   (a) Prove that if R is an integral domain, then 0 and 1 are the only idempotents in R.
   (b) Prove that if R is a commutative ring of characteristic 2 (not necessarily an integral domain), then the idempotents form a subring.
2. Prove that in $\mathbb{Z}_n$, every nonzero element is either a unit or a zero divisor.
3. Prove that if F is a field of order p^n for some prime p and some $n \in \mathbb{N}$, then the characteristic of F is p. (You may want to use Lagrange's Theorem.)
4. It turns out we can form a field with 4 elements as follows: Let j satisfy $j^2 + j + 1 = 0$. Consider the set $\mathbb{Z}_2[j] := \{a + bj : a, b \in \mathbb{Z}_2\}$. Determine the appropriate addition and multiplication tables for this set. (This should be along the lines of how $\mathbb{Z}_3[i]$ was defined, but the multiplication may be a bit more complicated.)

Fun Facts about 4 Hopefully, you should notice that each of $(\mathbb{Z}_2[j], +)$ and $(\mathbb{Z}_2[j] \setminus \{0\}, \cdot)$ form a latin square and that there are no zero divisors. You might also convince yourself that the operations are associative and distribution holds, so we indeed get a finite integral domain, and hence a field!