CHARACTERS OF π'-DEGREE WITH CYCLOTOMIC FIELDS OF VALUES

EUGENIO GIANNELLI, NGUYEN NGOC HUNG, A. A. SCHAEFFER FRY, AND CAROLINA VALLEJO

Abstract. We characterize finite groups that possess a nontrivial irreducible character of $\{p,q\}'$-degree with values in $\mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(e^{2\pi i/q})$, where p and q are primes. This extends previous work of Navarro-Tiep and of Giannelli-Schaeffer Fry-Vallejo. Along the way we completely describe the alternating groups possessing a nontrivial irreducible rational-valued character of $\{p,q\}'$-degree. A similar classification is obtained for solvable groups, when $p=2$.

1. Introduction

In 2006, G. Navarro and P.H. Tiep [NT06] confirmed a conjecture of R. Gow predicting that every group of even order has a nontrivial rational-valued irreducible character of odd degree. Later, in [NT08], they generalized their result by proving that every finite group of order divisible by a prime q admits a nontrivial irreducible character of q'-degree with values in $\mathbb{Q}(e^{2i\pi/q})$, the cyclotomic field extending the rational numbers by a primitive q-th root of unity. In [GSV19], the first, third, and fourth-named authors have recently shown that for any set π consisting of at most two primes, every nontrivial group has a nontrivial character of π'-degree (that is, a character of p'-degree, for all primes $p \in \pi$). How to extend these results, if possible, is the main topic under consideration in this article.

In Theorem A, we show that every finite group possesses a nontrivial irreducible character of $\{2,q\}'$-degree with values in $\mathbb{Q}(e^{2i\pi/q})$, an unexpected result that generalizes both [NT06] and [NT08] in the fashion of [GSV19].

Theorem A. *Let G be a finite group and q be a prime, and write $\pi = \{2,q\}$. Then G possesses a nontrivial π'-degree irreducible character with values in $\mathbb{Q}(e^{2\pi i/q})$ if, and only if,* $\gcd(|G|, 2q) > 1$.

A natural problem that arises in this context is to try to understand when the irreducible character identified by Theorem A can be chosen to be rational-valued. In other words, for a group G of even order and an odd prime q, we would like to characterize when G has a π'-degree rational character, where $\pi = \{2,q\}$. This is not always the case, in contrast to what happens if we allow small cyclotomic field extensions of $\mathbb{Q}$ as fields of values as described by Theorem A. For example, the only rational linear character of A_4 is the principal one. A complete answer to this problem appears difficult to achieve and at the time of this writing,

Date: February 13, 2020.

2010 *Mathematics Subject Classification.* Primary 20C15, 20C30, 20C33; Secondary 20D05.

Key words and phrases. character degrees, fields of values, rationality.

The third-named author acknowledges support from the National Science Foundation under Grant No. DMS-1801156. The fourth-named authort acknowledges support by Ministerio de Ciencia e Innovación PID2019-103854GB-I00 and FEDER funds.

we do not know what form such a classification would take. However, in the case where G is a solvable group (or an alternating group, see Theorem D below), we can completely solve this problem.

Theorem B. *Let G be a solvable group, q be a prime and set $\pi = \{2, q\}$. Then G admits a nontrivial rational irreducible character of π'-degree if, and only if, H/H' has even order, where $H \in \mathrm{Hall}_\pi(G)$.*

Our proof of Theorem A relies on the Classification of the Finite Simple Groups. In fact, for alternating groups and generic groups of Lie type, the arguments naturally extend from a pair $\{2, q\}$ of primes to any pair $\{p, q\}$. Hence we obtain Theorem A as a corollary of the following statement, which classifies finite groups admitting a π'-degree character with values in certain cyclotomic extensions of $\mathbb{Q}$, for any set π consisting of two primes.

Theorem C. *Let G be a finite group and $\pi = \{p, q\}$ be a set of primes such that either p or q divides $|G|$. Assume that:*

- *$\pi \neq \{3, 5\}$ or G does not have a composition factor isomorphic to the Tits group ${}^2F_4(2)'$.*
- *$\pi \neq \{23, 43\}, \{29, 43\}$ or G does not have a composition factor isomorphic to the Janko group J_4.*

Then G possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

To prove Thoerem C, we show that every finite nonabelian simple group admits a nontrivial irreducible character χ of $\{p, q\}'$-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$, with the only exceptions of the Tits group ${}^2F_4(2)'$ in the case $\pi = \{3, 5\}$ and the Janko group J_4 in the cases where $\pi = \{23, 43\}$ and $\pi = \{29, 43\}$. (See Theorem 2.1 below.)

As previously mentioned, a complete classification of groups admitting a nontrivial $\{2, q\}'$-degree rational irreducible characters seems out of reach at the present moment. Nevertheless, for alternating groups, we can solve this classification problem for any set π consisting of exactly two primes.

Theorem D. *Let $n \geqslant 5$ be a natural number and let p, q be distinct primes. Let $\pi = \{p, q\}$. The alternating group A_n admits a nontrivial rational-valued irreducible character of π'-degree for all those $n \in \mathbb{N}$ that do not satisfy any of the following conditions.*

(i) *$n = p^m = 2q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that m is odd.*
(ii) *$n = 2p^m = q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that k is odd.*

Moreover, in case (i) $\mathbb{Q}(\phi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ for all $\phi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$. On the other hand, in case (ii) $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(e^{2\pi i/q})$ for all $\psi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$.

This paper is structured as follows: In Section 2 we prove Theorems A and C assuming Theorem 2.1 on finite simple groups. In Section 3 we prove Theorem D, which in particular yields the alternating group case of Theorem 2.1. In Section 4 we prove Theorem 2.1 for sporadic groups and simple groups of Lie type, thus completing the proof of Theorem 2.1 by the Classification of Finite Simple Groups. Finally, in Section 5 we prove Theorem B.

2. Proofs of Theorems A and C

To prove Theorems A and C, we assume the following result on finite simple groups. This will be shown to hold Sections 3 and 4.

Theorem 2.1. *Let S be a nonabelian simple group and $\pi = \{p, q\}$ be a set of primes. Assume that $(S, \pi) \neq ({}^2F_4(2)', \{3, 5\})$, $(J_4, \{23, 43\})$, and $(J_4, \{29, 43\})$. Then there exists $\mathbf{1}_S \neq \chi \in \mathrm{Irr}(S)$ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.*

We start with a lemma.

