Odd-Degree Characters and Self-Normalizing Sylow 2-Subgroups: A Reduction to Simple Groups

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Abstract

Let G be a a finite group, p a prime, and P a Sylow p-subgroup of G. A recent refinement, due to G. Navarro, of the McKay conjecture suggests that there should exist a bijection between irreducible characters of p'-degree of G and $N_G(P)$ which commutes with certain Galois automorphisms. In this paper, we explore one of the consequences of this refinement, namely a way to read off from the character table of G whether a Sylow 2-subgroup of G is self-normalizing. We provide a reduction to finite simple groups and begin an investigation of the problem for simple groups.

Keywords: local-global conjectures, characters, McKay conjecture, self-normalizing Sylow subgroups, finite simple groups

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1 Introduction

For many years, much of the representation theory of finite groups has been devoted to proving various "local-global" conjectures, which relate certain invariants of a finite group with those of particular subgroups. One of the oldest and simplest (yet no less elusive) of these conjectures is often attributed to McKay [20], and asserts that for G a finite group, $\ell||G|$ a prime, and $P \in \mathrm{Syl}_\ell(G)$, the number $|\mathrm{Irr}_{\ell'}(G)|$ of irreducible complex characters of G with degree prime to ℓ is equal to the number $|\mathrm{Irr}_{\ell'}(N_G(P))|$ of such characters of $N_G(P)$.

Many refinements to the McKay conjecture have been proposed, and a reduction theorem has been proved in [16], with the hope of providing not only a method by which to prove it, but also a better understanding of the deeper underlying reason behind it. One such refinement is due to G. Navarro [22], and says that for the proper choices of Galois automorphisms σ in $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, the number of members of $\mathrm{Irr}_{\ell'}(G)$ fixed by σ should be the same as that for $\mathrm{Irr}_{\ell'}(N_G(P))$. Here for n a positive integer, we write $\mathbb{Q}_n$ for the extension field $\mathbb{Q}(e^{2\pi i/n})$ of $\mathbb{Q}$. That is, $\mathbb{Q}_n$ denotes the field obtained by adjoining a primitive nth root of unity.

In [22], Navarro shows (among other interesting consequences) that the validity of his conjecture would lead to a necessary and sufficient condition for a Sylow ℓ-subgroup to be self-normalizing. For odd primes, Navarro-Tiep-Turull [25] showed that this particular consequence is true without assuming the conjecture. Namely, they show that for ℓ an odd prime, $P \in \mathrm{Syl}_\ell(G)$ is self-normalizing if and only if there is no nontrivial irreducible ℓ-rational character of G of degree prime to ℓ.

For $\ell = 2$, this statement does not hold, and proving the correct corresponding statement requires a different approach than used in [25], which is made evident by the fact that the consequence in [22] of Navarro's conjecture takes a much different form when $\ell = 2$. Moreover, in the case of odd ℓ, it has been shown (see [13]) that a nonsolvable group can only have a self-normalizing Sylow ℓ-subgroup if $\ell = 3$ and it contains a composition factor isomorphic to $PSL_2(3^{3^f})$ for some $f \geq 1$, whereas there are many examples of simple groups with self-normalizing Sylow 2-subgroups. (See, for example, the treatment of Sylow 2-subgroups of simple groups in [17], [4], and [29]).

The problem of interest in this paper is the following, which is the corresponding consequence to Navarro's conjecture for $\ell = 2$:

Problem 1. Let G be a finite group and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ fixing 2-roots of unity and squaring $2'$-roots of unity. Then G has a self-normalizing Sylow 2-subgroup if and only if every irreducible complex character of G with odd degree is fixed by σ.

Aside from providing yet more evidence for the validity of Navarro's conjecture, and therefore the McKay conjecture, a direct consequence of Problem 1 would be that that one can read off from the character table of an arbitrary group whether or not a Sylow-2 subgroup is self-normalizing (which can already be done for odd primes by [25]).

The current paper provides a first look at this problem. The main result is a reduction to (quasi-)simple groups (see Theorem 3.7 below). In Section 2, we present the required statements for simple groups and define a group to be "SN2S-Good" if it satisfies these statements, and in Section 3, we show that, indeed, it suffices to prove that every nonabelian simple group is SN2S-Good. In Section 4, we begin an investigation into the validity of the required statements for these groups. We show that the statements hold for alternating and sporadic groups, as well as many finite groups of Lie type. Though we do not complete the proof of the statements in the case of groups of Lie type here, we discuss the progress and the various issues that arise, and hope to treat the remaining cases in a forthcoming paper. In particular, we nearly complete the case of groups of Lie type in characteristic 2, including type A, for which we have used results regarding generalized Gelfand-Graev characters.

