MTH 4110: Abstract Algebra 2 - Spring Semester 2019
Problem List 5

Prof: Mandi Schaeffer Fry

Due Friday 3/15/19

You are encouraged to ask questions during office hours. You are also encouraged to work through problems together and bounce ideas off of one another; however, the actual write up should be done on your own. This means your homework should not be identical to another person's.

NOTE: Late homework will NOT be accepted without the use of a "full redo". Solutions should be submitted by email to Dr. Mandi (aschaef6@msudenver.edu) as a **single PDF file** in clear writing, written neatly, using complete sentences. (This may require re-writing your final draft to turn in!) Recall that if you use LaTeX (including Overleaf) to typeset your homework, you'll get an extra "full redo".

1 Before Class On...

- Mon, 3/11: Read Chapter 16
- Wed, 3/13: Read Chapter 18

2 Notation, Definitions, Theorems to Know

An $*$ denotes Flashquiz - eligible items.

- field of quotients, prime subfield
- *first isomorphism theorem
- *Kernel
- *homomorphism of rings
- *isomorphism of rings
- *statement of cor3: every field contains a copy of $\mathbb{Z}_p$ or $\mathbb{Z}$.

3 For Practice...

- Ch. 15, 6th Edition: Problems 42, 43, 46, 51, 52, 54-56, 60
  OR 8th Edition: Problems 48, 49, 52, 57, 58, 60-62, 66
- Prove directly from the definition that $\ker\phi$ is an ideal for any ring homomorphism ϕ.

The problems to turn in are on the other side

4 To Turn In On 3/15/19

1. Let R be a commutative ring with prime characteristic p. Prove that the *Frobenius* map $\varphi_p \colon R \to R$ given by $x \mapsto x^p$ is a ring homomorphism.

2. Let R be an integral domain and let $S = \{(a, b) \in R \times R \mid b \neq 0\}$. Prove that the relation $(a, b) \sim (c, d) \iff ad = bc$ is an equivalence relation.

3. Let F be a field and let $S \leq F$ be a subring such that $S \cong \mathbb{Z}$. Prove that the set $\{ab^{-1} \mid a, b \in S, b \neq 0\}$ is a field isomorphic to the rationals.

4. Let $\phi \colon R \to S$ be a ring homomorphism. Prove directly from the definition that $\ker \phi = \{0\}$ if and only if ϕ is an injection.

Bonus Let F be a field of characteristic p a prime. Let $M_2(F)$ be defined analogously to $M_2(\mathbb{R})$, but with entries in F.

(a) Show that $GL_2(F) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in F; ad - bc \neq 0\}$ is a group under matrix multiplication. (In fact, this is the group of units of $M_2(F)$).

(b) Show that the map $\overline{\varphi}_p \colon GL_2(F) \to GL_2(F)$ given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \varphi_p(a) & \varphi_p(b) \\ \varphi_p(c) & \varphi_p(d) \end{bmatrix}$ is a group homomorphism. Here φ_p is the ring homomorphism (now field homomorphism) from problem 1. (When F is a finite field, this is actually an automorphism, called a *field automorphism* of $GL_2(F)$. It is very important in the theory of groups of Lie type, which is the class of groups that Dr. Mandi studies!)