Lemma 2.2. *Let $M \lhd G$ such that $|G : M| = r$ an odd prime. Let $\theta \in \mathrm{Irr}(M)$ with $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$ for some prime $p \neq r$. Then there exists $\chi \in \mathrm{Irr}(G)$ lying over θ with $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$.*

Proof. If $\mathrm{Stab}_G(\theta) = M$ then by Clifford correspondence, we have $\theta^G \in \mathrm{Irr}(G)$ with $\mathbb{Q}(\theta^G) \subseteq \mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$, as required. So we assume that θ is G-invariant.

Note that θ is r-rational. It follows from [Isa06, Theorem 6.30] that θ^G has a unique r-rational irreducible constituent χ. Indeed θ is extendible to χ, and hence $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$.

For each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}(\theta))$, obviously χ^σ is also an r-rational character of G lying over θ. Therefore, by the uniqueness of χ, we have $\chi^\sigma = \chi$. So χ is $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}(\theta))$-fixed, which implies that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\theta)$. We have shown that $\mathbb{Q}(\chi) = \mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$, as desired. □

Theorem 2.3. *Let G be a finite group and $\pi = \{p, q\}$ be a set of primes. Then G possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$ if, and only if, $\gcd(|G|, 2pq) > 1$, provided that we are not in one of the following situations:*

- *G has a composition factor isomorphic to the Tits group ${}^2F_4(2)'$ and $\pi = \{3, 5\}$.*
- *G has a composition factor isomorphic to the Janko group J_4 and $\pi = \{23, 43\}$ or $\pi = \{29, 43\}$.*

Proof. First assume that G is a finite group with $\gcd(|G|, 2pq) = 1$. Let $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/|G|})$, we have χ is rational-valued. As G is of odd order, it follows from Burnside's theorem that χ is trivial.

Next we assume that $\gcd(|G|, 2pq) > 1$. We aim to show that G has a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

Let $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = 1$ be a composition series of G and let $0 \leqslant k \leqslant n-1$ be the smallest such that G_k/G_{k+1} is either nonabelian simple or cyclic of order $2, p$ or q. In particular, G_i/G_{i+1} is cyclic of order coprime to $2pq$ for every $i < k$.

When G_k/G_{k+1} is cylic of order $2, p$ or q then obviously G_k/G_{k+1} has a nontrivial irreducible character θ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. On the other hand, when $G_k/G_{k+1} = S$ is nonabelian simple then Theorem 2.1 implies that there exists $\mathbf{1}_S \neq \theta \in \mathrm{Irr}(S)$ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

Viewing the above θ as a character of G_k, we now know that G_k possesses a nontrivial irreducible character θ_k of π'-degree such that $\mathbb{Q}(\theta_k) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta_k) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Using Lemma 2.2, we obtain $\theta_{k-1} \in \mathrm{Irr}(G_{k-1})$ lying over θ_k with $\mathbb{Q}(\theta_{k-1}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta_{k-1}) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Moreover, following the proof of Lemma 2.2, we see that $\theta_{k-1}(1) = \theta_k(1)$ or $\theta_{k-1}(1) = |G_{k-1} : G_k|\theta_k(1)$, which guarantees that θ_{k-1} is of π'-degree. Repeating this process k times, we can produce a nontrivial irreducible character $\chi := \theta_0$ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. □

Theorems A and C follow immediately from Theorem 2.3.

3. Alternating groups

The aim of this section is to prove Theorem 2.1 for alternating groups. In order to do so, we completely describe alternating groups possessing a rational valued π'-character. This is done by proving Theorem D of the Introduction, which might be of independent interest.

We begin by recalling that irreducible characters of the symmetric group S_n are labelled by partitions of n. We denote by χ^λ the irreducible character of S_n corresponding to the partition λ of n. We will sometimes use the notation $\lambda \vdash n$ to mean that λ is a partition of n. Similarly we will write $\lambda \vdash_{p'} n$ to say that $\chi^\lambda(1)$ is coprime to p. Given a partition λ of n we denote by λ' its conjugate. If $\lambda \neq \lambda'$ then $(\chi^\lambda)_{\mathsf{A}_n} \in \mathrm{Irr}(\mathsf{A}_n)$. On the other hand, if $\lambda = \lambda'$ then $(\chi^\lambda)_{\mathsf{A}_n} = \phi + \phi^g$ for some $\phi \in \mathrm{Irr}(\mathsf{A}_n)$ and $g \in \mathsf{S}_n \smallsetminus \mathsf{A}_n$.

Assuming that the reader is familiar with the basic combinatorial concepts involved in the representation theory of symmetric groups (as explained for instance in [Ol94, Chapter 1]), we recall some important facts that will play a crucial role in our proofs. Given $\lambda \vdash n$, $i, j \in \mathbb{N}$ we denote by $h_{ij}(\lambda)$ the length of the *hook of* λ corresponding to node (i, j). For $e \in \mathbb{N}$ we let $\mathcal{H}^e(\lambda)$ be the set consisting of all those nodes (i, j) of λ such that e divides $h_{ij}(\lambda)$. Moreover, we let $C_e(\lambda)$ denote the e-core of λ.

The following lemma follows from [Ol94, Proposition 6.4].

Lemma 3.1. *Let p be a prime and let n be a natural number with p-adic expansion $n = \sum_{j=0}^{k} a_j p^j$. Let λ be a partition of n. Then $\nu_p(\chi^\lambda(1)) = 0$ if, and only if, $|\mathcal{H}^{p^k}(\lambda)| = a_k$ and $C_{p^k}(\lambda) \vdash_{p'} n - a_k p^k$.*

For a natural number m, in Lemma 3.1 we denoted by $\nu_p(m)$ the exponent of the maximal power of p dividing m.

A consequence of Lemma 3.1 is highlighted by the following statement.

Lemma 3.2. *Let p be a prime and let $n = p^k + \varepsilon$ for some $\varepsilon \in \{0, 1\}$. Let $\lambda \vdash n$ such that $\chi^\lambda(1) > 1$. Then χ^λ is an irreducible character of p'-degree of S_n, if and only if $h_{11}(\lambda) = p^k$.*

A second useful consequence of [Ol94, Proposition 6.4] is stated in the following lemma.

Lemma 3.3. *Let $n = 2^k + \varepsilon$ for some $\varepsilon \in \{0, 1\}$, and let $\lambda \vdash n$. Then $\nu_2(\chi^\lambda(1)) = 1$ if and only if $\mathcal{H}^{2^k}(\lambda) = \varnothing$ and $|\mathcal{H}^{2^{k-1}}(\lambda)| = 2$.*

We conclude this brief background summary by recalling a well-known fact on cyclotomic extensions of the rational numbers.