1.1 Notation

Throughout, σ will always denote the Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ as in Problem 1. That is, σ fixes 2-roots of unity and squares $2'$-roots of unity. We will write $\mathrm{Syl}_2(X)$ for the set of Sylow 2-subgroups of the finite group X.

As usual, $\mathrm{Irr}(X)$ will denote the set of irreducible ordinary characters of the group X. Given a finite group X and a subgroup $Y \leq X$, the restriction of $\chi \in \mathrm{Irr}(X)$ to Y will be denoted $\chi|_Y$ and for $\theta \in \mathrm{Irr}(Y)$, θ^X will denote the induced character of θ to X. If $\chi|_Y = \theta \in \mathrm{Irr}(Y)$, we say θ extends (or is extendible) to X. $\mathrm{Irr}_{2'}(X)$ will denote the subset of $\mathrm{Irr}(X)$ comprised of irreducible characters of odd degree. Moreover, we will denote by $\mathrm{Irr}(X|\theta)$, (resp. $\mathrm{Irr}_{2'}(X|\theta)$), the subset of $\mathrm{Irr}(X)$ (resp. $\mathrm{Irr}_{2'}(X)$) of characters containing θ as a constituent when restricted to X. That is, by Frobenius reciprocity, $\mathrm{Irr}(X|\theta)$ is the set of irreducible constituents of θ^X. If the group X acts on a set Ω, we write $\mathrm{stab}_X(\alpha)$ for the stabilizer in X of $\alpha \in \Omega$.

2 Statements for Simple Groups

We present here the conjectures for simple and quasisimple groups that will suffice for proving Problem 1. The two conditions that we require for simple groups are as follows:

Condition 2.1. *Let G be a finite quasisimple group with center $Z := Z(G)$ and Q a finite 2-group acting on G as automorphisms. Assume $P/Z \in \mathrm{Syl}_2(G/Z)$ is Q-invariant and $C_{N_G(P)/P}(Q) = 1$. Then for any Q-invariant, σ-fixed $\lambda \in \mathrm{Irr}(Z)$, we have $\chi^\sigma = \chi$ for any Q-invariant $\chi \in \mathrm{Irr}_{2'}(G|\lambda)$.*

We note that the condition $C_{N_G(P)/P}(Q) = 1$ is equivalent to GQ/Z having a self-normalizing Sylow 2-subgroup (see, for example, [25, Lemma 2.1 (ii)]).

Condition 2.2. *Let G be a finite nonabelian simple group, Q a finite 2-group acting on G as automorphisms, and $P \in \mathrm{Syl}_2(G)$ be Q-invariant. If every Q-invariant $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ, then $C_{N_G(P)/P}(Q) = 1$.*

Note that Condition 2.1 provides a converse to Condition 2.2, taking $Z = 1$, $\lambda = 1_Z$. We combine the conditions in the following definition:

Definition 1. Let S be a finite nonabelian simple group. We will say S is "SN2S-Good" if the following hold:

- S satisfies Condition 2.2 and
- if G is a quasisimple group with $G/Z(G) \cong S$, then G satisfies Condition 2.1.

3 Reductions

In this section, we reduce Problem 1 to the case of simple groups. Namely, we show that it suffices to prove that every simple group is SN2S-Good, as defined in Section 2. We begin with a lemma following directly as a consequence of SN2S-Goodness.

Lemma 3.1. *Let G be a direct product of nonabelian simple groups which are SN2S-Good. Let Q be a finite 2-group acting on G as automorphisms, and P a Q-invariant Sylow 2-subgroup of G. Then $C_{N_G(P)/P}(Q) = 1$ if and only if every Q-invariant $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ.*

Proof. We largely follow [25, Theorem 4.2]. We induct on $|G|$. As in [25, Theorem 4.2], we write G as the direct product $G = X_1 \times ... \times X_n$, where each X_i is the direct product of simple groups in the Q-orbit of some T_j in the set $\mathcal{U} = \{T_1, ..., T_a\}$ of simple normal subgroups of G which are SN2S-Good. Then each X_i is Q-invariant, so we may inductively assume that Q is transitive on the elements of $\mathcal{U}$. Then by [25, Theorem 4.1(ii)], we have $\psi \in \mathrm{Irr}(T_1)$ is Q_1-invariant if and only if $\psi^{u_1} \cdot ... \cdot \psi^{u_a}$ is Q-invariant, where $Q_1 := \mathrm{stab}_Q(T_1)$ and $u_1, ..., u_a$ is a transversal for the right cosets of Q_1 in Q with $T_i = T_1^{u_i}$.

Now, assume every Q-invariant $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ and let $\psi \in \mathrm{Irr}_{2'}(T_1)$ be Q_1-invariant. Then in particular, $\psi^{u_1} \cdot ... \cdot \psi^{u_a}$ is σ-fixed, and hence so is ψ. So every Q_1-invariant $\psi \in \mathrm{Irr}_{2'}(T_1)$ is σ-fixed, and we know by Condition 2.2 that $C_{N_{T_1}(P_1)/P_1}(Q_1) = 1$, where $P_1 := P \cap T_1$, and hence $C_{N_G(P)/P}(Q) = 1$ by [25, Theorem 4.1(i)].

Conversely, suppose that $C_{N_G(P)/P}(Q) = 1$, so that $C_{N_{T_1}(P_1)/P_1}(Q_1) = 1$ by [25, Theorem 4.1(i)]. Then by Condition 2.1, we know that every Q_1-invariant odd character of T_1 is fixed by σ. Now let $\chi \in \mathrm{Irr}_{2'}(G)$ be Q-invariant. Then we can write $\chi = \psi_1 \cdot ... \cdot \psi_a$ for $\psi_i \in \mathrm{Irr}_{2'}(T_i)$. Let $q_1 \in Q_1 = \mathrm{stab}_Q(T_1)$. Then $z\psi_1^{q_1} = (\chi|_{T_1})^{q_1} = \chi^{q_1}|_{T_1} = \chi|_{T_1} = z\psi_1$, where $z = \psi_2(1) \cdot ... \cdot \psi_a(1) \in \mathbb{Z}_{\geq 0}$. Then we see that ψ_1 is Q_1-invariant, and hence $\psi_1^\sigma = \psi_1$. Now, by reordering the T_i's, we see by repeating the argument that each $\psi_i^\sigma = \psi_i$, and hence $\chi^\sigma = \chi$, as desired.

□

The following theorem is crucial in one direction of our reduction, namely the proof of Theorem 3.5 below. It follows directly from a result of Benard and Schacher [1], whose proof requires significant machinery.

Theorem 3.2. *Let $\chi \in \mathrm{Irr}_{2'}(G)$ be fixed by σ, and let $\mathbb{K}$ be the field $\mathbb{Q}_{|G|} \supseteq \mathbb{K} \supseteq \mathbb{Q}(\chi) \supseteq \mathbb{Q}$ of fixed points of $\mathbb{Q}_{|G|}$ under σ. Then χ can be afforded by an absolutely irreducible $\mathbb{K}$-representation.*

Proof. Write $m := m_{\mathbb{Q}}(\chi)$ for the Schur index of χ over $\mathbb{Q}$, so that by [15, Theorem (10.2) (d) and (f)] $m\chi$ is afforded by an irreducible $\mathbb{Q}(\chi)$-representation and for any field $\mathbb{C} \supseteq F \supseteq \mathbb{Q}$, $m_F(\chi)|m$. (For a discussion of Schur indices, see for example [15, Chapter 10].) Note that since $\chi(1)$ is odd, it must be that m is also odd (see, for example, [15, (10.2)(h)]). Moreover, by [1, Theorem 1'], $\mathbb{Q}(\chi)$ contains a primitive m'th root of unity, ζ_m. Then since m is odd, we have $\zeta_m^\sigma = \zeta_m^2$ by the definition of σ. We therefore see that $m = 1$, as $\mathbb{Q}(\chi)$ is comprised of fixed points under σ. Hence χ is afforded by an absolutely irreducible $\mathbb{Q}(\chi)$ representation, which completes the proof. □

As is often the case, it will be useful in our reduction (see Theorem 3.5 below) to replace a character triple (G, N, θ) (i.e. $N \lhd G$ with $\theta \in \mathrm{Irr}(N)$ invariant under G) with a more convenient "isomorphic" triple which does not change the isomorphism class of G/N or certain aspects of the character theory for G over θ, but for which N is central. (See [15, Chapter 11] for details on character triples and isomorphisms of character triples.) In our reduction, we hope that this can be done in such a way that σ-invariance is not affected. The following theorem will allow us to do this.

Theorem 3.3. *Let (Γ, N, θ) be a character triple such that $\theta \in \mathrm{Irr}_{2'}(N)$ is σ-fixed. Suppose $N < G \lhd \Gamma$ with G/N perfect. Then there is a character triple (G^*, N^*, θ^*) isomorphic to (G, N, θ) satisfying*

(i) Γ acts as automorphisms on G^* and centralizes N^*, with $((gN)^*)^\gamma = (g^\gamma N)^*$ for every $g \in G, \gamma \in \Gamma$;

(ii) G^* is perfect and $N^* \leq Z(G^*)$;

(iii) θ^* is σ-fixed;

(iv) $(\chi^*)^\gamma = (\chi^\gamma)^*$ for every $\gamma \in \Gamma$ and $\chi \in \mathrm{Irr}(G|\theta)$;

(v) If $\chi \in \mathrm{Irr}(G|\theta)$, then χ is σ-fixed if and only if χ^* is σ-fixed.