Lemma 3.4 (Gauss)**.** *If p is and odd prime number, then $\mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ if and only if $p \equiv 1 \bmod 4$, and $\mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ if and only if $p \equiv 3 \bmod 4$.*

We are now ready to prove the main result of this section, which is Theorem D in the introduction. We restate it here for the reader's convenience.

Theorem D. *Let $n \geqslant 5$ be a natural number and let p, q be distinct primes. Let $\pi = \{p, q\}$. The alternating group A_n admits a nontrivial rational-valued irreducible character of π'-degree for all those $n \in \mathbb{N}$ that do not satisfy any of the following conditions.*

(i) *$n = p^m = 2q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that m is odd.*
(ii) *$n = 2p^m = q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that k is odd.*

Moreover, in case (i) $\mathbb{Q}(\phi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ for all $\phi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$. On the other hand, in case (ii) $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(e^{2\pi i/q})$ for all $\psi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$.

Proof. Let $n = \sum_{i=1}^{t} a_i p^{m_i} = \sum_{j=1}^{r} b_j q^{k_j}$ be the p-adic and respectively q-adic expansions of n. Here $m_1 > m_2 > \cdots > m_t \geqslant 0$ and $k_1 > k_2 > \cdots k_r \geqslant 0$. Without loss of generality we can assume that $b_1 q^{k_1} < a_1 p^{m_1}$. We consider $\lambda \in \mathcal{P}(n)$ to be defined by:

$$\lambda = (n - b_1 q^{k_1}, n - a_1 p^{m_1} + 1, 1^{b_1 q^{k_1} - (n - a_1 p^{n_1} + 1)}).$$

As done in the proof of [GSV19, Theorem 2.8] we observe that $\chi^\lambda \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ and that $\chi^\lambda(1) \neq 1$ unless $n = a_1 p^{m_1} = b_1 q^{k_1} + 1$. We also claim that $\lambda \neq \lambda'$. This follows by observing that $\lambda = \lambda'$ would imply that

$$b_1 q^{k_1} - (n - a_1 p^{m_1}) = n - b_1 q^{k_1} - 1 \text{ and that } n - a_1 p^{m_1} \in \{0, 1\}.$$

Then we would have that $b_1 q^{k_1} = n - b_1 q^{k_1} - 1$ if $n - a_1 p^{m_1} = 0$ or that $b_1 q^{k_1} = n - b_1 q^{k_1}$ if $n - a_1 p^{m_1} = 1$. Both these situations can not occur. We conclude that $\chi := (\chi^\lambda)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ and that $\mathbb{Q}(\chi) = \mathbb{Q}$.

Let us now consider the case where $n = ap^m = bq^k + 1$, for some $m, k \in \mathbb{N}$, some $1 \leqslant a \leqslant p - 1$ and some $1 \leqslant b \leqslant q - 1$.

If $b \geqslant 3$, then we consider $\mu = ((b-1)q^k + 1, 1^{q^k})$. Since $h_{11}(\mu) = ap^m$, $h_{12}(\mu) = (b-1)q^k$ and $h_{21}(\mu) = q^k$, we deduce that $\chi^\mu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ by Lemma 3.1. Since $b \geqslant 3$ we also have that $\mu \neq \mu'$ and hence that $\chi := (\chi^\mu)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ is nontrivial and such that $\mathbb{Q}(\chi) = \mathbb{Q}$.

If $b \in \{1, 2\}$ and $a \geqslant 3$ then we consider $\nu = ((a-1)p^m, 2, 1^{p^m - 2})$. Since $h_{11}(\nu) = bq^k$, $h_{12}(\nu) = (a-1)p^m$ and $h_{21}(\nu) = p^m$, we deduce that $\chi^\nu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ by Lemma 3.1. As above, $a \geqslant 3$ implies that $\nu \neq \nu'$ and hence that $\chi := (\chi^\nu)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ is nontrivial and such that $\mathbb{Q}(\chi) = \mathbb{Q}$.

Let us now study the situation where $a, b \in \{1, 2\}$. Since $ap^m = bq^k + 1$ we observe that the only cases to consider are $(a, b) \in \{(1, 2), (2, 1)\}$.

If $a = 1$ then $n = p^m = 2q^k + 1$ and hence $p \neq 2$. By Lemma 3.2 we deduce that $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$ if and only if $\lambda = (d, 1^{n-d})$ is a hook partition. Moreover, if q is odd, again from Lemma 3.1 we observe that the only hook partitions of n that label characters of S_n of degree coprime to q are $(n), (1^n)$ and $\zeta = (1 + q^k, 1^{q^k}) = \zeta'$. We also observe that in this situation m must be odd, as $p^m = 2q^k + 1 \equiv 3 \bmod 4$. It follows that A_n admits exactly two distinct nontrivial irreducible characters of π'-degree: the two irreducible constituents ϕ_1, ϕ_2 of $(\chi^\zeta)_{\mathsf{A}_n}$. By [JK81, 2.5.13] we observe that their fields of values are equal to $\mathbb{Q}(\sqrt{-p^m})$ and strictly contain $\mathbb{Q}$. Moreover, since m is odd then $p \equiv p^m = 2q^k + 1 \equiv 3 \bmod 4$. Hence using Lemma 3.4 we observe that for all $i \in \{1, 2\}$ we have

$$\mathbb{Q}(\phi_i) = \mathbb{Q}(\sqrt{-p^m}) = \mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(e^{2\pi i/p}).$$

Similarly, if $q = 2$ then Lemma 3.2 shows that $(n), (1^n)$ are the only hook partitions labelling an odd-degree character of S_n. Moreover, using Lemma 3.3 we observe that the hook ζ defined above, is the only hook partition of n such that $\nu_2(\chi^\zeta(1)) = 1$. Again we deduce that the two irreducible constituents ϕ_1 and ϕ_2 of $(\chi^\zeta)_{\mathsf{A}_n}$ are the only nontrivial irreducible characters of π'-degree of A_n. By [JK81, 2.5.13] we observe that $\mathbb{Q}(\phi_1) = \mathbb{Q}(\phi_2) = \mathbb{Q}(\sqrt{p^m})$. Hence $\mathbb{Q}(\phi_1)$ (and $\mathbb{Q}(\phi_2)$) strictly contain $\mathbb{Q}$ if and only if m is odd. In this case, $p \equiv p^m \equiv 2^{k+1} + 1 \equiv 1 \bmod 4$. Therefore Lemma 3.4 implies that for all $i \in \{1, 2\}$ we have that

$$\mathbb{Q}(\phi_i) = \mathbb{Q}(\sqrt{p^m}) = \mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(e^{2\pi i/p}).$$