Here we use $*$ to denote both the isomorphism $*\colon G/N \to G^*/N^*$ and the bijection $*\colon \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$.

Proof. Let $\mathbb{K}$ be as in Theorem 3.2 with respect to θ. Then θ can be afforded by an absolutely irreducible representation $\mathfrak{Y}$ for N over $\mathbb{K}$. An argument nearly identical to the proof of [25, Theorem 5.1] with F replaced by $\mathbb{K}$ now yields the result, but we include the argument for completeness.

By the argument in [15, Theorem (11.2)], there is a projective $\mathbb{K}\Gamma$-representation $\mathfrak{X}$ such that $\mathfrak{X}(n) = \mathfrak{Y}(n)$, $\mathfrak{X}(ng) = \mathfrak{X}(n)\mathfrak{X}(g)$, and $\mathfrak{X}(gn) = \mathfrak{X}(g)\mathfrak{X}(n)$ for all $n \in N, g \in \Gamma$. Denote by α the factor set for $\mathfrak{X}$, so that $\alpha(g, n) = \alpha(n, g) = 1$ and $\alpha(gn, hm) = \alpha(g, h)$ for all $g, h \in \Gamma$ and $n, m \in N$. Writing $\widetilde{\Gamma} := \Gamma \times \mathbb{K}^\times$ with multiplication $(g_1, k_1)(g_2, k_2) = (g_1g_2, \alpha(g_1, g_2)k_1k_2)$ yields a group with subgroups $\widetilde{H} = H \times \mathbb{K}^\times$ for each $H \leq \Gamma$.

The natural surjection $\pi\colon \widetilde{\Gamma} \to \Gamma$ given by $(g, k) \mapsto g$ is a homomorphism with kernel $\ker \pi = \widetilde{1} := 1 \times \mathbb{K}^\times$, so $\widetilde{\Gamma}/\widetilde{1} \cong \Gamma$. Then by the isomorphism theorems, it follows that $\widetilde{1}, \widetilde{G}, \widetilde{N}$, and $N \times 1$ (which we call N, with an abuse of notation) are all normal in $\widetilde{\Gamma}$. Also, note that $\widetilde{N} = N \times \widetilde{1}$, $\widetilde{1} \leq Z(\widetilde{\Gamma})$, and $\widetilde{N}/N \leq Z(\widetilde{\Gamma}/N)$, where here we identify $N \leq \widetilde{\Gamma}$ with $N \times 1$.

Now, let $G_1 \leq \widetilde{G}$ such that $G_1/N = (\widetilde{G}/N)'$, so that $G_1\widetilde{N} = \widetilde{G}$ and G_1/N is perfect, since $\widetilde{G}/\widetilde{N} \cong G/N$ is perfect.

Now, G_1/N is finite by Schur's lemma, so G_1 is finite. (Indeed, we have that X' is finite if $X/Z(X)$ is finite. But $(\widetilde{G}/N)/(\widetilde{N}/N) \cong \widetilde{G}/\widetilde{N} \cong G/N$ is finite, and $\widetilde{N}/N$ is central.) Notice that

G_1 is normal in $\widetilde{\Gamma}$, so $\widetilde{\Gamma}$ acts on G_1 by conjugation. Moreover, $\widetilde{1}$ is in the kernel of this action, so $\Gamma \cong \widetilde{\Gamma}/\widetilde{1}$ acts on G_1.

For $g \in \Gamma, k \in \mathbb{K}$, define $\widetilde{\mathfrak{X}}(g,k) := k\mathfrak{X}(g)$, so that $\widetilde{\mathfrak{X}}$ is a $\mathbb{K}$-representation for $\widetilde{\Gamma}$ satisfying $\widetilde{\mathfrak{X}}(n) = \mathfrak{Y}(n)$ for $n \in N$. Let τ be the character of G_1 afforded by $\widetilde{\mathfrak{X}}|_{G_1}$. Then τ is certainly $\widetilde{\Gamma}$-invariant. Moreover, τ takes values in $\mathbb{K}$, so is fixed by σ.

Write $N_1 := G_1 \cap \widetilde{N}$, $\mathbb{K}_1 := N_1 \cap \widetilde{1}$, so that $N_1 = N \times \mathbb{K}_1$, and define $\lambda \in \mathrm{Irr}(N_1)$ by $\lambda(n,k) = k$ for $k \in \mathbb{K}_1$, $n \in N$ and $\theta_1 := \theta \times 1 \in \mathrm{Irr}(N_1)$. Note that both λ and θ_1 are fixed by σ (since θ is fixed by σ and λ takes values in $\mathbb{K}$) and that $\tau|_{N_1} = \lambda\theta_1$ and $\tau|_N = \theta$. Since $\theta \in \mathrm{Irr}(N)$, it follows that $\tau \in \mathrm{Irr}(G_1)$.

Now, note that the map $G_1 \to G$ given by $g \mapsto \pi(g)$ is a surjective homomorphism with kernel $\mathbb{K}_1$. Note that $\mathbb{K}_1 \leq \ker\theta_1$, so $(G_1, N_1, \theta_1) \cong (G, N, \theta)$ are isomorphic character triples, by [15, Theorem (11.26)]. Moreover, the construction of this isomorphism of character triples preserves the field of values of characters (and hence σ-invariance) and commutes with conjugation by Γ.

Moreover, by the remark after [15, Theorem (11.27)], $(G_1, N_1, \lambda) \cong (G_1, N_1, \theta_1)$ is an isomorphism of character triples, since $\lambda\theta_1$ is extendible to $\tau \in \mathrm{Irr}(G_1)$. Combining these, we have an isomorphism $\star\colon (G, N, \theta) \to (G_1, N_1, \lambda)$, where for $\chi \in \mathrm{Irr}(G|\theta)$, the corresponding character in $\mathrm{Irr}(G_1|\lambda)$ is $\chi^\star = \chi\tau$. Since $\tau^\sigma = \tau$, it is clear that $\chi^\star$ is fixed by σ if and only if χ is. Moreover, the bijection $\chi \mapsto \chi^\star$ commutes with the action of Γ, since τ is Γ-invariant.

Since λ is trivial on N, applying [15, Theorem (11.26)] again yields an isomorphism $(G_1, N_1, \lambda) \cong (G_1/N, N_1/N, \lambda)$, which preserves the field of values and commutes with the action of Γ. Hence writing $G^* := G_1/N, N^* := N_1/N$ and $\theta^* := \lambda$, we have an isomorphic character triple (G^*, N^*, θ^*) with the desired properties.

□

Lemma 3.4. *Let G be a finite group and $N \lhd G$ such that G/N is an abelian 2-group. Suppose $\theta \in \mathrm{Irr}_{2'}(N)$ is G-invariant and fixed by σ. Then every $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ is also fixed by σ.*

Proof. Let G, N, and θ be as in the statement, and let $\chi \in \mathrm{Irr}_{2'}(G|\theta)$. Recall that by Clifford theory, $\chi|_N = e\theta$ for some positive integer e which divides $[G:N]$. Since $\chi(1)$ and $\theta(1)$ are both odd and $[G:N]$ is a power of 2, it follows that $\chi|_N = \theta$. Note that $\lambda := \det\theta$ is then also fixed by σ and extendible to G. That is, $\lambda^\sigma = \lambda$ and there is some $\mu \in \mathrm{Irr}(G)$ with $\mu|_N = \lambda$. (Here $\det\theta$ is as in [15, Problem (2.3)].) Then by Gallagher's theorem, $\mathrm{Irr}(G|\theta) = \{\beta\chi | \beta \in \mathrm{Irr}(G/N)\}$ and $\mathrm{Irr}(G|\lambda) = \{\beta\mu | \beta \in \mathrm{Irr}(G/N)\}$.

Let $g \in G$. As G can be written $G = P_1N$ for $P_1 \in \mathrm{Syl}_2(G)$, we may write $g = hn$ for $h \in P_1, n \in N$. Then

$$\mu(g) = \mu(hn) = \mu(h)\mu(n) = \mu(h)\lambda(n)$$

and therefore

$$\mu^\sigma(g) = \mu(h)^\sigma\lambda(n)^\sigma = \mu(h)\lambda(n) = \mu(g),$$

where the second-to-last equality follows from the observation that λ is fixed by σ and the fact that $h \in P_1$ is a 2-element (and hence $\mu(h)$ is a 2-power root of unity, so is fixed by σ by definition). It follows that $\mu^\sigma = \mu$, so $\beta\mu$ is σ-fixed for each $\beta \in \mathrm{Irr}(G/N)$, since the linear character β again takes values which are 2-power roots of unity. Hence we see that every member of $\mathrm{Irr}(G|\lambda)$ is σ-fixed. In particular, $\det\chi$ lies above $\det\theta = \lambda$, so is σ-fixed.