If $b = 1$ then $n = 2p^m = q^k + 1$ and hence $q \neq 2$. The situation is similar to the one described above. Using Lemma 3.2 we notice that the non-linear irreducible characters of S_n

of degree coprime to q are labelled by all partitions λ such that $h_{11}(\lambda) = q^k$ and $h_{22}(\lambda) = 1$. Among these, when $p \neq 2$, the only one that labels an irreducible character of S_n of degree coprime to p is $\eta = (1{+}p^m, 2, 1^{p^m-2}) = \eta'$. On the other hand, when $p = 2$ then we notice that k is necessarily odd. Moreover, Lemma 3.2 implies that the only $\{2, q\}'$-degree irreducible characters of S_n are the linear ones. On the other hand, Lemma 3.3 shows that η is the only partition labelling a q'-degree irreducible character of S_n such that $\nu_2(\chi^\eta(1)) = 1$. Exactly as before we deduce that A_n admits exactly two distinct nontrivial irreducible characters of π'-degree: the two irreducible constituents ψ_1 and ψ_2 of $(\chi^\eta)_{\mathsf{A}_n}$. By [JK81, 2.5.13] we observe that for all $i \in \{1, 2\}$ we have that

$$\mathbb{Q}(\psi_i) = \begin{cases} \mathbb{Q}(\sqrt{q^k}) & \text{if } p \neq 2, \\ \mathbb{Q}(\sqrt{-q^k}) & \text{if } p = 2. \end{cases}$$

It follows that for any $i \in \{1, 2\}$, $\mathbb{Q}(\psi_i)$ strictly contains $\mathbb{Q}$ if and only if k is odd. In this case, for all $i \in \{1, 2\}$ we have that

$$\mathbb{Q}(\psi_i) = \begin{cases} \mathbb{Q}(\sqrt{q}) & \text{if } p \neq 2, \\ \mathbb{Q}(\sqrt{-q}) & \text{if } p = 2. \end{cases}$$

Moreover, if $p \neq 2$ then $q \equiv 1 \bmod 4$. On the other hand $q \equiv 3 \bmod 4$, when $p = 2$. Therefore $\mathbb{Q}(\psi_i) \subseteq \mathbb{Q}(e^{2\pi i/q})$, by Lemma 3.4. ☐

A straightforward consequence of Theorem D is that Theorem 2.1 holds for alternating groups.

Corollary 3.5. *Let $\pi = \{p, q\}$ be a set of two primes. Then A_n possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.*

Proof. If n does not satisfy conditions (i) and (ii) of Theorem 3, then A_n has a nontrivial rational character. If n satisfies condition (i), then there exists $\phi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ such that $\phi(1) > 1$ and such that $\mathbb{Q}(\phi) \subseteq \mathbb{Q}(e^{2\pi i/p})$. On the other hand, if n satisfies condition (ii), then there exists $\psi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ such that $\psi(1) > 1$ and such that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. ☐

4. Simple groups of Lie type

Here we prove Theorem 2.1 for simple groups of Lie type and sporadic simple groups. The following reduces us to the case of simple groups of Lie type with non-exceptional Schur multipliers.

Proposition 4.1. *Let S be a simple group of Lie type with an exceptional Schur multiplier, or let S be a sporadic group. Assume that S is not the Janko group J_4 or the Tits group ${}^2F_4(2)'$. Then S satisfies Theorem 2.1. Further, the Tits group ${}^2F_4(2)'$ satisfies Theorem 2.1 for $\pi \neq \{3, 5\}$, and the Janko group J_4 satisfies Theorem 2.1 for $\pi \notin \{\{23, 43\}, \{29, 43\}\}$.*

Proof. This can be seen using GAP and the Atlas [GAP, Atl]. ☐

When S is a simple group of Lie type, in some cases the required character $\chi \in \mathrm{Irr}(S)$ of π'-degree we produce will be a *semisimple character*. Let us recall some background on these characters.

Let $\mathcal{G}$ be a connected reductive algebraic group in characteristic p and F a Frobenius endomorphism of $\mathcal{G}$. For each rational maximal torus $\mathcal{T}$ of $\mathcal{G}$ and a character $\theta \in \mathrm{Irr}(\mathcal{T}^F)$, using Harish-Chandra induction $R^{\mathcal{G}}_{\mathcal{T}}$, one can define the Deligne-Luszlig character $R^{\mathcal{G}}_{\mathcal{T}}(\theta)$.

Let $\mathcal{G}^*$ be an algebraic group with a Frobenius endomorphism F^* such that $(\mathcal{G}, F)$ is dual to $(\mathcal{G}^*, F^*)$. Set $G := \mathcal{G}^F$ and $G^* := (\mathcal{G}^*)^{F^*}$.

Recall that if $(\mathcal{T}, \theta)$ is G-conjugate to $(\mathcal{T}', \theta')$, then $R^{\mathcal{G}}_{\mathcal{T}}(\theta) = R^{\mathcal{G}}_{\mathcal{T}'}(\theta')$. Moreover, by Proposition 13.13 of [DM91], the G-conjugacy classes of pairs $(\mathcal{T}, \theta)$ are in one-to-one correspondence with the G^*-conjugacy classes of pairs $(\mathcal{T}^*, s)$ where s is a semisimple element of G^* and $\mathcal{T}^*$ is a rational maximal torus containing s. Due to this correspondence, we can use the notation $R^{\mathcal{G}}_{\mathcal{T}^*}(s)$ for $R^{\mathcal{G}}_{\mathcal{T}}(\theta)$. For each conjugacy class (s) of semisimple elements in G^* such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, one can define a so-called semisimple character of G as follow:

$$\chi_{(s)} := \frac{1}{|W(s)|} \sum_{w \in W(s)} \varepsilon_{\mathcal{G}} \varepsilon_{\mathcal{T}^*_w} R^{\mathcal{G}}_{\mathcal{T}^*_w}(s),$$

where $W(s)$ is the Weyl group of $\mathbf{C}_{\mathcal{G}^*}(s)$, $\mathcal{T}^*_w$ is a torus of $\mathcal{G}^*$ of type w, and $\varepsilon_{\mathcal{G}} = \pm 1$ depending on whether the relative rank of $\mathcal{G}$ is even or odd, see Definition 14.40 of [DM91]. Moreover,

$$\chi_{(s)}(1) = |G^* : \mathbf{C}_{G^*}(s)|_{p'},$$

where we recall that p is the defining characteristic of G and $n_{p'}$ denote the p'-part of a positive integer n.

Lemma 4.2. *With the notation as above, let $s \in G^*$ be a semisimple element such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected. If s has order k, then $\mathbb{Q}(\chi_{(s)}) \subseteq \mathbb{Q}(e^{2\pi i/k})$.*

Proof. Let $\mathcal{T}^*$ be a rational maximal torus of $\mathcal{G}^*$ containing s. Let $\mathcal{T}$ be a rational maximal torus of $\mathcal{G}$ and $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ such that the G-conjugacy class of $(\mathcal{T}, \theta)$ corresponds to the G^*-conjugacy class of $(\mathcal{T}^*, s)$ under the correspondence described above. The multiplicative order of θ (in the group $\mathrm{Irr}(\mathcal{T}^F)$) is the same as the order of s. Therefore, the values of θ are in $\mathbb{Q}(e^{2\pi i/k})$.

We recall the character formula for $R^{\mathcal{G}}_{\mathcal{T}'}(\theta)$, which we simplify as $R_{\mathcal{T}',\theta}$:

$$R_{\mathcal{T}',\theta}(g) = \frac{1}{|\mathbf{C}^0_{\mathcal{G}}(t)^F|} \sum_{x \in \mathcal{G}^F} \theta(x^{-1}tx) Q^{\mathbf{C}^0_{\mathcal{G}}(t)}_{x\mathcal{T}'x^{-1}}(u),$$

where t is semisimple, u is unipotent, and $g = tu = ut$ is the Jordan decomposition of $g \in G$. Also, $\mathbf{C}^0_{\mathcal{G}}(t)$ is the connected component of $\mathbf{C}_{\mathcal{G}}(t)$ and $Q^{\mathbf{C}^0_{\mathcal{G}}(t)}_{x\mathcal{T}x^{-1}}$ are Green functions of $\mathbf{C}^0_{\mathcal{G}}(t)$, see [Ca85, Theorem 7.2.8]. As θ is $\mathbb{Q}(e^{2\pi i/k})$-valued and the Green functions are rational-valued, we have $\mathbb{Q}(R_{\mathcal{T}',\theta}) \subseteq \mathbb{Q}(e^{2\pi i/k})$ for every rational maximal torus $\mathcal{T}'$ of $\mathcal{G}$. The conclusion now follows from the definition of $\chi_{(s)}$. $\square$

More generally, for the conjugacy class corresponding to a semisimple element $s \in G^*$, we will denote by χ_s the character corresponding to the pair $(s, 1_{C_{G^*}(s)})$ under a fixed Jordan decomposition $\mathcal{E}(G, s) \leftrightarrow \mathcal{E}(\mathbf{C}_{G^*}(s), 1)$ of characters for G. Here $\mathcal{E}(G, s)$ denotes the rational Lusztig series of G corresponding to s and $\mathcal{E}(\mathbf{C}_{G^*}(s), 1)$ is the set of characters lying above unipotent characters of $\mathbf{C}^\circ_{G^*}(s) := (\mathbf{C}^\circ_{\mathcal{G}^*}(s))^F$. In this case, we still have

$$\chi_s(1) = |G^* : \mathbf{C}_{G^*}(s)|_{p'}.$$

However, we remark that since here $\mathbf{C}_{\mathcal{G}^*}(s)$ is no longer assumed to be connected, this indeed depends on a choice of the Jordan decomposition. Hence χ_s is only unique up to the orbit of characters corresponding to those lying over $1_{\mathbf{C}^\circ_{G^*}(s)}$.

Proposition 4.3. *Let $S \neq {}^2F_4(2)'$ be a simple group of Lie type. Then Theorem 2.1 holds for S.*

Proof. We may assume S is not one of the groups listed in Proposition 4.1 nor isomorphic to an alternating group. Further, thanks to [NT06, NT08], we may assume that $p \neq q$.

Let S be of the form $G/\mathbf{Z}(G)$, where $G = \mathcal{G}^F$ is the set of fixed points of a connected reductive algebraic group of simply connected type defined in characteristic r, under a Frobenius endomorphism F. Note that the Steinberg character St_G of G has degree a power of r, is rational-valued, and is trivial on $\mathbf{Z}(G)$. Hence, we may assume that $r = p$ is one of the primes in π.

Throughout, let $\eta \in \{\pm 1\}$ be such that $p \equiv \eta \pmod 4$, and note that $\mathbb{Q}(\sqrt{\eta p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$, using Lemma 3.4. In [GSV19, Lemma 3.3 and Theorem 3.5], a character in $\mathrm{Irr}_{\pi'}(S)$ is constructed from a character χ_s of G trivial on the center, using a semisimple element $s \in G^*$ of q-power order. In fact, we may choose s specifically to have order q. In most cases, we will see that this s of order q can further be chosen such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, yielding that $\mathbb{Q}(\chi_s) \subseteq \mathbb{Q}(e^{2\pi i/q})$, as desired, by Lemma 4.2. Throughout, let $Q \in \mathrm{Syl}_q(G^*)$.

By [MT11, Exercise 20.16], we see that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected whenever $|s|$ is relatively prime to $|\mathbf{Z}(\mathcal{G})|_{p'}$. Then if $q \nmid |\mathbf{Z}(\mathcal{G})|_{p'}$, we may choose $s \in \mathbf{Z}(Q)$ as in [GSV19, Lemma 3.3], but so that $|s| = q$, and we are done. Hence we assume $q \mid |\mathbf{Z}(\mathcal{G})|$.

Let $\mathcal{G}$ be of type A_{n-1}. Then $\mathcal{G} = \mathrm{SL}_n$ with $n = a_1 q + \cdots a_t q^t$ with $0 \leqslant a_i < q$ for $1 \leqslant i \leqslant t$. We will write $\widetilde{G} = \mathrm{GL}_n^\epsilon(p^a)$ and $G = \mathrm{SL}_n^\epsilon(p^a)$, where $\epsilon \in \{\pm 1\}$, $\epsilon = 1$ is the untwisted version $\mathrm{SL}_n(p^a)$, and $\epsilon = -1$ is the twisted version $\mathrm{SU}_n(p^a)$. Further, note that $\mathbf{Z}(G) = G \cap \mathbf{Z}(\widetilde{G})$, $\widetilde{G}^* \cong \widetilde{G}$, $S = \mathrm{PSL}_n^\epsilon(p^a) \cong [G^*, G^*]$, $G = [\widetilde{G}, \widetilde{G}] \cong [\widetilde{G}^*, \widetilde{G}^*]$, and $G^* \cong \widetilde{G}/\mathbf{Z}(\widetilde{G}) = \mathrm{PGL}_n^\epsilon(p^a)$. Throughout, we will make these identifications. Let $\widetilde{Q} \in \mathrm{Syl}_q(\widetilde{G})$. Then by [CF64, We55], we have $\widetilde{Q} = \prod_{i=1}^t Q_i^{a_i}$, where the $Q_i \in \mathrm{Syl}_q(\mathrm{GL}^\epsilon_{q^i}(p^a))$ are embedded diagonally in $\widetilde{G}$. Let $k = \min\{i | a_i > 0\}$, so that $n_q = q^k$.