Since $\chi^\sigma \in \mathrm{Irr}(G|\theta^\sigma) = \mathrm{Irr}(G|\theta)$, we see $\chi^\sigma = \beta\chi$ for some $\beta \in \mathrm{Irr}(G/N)$. Then as β is linear,

$$\det\chi = (\det\chi)^\sigma = \det(\chi^\sigma) = \det(\beta\chi) = (\det\beta)^{\chi(1)}(\det\chi)^{\beta(1)} = \beta^{\chi(1)}\det\chi.$$

Hence since $\chi(1)$ is odd and the values of β are all 2-power roots of unity, we see that $\beta = 1_G$. This shows $\chi^\sigma = \chi$, completing the proof. □

We are now ready to prove the first direction of our reduction. We note that the basic structure of the proof will be analogous to [25, Theorem 6.1], though many of the details are quite different, requiring the above lemmas. Recall that a group X is said to be *involved* in a group G if X is isomorphic to H/K for some subgroups H and K of G such that K is a normal subgroup of H.

Theorem 3.5. *Let G be a finite group and Q a finite 2-group acting on G as automorphisms. Let $N \lhd G$ be a normal Q-invariant subgroup and assume Condition 2.1 for every finite quasisimple group X such that $X/Z(X)$ is involved in G/N. Let $P/N \in \mathrm{Syl}_2(G/N)$ be Q-invariant and assume that $C_{N_G(P)/P}(Q) = 1$. Let $\theta \in \mathrm{Irr}_{2'}(N)$ be σ-fixed and PQ-invariant. Then every Q-invariant $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ is also σ-fixed.*

Proof. We proceed by induction on $[G : N]$.

1) *Claim: It suffices to assume θ is G-invariant*

Let $T := \mathrm{stab}_G(\theta)$ denote the stabilizer in G of θ. Note that $P \leq T$, T is Q-invariant, and as $N_T(P) \leq N_G(P)$, we have $C_{N_T(P)/P}(Q) = 1$. Suppose that $T < G$. Then by induction, every Q-invariant $\varphi \in \mathrm{Irr}_{2'}(T|\theta)$ is σ-fixed. Let $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ be Q-invariant. By Clifford correspondence (see, for example, [15, Theorem (6.11)]), we can write $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(T|\theta)$. Since $\chi(1) = \psi^G(1) = \psi(1)[G : T]$ is odd, we see that in fact $\psi \in \mathrm{Irr}_{2'}(T|\theta)$. Moreover, ψ is Q-invariant, so is σ-fixed. (Indeed, since χ is Q-invariant, we can write $\chi = (\psi^\alpha)^G$ for any $\alpha \in Q$. But $\psi^\alpha \in \mathrm{Irr}_{2'}(T|\theta^\alpha) = \mathrm{Irr}_{2'}(T|\theta)$ since θ is Q-invariant, so by the uniqueness of Clifford correspondence, $\psi^\alpha = \psi$ for each $\alpha \in Q$, and ψ is Q-invariant as well.) Therefore, it follows that $\chi = \psi^G$ is σ-fixed. Hence, we may assume that $T = G$, so θ is G-invariant.

2) *Claim: It suffices to assume G/N is a chief factor of GQ*

Suppose $N < M \lhd G$ with $M \neq G$ stabilized under Q. Notice that $PM/M \in \mathrm{Syl}_2(G/M)$ and $(P \cap M)/N \in \mathrm{Syl}_2(M/N)$ are also stabilized by Q. Further, $C_{N_M(P\cap M)/(P\cap M)}(PQ) = 1$ and $C_{N_G(PM)/PM}(Q) = 1$ by [25, Lemma 2.1]. We may therefore apply the induction hypothesis and see that every PQ-invariant $\psi \in \mathrm{Irr}_{2'}(M|\theta)$ is σ-fixed and every Q-invariant $\tau \in \mathrm{Irr}_{2'}(G|\psi)$ is σ-fixed for such ψ.

Let $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ be Q-invariant. Then $\chi|_N = e\theta$ for some positive integer e. By [25, Lemma 2.2], $\chi|_M$ has a unique PQ-invariant irreducible constituent, say ϕ. But since θ is the unique irreducible constituent of $\chi|_N$, we see that $\phi \in \mathrm{Irr}(M|\theta)$. Moreover, $\phi(1)$ is odd, as $\phi(1)|\chi(1)$. Hence we see $\phi \in \mathrm{Irr}_{2'}(M|\theta)$ is PQ-invariant and $\chi \in \mathrm{Irr}_{2'}(G|\phi)$, so by the previous paragraph it follows that χ is σ-fixed.

Therefore, we may assume that G/N is a chief factor of GQ. In particular, G/N is the direct product of isomorphic simple groups transitively permuted by Q. If G/N is a 2-group or a $2'$-group, then by applying the odd-order theorem, we see G/N must be an elementary abelian p-group for some prime p. Otherwise, G/N is a product of nonabelian simple groups transitively permuted by Q.

3) *Claim: It suffices to assume G/N is a product of nonabelian simple groups*

First, note that if G/N is a 2-group, then in particular it is an elementary abelian 2-group, so by Lemma 3.4, we have $\chi^\sigma = \chi$ for any $\chi \in \mathrm{Irr}_{2'}(G|\theta)$. So suppose G/N is a $2'$-group, so that G/N is an elementary abelian p-group for some prime $p \neq 2$. As $|P/N| = |G/N|_2 = 1$, we see that $P = N$, and hence $N_G(P)/P = N_G(N)/N = G/N$ since N is normal. Then the assumption that $C_{N_G(P)/P}(Q) = 1$ yields that $C_{G/N}(Q) = 1$.

Let $\Gamma := GQ$ and note that $N \lhd \Gamma$ since N is Q-invariant. Since Q is a 2-group and G/N is a p-group, note that both Γ/G and G/N are solvable and that $([\Gamma : G], [G : N]) = 1$. Moreover, NQ/N is a complement for G/N in Γ/N, so by [15, Problem (13.10)], there is a unique Γ-invariant $\chi \in \mathrm{Irr}(G|\theta)$. That is, there is a unique Q-invariant $\chi \in \mathrm{Irr}(G|\theta)$. Note that this χ is actually a member of $\mathrm{Irr}_{2'}(G|\theta)$, since by Clifford theory $\chi(1) = e\theta(1)$, where $e|[G:N]$, so both e and $\theta(1)$ are odd.

But notice that χ^σ is also Q-invariant and $\chi^\sigma \in \mathrm{Irr}(G|\theta)$ as well, since $\theta^\sigma = \theta$. Then by uniqueness, we see $\chi^\sigma = \chi$, and hence every Q-invariant $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ is σ-fixed. Thus we may assume that G/N is not a $2'$-group, and hence G/N is a product of nonabelian simple groups transitively permuted by Q.

4) *Claim: It suffices to assume $N = Z(G)$ and is centralized by Q*

Let $\Gamma := GQ$, so (Γ, N, θ) is a character triple. Since G/N is a product of nonabelian simple groups, and is therefore perfect, Theorem 3.3 implies that we may replace the character triple (G, N, θ) with an isomorphic triple (G^*, N^*, θ^*) such that $N^* \leq Z(G^*)$, Q acts on G^*, centralizing N^*, such that the actions on G/N and G^*/N^* are isomorphic, θ^* is also σ-fixed, and the bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ preserves σ-invariance and commutes with the action of Q. Hence it suffices to assume $N = Z(G)$ and Q centralizes N.

5) Now, write $N := Z = Z(G)$ and write G/Z as a direct product $G/Z = \prod_{i=1}^a T_i/Z$ where the T_i/Z are simple and transitively permuted by Q. Then $[T_i, T_j] = 1$ for $i \neq j$, $S_i := (T_i)'$ is perfect, and $T_i = S_iZ$. (Indeed, $S_iZ/Z = (T_i)'Z/Z = (T_i/Z)' = T_i/Z$. Similarly, $S_i'Z/Z = (S_iZ/Z)' = T_i/Z = S_iZ/Z$ so $S_i'Z = S_iZ$. So $T_i/S_i' = S_iZ/S_i' = S_i'Z/S_i' \cong Z/(Z \cap S_i')$ is abelian, so $S_i \leq S_i'$.) In particular, S_i is quasisimple.

Write $P_i := T_i \cap P$, $R_i := S_i \cap P = S_i \cap P_i$, and $Z_i := Z \cap S_i$, so $S_i/Z_i \cong T_i/Z$ and $R_i/Z_i \in \mathrm{Syl}_2(S_i/Z_i)$.

Let $Q_i := \mathrm{stab}_Q(T_i)$ for each $1 \leq i \leq a$. Then by [25, Lemma 4.1(i)], $C_{N_{T_i}(P_i)/P_i}(Q_i) = 1$, since we have assumed $C_{N_G(P)/P}(Q) = 1$. Then we also have $C_{N_{S_i}(R_i)/R_i}(Q_i) = 1$, since $N_{T_i}(P_i)/P_i$ and $N_{S_i}(R_i)/R_i$ are Q_i-isomorphic.