First, assume that n is not a power of q. Let $s' \in \mathbf{Z}(Q_k)$ have order q. If $n \neq 2q^k$, define $\widetilde{s} \in \mathbf{Z}(\widetilde{Q})$ to be of the form $\mathrm{diag}(s', I_{n-q^k})$. If $q \mid (p^a - \epsilon)$, then s' may further be chosen to be of the form $\mu I_{q^k} \in \mathbf{Z}(\mathrm{GL}^\epsilon_{q^k}(p^a))$, where $\mu \in C_{p^a-\epsilon} \leqslant \mathbb{F}^\times_{p^{2a}}$ has order q. Then $\det(\widetilde{s}) = \det(s') = \mu^{q^k} = 1$. Otherwise, $q \nmid \widetilde{G}/G$, so $\widetilde{Q} \leqslant G$. In either case, $\widetilde{s} \in G = [\widetilde{G}^*, \widetilde{G}^*]$, so the corresponding semisimple character $\chi_{\widetilde{s}}$ of $\widetilde{G}$ is trivial on $\mathbf{Z}(\widetilde{G})$, by [NT13, Lemma 4.4]. If q is odd and $n = 2q^k$, we may instead let $\widetilde{s} \in \mathbf{Z}(\widetilde{Q})$ be of the form $\mathrm{diag}(\mu I_{q^k}, \mu^{-1} I_{q^k})$ if $q \mid (p^a - \epsilon)$ and $\mathrm{diag}(s', I_{q^k})$ otherwise, and we again see that $\widetilde{s} \in G$. Further, since the conjugacy classes of semisimple elements of $\widetilde{G}$ are determined by their eigenvalues, we see $\widetilde{s}$ is not $\widetilde{G}$-conjugate to $\widetilde{s}z$ for any nontrivial $z \in \mathbf{Z}(\widetilde{G}^*)$. But the characters of $\widetilde{G}/G$ are in bijection with elements of $\mathbf{Z}(\widetilde{G}^*)$, and $\chi_{\widetilde{s}} \otimes \widehat{z} = \chi_{\widetilde{s}z}$ for $\widehat{z} \in \mathrm{Irr}(\widetilde{G}/G)$ corresponding to $z \in \mathbf{Z}(\widetilde{G}^*)$ (see [DM91, 13.30]). Hence $\chi_{\widetilde{s}}$ is also irreducible when restricted to G. By Lemma 4.2, $\chi_{\widetilde{s}}$ has values in $\mathbb{Q}(e^{2\pi i/q})$, so this yields a character of S with degree prime to both p and q and with values in $\mathbb{Q}(e^{2\pi i/q})$, as desired.

Now assume that $n = q^k$. Then [GSV19, Lemma 3.4] yields that any character of G with degree prime to q is trivial on the center, which has size $\gcd(n, p^a - \epsilon)$. Hence it suffices to show there exists a character of G with degree prime to p and to q whose values lie in $\mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(e^{2\pi i/q})$.

Let $s \in \mathbf{Z}(Q)$ have order q, where $Q = \widetilde{Q}\mathbf{Z}(\widetilde{G})/\mathbf{Z}(\widetilde{G}) \in \mathrm{Syl}_q(G^*)$, and let $\widetilde{s} \in \widetilde{Q}\mathbf{Z}(\widetilde{G})$ be such that $\widetilde{s}\mathbf{Z}(\widetilde{G}) = s$. Then notice $\widetilde{s}^q \in \mathbf{Z}(\widetilde{G})$. Let $\zeta = e^{2\pi i/|\widetilde{s}|}$, so that the semisimple character $\widetilde{\chi}_{\widetilde{s}}$ of $\widetilde{G}$ corresponding to $\widetilde{s}$ takes its values in $\mathbb{Q}(\zeta)$, by Lemma 4.2, and lies over the $\{p,q\}'$-degree character χ_s of G. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}(e^{2\pi i/q}))$. Then σ maps ζ to ζ^m for some m with $\gcd(m, |\widetilde{s}|) = 1$. Further, $m \equiv 1 \pmod{q}$, since σ fixes qth roots of unity. In particular, $\widetilde{s}^m = \widetilde{s}z$ for some $z \in \mathbf{Z}(\widetilde{G})$. Then using [SFT18, Lemma 3.4], we have $\widetilde{\chi}_{\widetilde{s}}^\sigma = \widetilde{\chi}_{\widetilde{s}^m} = \widetilde{\chi}_{\widetilde{s}z}$, and hence $\widetilde{\chi}_{\widetilde{s}}^\sigma$ also lies over χ_s. In particular, $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}})^\sigma = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}}^\sigma) = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}z}) = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}} \otimes \widehat{z}) = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}})$. So, $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}})$ is fixed by each such σ, and therefore has values in $\mathbb{Q}(e^{2\pi i/q})$.

If q is odd and $\chi \in \mathrm{Irr}(G)$, then [SFV19, Theorem 6.1] yields that $\mathbb{Q}(\chi) = \mathbb{Q}(\mathrm{Res}^{\widetilde{G}}_G\widetilde{\chi})$ for any $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ lying over χ. Hence in this case, $\mathbb{Q}(\chi_s) \subseteq \mathbb{Q}(e^{2\pi i/q})$ as well.

If $q = 2$, then the above yields $\mathbb{Q}(\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_{\widetilde{s}})) = \mathbb{Q}$, so [SFV19, Theorem 6.1] yields that $\mathbb{Q}(\chi_s) \subseteq \mathbb{Q}(\sqrt{\eta p})$, completing the proof in this case.

Hence we may assume $\mathcal{G}$ is not of type A_{n-1}, and therefore $|\mathbf{Z}(\mathcal{G})|_{p'}$ is a power of q. Here if $q \neq 2$, we have $(\mathcal{G}, q) = (E_6, 3)$.

In the latter case, G is the simply connected type group $E_6^\epsilon(p^a)_{sc}$, where $\epsilon \in \{\pm 1\}$, $\epsilon = 1$ corresponds to the untwisted version, and $\epsilon = -1$ corresponds to the twisted version. Let $\chi_s \in \mathcal{E}(G, s)$ be the character constructed in [GSV19] but so that s has order $q = 3$. Using [SFT18, Lemma 3.4], we have $\mathcal{E}(G, s)$ is stable under any Galois automorphism σ that fixes the field $\mathbb{Q}(e^{2\pi i/3})$. Now, using [TZ04, Theorem 1.8 and Lemma 2.6], we see that any character of G takes integer values on unipotent elements, and hence any Gelfand-Graev character of G is rational-valued, since they are unipotently supported. Hence by [SFT18, Proposition 3.8], χ_s is fixed by σ as well, and hence has values in $\mathbb{Q}(e^{2\pi i/3})$.

We may therefore take $q = 2$, p odd, and $\mathcal{G}$ to be of type B_n, C_n, D_n, or E_7. Again recall that by [GSV19], there exists a character of degree prime to $\{2, p\}$. So, we aim to show that such a character can be found with the values as stated. The data available in CHEVIE and [Lüb07] yield that the odd-degree characters of ${}^3D_4(p^a)$ and $E_7(p^a)$ are rational-valued, and hence we assume S is $B_n(p^a)$ with $n \geqslant 2$, $C_n(p^a)$ with $n \geqslant 3$, $D_n(p^a)$ with $n \geqslant 4$, or ${}^2D_n(p^a)$ with $n \geqslant 4$.

If S is $C_n(p^a)$, then we may take $G = \mathrm{Sp}_{2n}(p^a)$ and $S = G/\mathbf{Z}(G)$. By [MS16, Theorem 7.7], any odd-degree irreducible character χ is either in the principal series corresponding to a pair (T, λ) where T is a maximally split torus of G and $\lambda \in \mathrm{Irr}(T)$ satisfies $\lambda^2 = 1$, or $q \equiv 3 \pmod{4}$ and χ is in a Harish-Chandra series corresponding to (L, λ), where $L \cong \mathrm{Sp}_2(p^a) \times T_1$ with T_1 a maximally split torus of $\mathrm{Sp}_{2(n-1)}(p^a)$. In the latter case, we further have $\lambda = \psi \otimes \lambda_1$, where $\lambda_1^2 = 1$ and ψ is one of the two characters of degree $\frac{q-1}{2}$ of $\mathrm{Sp}_2(p^a)$, which take values in $\mathbb{Q}(\sqrt{\eta p})$. In either case, [SFT20, Theorem B] combined with [SF19, Theorem 3.8] yield that any character of odd degree of G has values in $\mathbb{Q}(\sqrt{\eta p})$, and hence in $\mathbb{Q}(e^{2\pi i/p})$, so we are done in this case.

Finally, if S is $B_n(p^a)$, $D_n(p^a)$, or ${}^2D_n(p^a)$, we adjust the argument from the case of E_6 above. In particular, the character constructed in [GSV19] may be chosen to come from a semisimple character $\chi_s \in \mathcal{E}(G, s)$, where $s^2 = 1$. Then $\mathcal{E}(G, s)$ is stable under any element of $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$ using [SFT18, Lemma 3.4]. Further, using [TZ04, Corollary 8.3 and Lemma 2.6] and arguing as in the case of E_6, we see that every Gelfand-Graev character of G takes its values in $\mathbb{Q}(\sqrt{\eta p})$. Then [SFT18, Proposition 3.8] yields that χ_s takes its values in $\mathbb{Q}(\sqrt{\eta p})$ as well, completing the proof. $\square$

5. Rational characters of π'-degree in solvable groups

In this Section we prove Theorem B, namely we characterize when a solvable group G has a π'-degree rational character, where $\pi = \{2, q\}$ is a pair of primes. We first note that if G has a normal Hall $\{2, q\}$-subgroup, then the solution to the characterization problem is pretty simple. We thank G. Navarro for pointing out to us a simplified version of a previous argument.

Lemma 5.1. *Let G be a finite group and $p < q$ be two primes. Set $\pi = \{p, q\}$. Suppose that $H \lhd G$ where $H \in \mathrm{Hall}_\pi(G)$ and G/H has odd order. Then G has a nontrivial irreducible character of π'-degree with values in $\mathbb{Q}(e^{2\pi i/p})$ if, and only if, H/H' has order divisible by p.*

Proof. Notice that to prove both implications we may assume that H is abelian. If H has order divisible by p, then let $\mathbf{1}_H \neq \lambda \in \mathrm{Irr}(H)$ be linear with $o(\lambda)$ equal to p. By [Isa06, Corollary 6.27], let $\hat{\lambda} \in \mathrm{Irr}(G_\lambda)$ be the only extension of λ such that $o(\hat{\lambda}) = o(\lambda)$. In particular, $\mathbb{Q}(\hat{\lambda}) = \mathbb{Q}(e^{2\pi i/p})$ and hence $\mathbf{1}_G \neq \chi = (\hat{\lambda})^G \in \mathrm{Irr}(G)$ has values in $\mathbb{Q}(e^{2\pi i/p})$. Since G/H is a π'-group, then χ has π'-degree, as wanted.

Suppose now that $\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)$ has π'-degree and $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ and p does not divide $|H|$. If $p = 2$, then χ is a nontrivial rational characters of an odd order group. This would contradict Burnside's theorem. If $p > 2$, then $|H| = q^b$ and since G/H is a π'-group, then G is a p'-group. In particular $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p}) \cap \mathbb{Q}(e^{2\pi i/|G|}) = \mathbb{Q}$. Since G/H is an odd order group by hypothesis, and $q > p > 2$ also by hypothesis, then we get again a contradiction with Burnside's theorem. □

Remark 5.2. We note that if $p = 2$, then the condition on the order of G/H is trivially satisfied. However, it is a necessary condition in general: if p is an odd prime then the group $G = \mathsf{C}_\mathsf{p} \rtimes \mathsf{C}_{p-1}$ where the action is faithful has a π'-degree rational irreducible character for every $\pi = \{p, q\}$ with q a divisor of $p - 1$.

For an arbitrary group G and pair of primes $\pi = \{p, q\}$ we will denote by $\mathcal{X}_{\pi',p}(G)$ the set of π'-degree irreducible characters of G with values in $\mathbb{Q}(e^{2\pi i/p})$. If G is π-separable, then it is shown in [NV12] that the set $\mathcal{X}_{\pi',p}(G)$ consists entirely of monomial characters given that $|\mathbf{N}_G(H)/H|$ is odd for $H \in \mathrm{Hall}_\pi(G)$. (In [NV12] this fact is proven in the case where π consists of a single prime, but it easily generalizes to any set of primes, see [Val16, Remark 2.9].)

Theorem 5.3. *Let G be a π-separable group, where $\pi = \{p, q\}$ is a pair of primes. Let $H \in \mathrm{Hall}_\pi(G)$. Write $N = \mathbf{N}_G(H)$ and suppose that N/H has odd order. Define a map*

$$\Omega\colon \mathcal{X}_{\pi',p}(G) \to \mathcal{X}_{\pi',p}(N)$$

in the following way: If $\chi \in \mathcal{X}_{\pi',p}(G)$, choose a pair (U, λ) where $H \leqslant U \leqslant G$ and $\lambda \in \mathrm{Irr} U$ linear such that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}(e^{2\pi i/p})$ and $\lambda^G = \chi$, then set $\Omega(\chi) = (\lambda_{U\cap N})^N$. Then Ω is a bijection.

Proof. The existence of the pair (U, λ) is guaranteed by Theorem A of [NV12] (see [Val16, Remark 2.9]). Then use [Isa90, Theorem C] to construct Ω. If Ω is well-defined then [Isa90, Theorem C] assures Ω is injective. It is not difficult to show that Ω is well-defined. To prove that Ω is surjective use the π-version of [IN08, Theorem 3.3]; note that such theorem admits a π-version for solvable groups as [IN08, Theorem 2.1 and Corollary 2.2] also do. (For more details see the proof of [Val16, Theorem 2.13].) □

As an immediate consequence of Lemma 5.1 and Theorem 5.3, we can derive the following result.

Corollary 5.4. *Let $p < q$ be two primes and set $\pi = \{p, q\}$. Let G be a π-separable group and $H \in \mathrm{Hall}_\pi(G)$. Assume that $\mathbf{N}_G(H)/H$ has odd order. Then G has a nontrivial irreducible character of π'-degree with values in $\mathbb{Q}(e^{2\pi i/p})$ if, and only if, H/H' has order divisible by p.*

If we let $p = 2$ in Corollary 5.4 then we obtain precisely Theorem B of the Introduction since the condition on the order of $\mathbf{N}_G(H)/H$ becomes superfluos and π-separability of G is equivalent to solvability of G by Burnside's $p^a q^b$ and Feit-Thompson's odd-order theorems.

References

[Ca85] R. W. Carter, *'Finite groups of Lie type. Conjugacy classes and complex characters'*, Wiley and Sons, New York et al, 1985, 544 pp.

[CF64] R. Carter and P. Fong, The Sylow 2-subgroups of the finite classical groups. *J. Algebra*, 1 (1964), 139–151.

[GAP] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4; 2004, `http://www.gap-system.org.`

[Atl] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *'Atlas of finite groups'*, Clarendon Press, Oxford, 1985.

[DM91] F. Digne and J. Michel, *'Representations of finite groups of Lie type'*, London Mathematical Society Student Texts 21, 1991, 159 pp.

[GSV19] E. Giannelli, A. A. Schaeffer Fry, and C. Vallejo Rodríguez, Characters of π'-degree, *Proc. Amer. Math. Soc.* **147** (2019), 4697–4712.

[Isa90] I. M. Isaacs. Hall subgroup normalizers and character correspondences in M-groups. *Proc. Amer. Math. Soc.* **109**, no. 3 (1990), 647–651.

[Isa06] I. M. Isaacs, '*Character theory of finite groups*', AMS Chelsea Publishing, Providence, Rhode Island, 2006.

[IN08] I. M. Isaacs, G. Navarro. Character sums and double cosets. *J. Algebra* **320** (2008), 3749–3764.

[JK81] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[Lüb07] F. Lübeck, Character degrees and their multiplicities for some groups of Lie type of rank < 9,`http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/DegMult/index.html`.

[MS16] G. Malle and B. Späth, Characters of odd degree, *Ann. of Math.* (2) **184** 3 (2016), 869–908.

[MT11] G. Malle and D. Testerman. *Linear algebraic groups and finite groups of Lie type.* Cambridge University Press, Cambridge, 2011.

[NT06] G. Navarro and P. H. Tiep, Characters of p'-degree with cyclotomic field of values, *Proc. Amer. Math. Soc.* **134** (2006), 2833–2837.

[NT08] G. Navarro and P. H. Tiep, Rational irreducible characters and rational conjugacy classes in finite groups, *Trans. Amer. Math. Soc.* **360** (2008), 2443–2465.

[NT13] G. Navarro and P. H Tiep, Characters of relative p'-degree over normal subgroups, *Ann. of Math. (2)* **178**, No. 3 (2013), 1135–1171.

[NV12] G. Navarro and C. Vallejo, Certain monomial characters of p'-degree, *Arch. Math.* **99** (2012), 407–411.

[Ol94] J. B. Olsson, *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 20, 1994.

[SF19] A. A. Schaeffer Fry, Action of Galois automorphisms on Harish-Chandra series and Navarro's self-normalizing Sylow 2-subgroup conjecture, *Trans. Amer. Math. Soc.* **372** no. 1 (2019), pp. 457–483.

[SFT18] A. A. Schaeffer Fry and J. Taylor, On self-normalizing Sylow 2-subgroups in type A, *J. Lie Theory* (2018), 28:1, 139-168.

[SFT20] A. A. Schaeffer Fry and J. Taylor, Galois Automorphisms and classical groups, Preprint.

[SFV19] A. A. Schaeffer Fry and C. R. Vinroot, Fields of character values for finite special unitary groups, *Pacific J. Math.* **300** (2019), No. 2, 473–489.

[TZ04] P. H. Tiep and A. E. Zalesskii, Unipotent elements of finite groups of Lie type and realization fields of their complex representations, *J. Algebra* **271**:1 (2004), 327–390.

[Val16] C. Vallejo Rodríguez. *Characters, correspondences and fields of values of finite groups*. Ph.D. Thesis (2016), available at `http://roderic.uv.es/handle/10550/53934`.

[We55] A. J. Weir, Sylow p-Subgroups of the classical groups over finite fields with characteristic prime to p, *Proc. Amer. Math. Soc.* **6**, No. 4 (1955), 529–533.

Dipartimento di Matematica e Informatica U. Dini, Viale Morgagni 67/a, Firenze, Italy
Email address: `eugenio.giannelli@unifi.it`

Department of Mathematics, The University of Akron, Akron, OH 44325, USA
Email address: `hungnguyen@uakron.edu`

Dept. Mathematical and Computer Sciences, MSU Denver, Denver, CO 80217, USA
Email address: `aschaef6@msudenver.edu`

Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain
Email address: `carolina.vallejo@uam.es`