It follows by Condition 2.1 that every Q_i-invariant $\eta \in \mathrm{Irr}_{2'}(S_i|\theta_i)$ is σ-fixed, where $\theta_i := \theta|_{Z_i}$. Given such an η, define $\psi := \eta\theta \in \mathrm{Irr}(S_iZ)$, so that $\psi \in \mathrm{Irr}_{2'}(T_i|\theta)$. (Here we mean the character as in [15, Problem (4.4)(b)].) Note that ψ is Q_i-invariant and fixed by σ, since both η and θ are.

We claim that any Q_i-invariant $\varphi \in \mathrm{Irr}_{2'}(T_i|\theta)$ can be written in this way, and is hence σ-fixed. Indeed, as $T_i = S_iZ$ is a central product, φ must be of the form $\varphi = \rho\theta$ for $\rho \in \mathrm{Irr}_{2'}(S_i|\theta_i)$. Since both φ, θ are Q_i-invariant and θ is linear, it follows that ρ is also Q_i-invariant, which proves the claim.

Now, the map $\mathrm{Irr}_{2'}(T_1|\theta) \times \ldots \times \mathrm{Irr}_{2'}(T_a|\theta) \to \mathrm{Irr}_{2'}(G|\theta)$ given by $(\psi_1, \ldots, \psi_a) \mapsto \psi_1 \cdot \ldots \cdot \psi_a$ is a bijection (by, for example, [25, Proposition 4.1(ii)]). Let $\chi \in \mathrm{Irr}_{2'}(G|\theta)$ be Q-invariant. Writing $\chi = \psi_1 \cdot \ldots \cdot \psi_a$ as above, note that $\chi|_{T_i} = r_i\psi_i$ for some positive integer r_i. Then since χ is Q-invariant, ψ_i is certainly Q-invariant. Hence we see that each ψ_i is Q_i-invariant, and is therefore σ-fixed by the preceding paragraph. Then $\chi^\sigma = \chi$, which proves the theorem.

□

Before proving the appropriate converse, we require another lemma:

Lemma 3.6. *Let G be a finite group with $P \in \mathrm{Syl}_2(G)$, and let $N \lhd G$. Suppose that every $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ and that $N_G(PN) = PN$. Suppose further that $\psi \in \mathrm{Irr}_{2'}(N)$ is extendible to PN. Then $\psi^\sigma = \psi$.*

Proof. Let $\varphi \in \mathrm{Irr}_{2'}(PN)$ with $\varphi|_N = \psi$. Then φ^G has odd degree, so at least one irreducible constituent of φ^G, say ϕ, has odd degree. But by [25, Lemma 2.2] (with $Q := 1$), ψ is the unique P-invariant constituent of ϕ_N, so since ψ^σ is also P-invariant and $\phi^\sigma = \phi$ by assumption, this yields $\psi^\sigma = \psi$, as desired. □

We are now prepared to prove our main theorem.

Theorem 3.7. *Let G be a finite group and $P \in \mathrm{Syl}_2(G)$. Assume that every finite nonabelian simple group involved in G is SN2S-Good (see Definition 1). Then $P = N_G(P)$ if and only if every $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ.*

Proof. First, note that if $P = N_G(P)$, then every $\chi \in \mathrm{Irr}_{2'}(G)$ is σ-fixed by Theorem 3.5 with $N := 1 =: Q$.

Conversely, suppose every $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ, and take G to be a minimal counterexample to the statement. Let $N \lhd G$ be a minimal normal subgroup, so that N is characteristically simple. In particular, N is the direct product of isomorphic simple groups. Hence by the Feit-Thompson odd-order theorem, N is either an elementary abelian 2-group, an elementary abelian p-group for p an odd prime, or the direct product of nonabelian simple groups transitively permuted by G.

Note that every member of $\mathrm{Irr}_{2'}(G/N)$ is σ-fixed, as we can view such characters as characters of G. Hence by the minimality of G, PN/N is self-normalizing in G/N. This yields that $C_{N_G(PN)/PN}(1) = 1$, so that $C_{N_G(PN)/PN}(P) = 1$ and $N_G(PN) = PN$, by [25, Lemma 2.1 (ii)](taking $Q := 1$ and $P := PN$). Write $R := P \cap N$.

First, suppose that N is a 2-group. Then $PN = P$, and the above discussion yields $N_G(P) = P$, completing the proof in this case.

Now, suppose N is an odd subgroup, so that we may write $N = (\mathbb{Z}/p)^r$ for some odd prime p and positive integer r. Let $\varphi \in \mathrm{Irr}_{2'}(PN)$. By Clifford theory, $\varphi(1) = et\psi(1)$ where ψ is a constituent of $\varphi|_N$ and both e, t divide $[PN : N]$. Hence $e = t = 1$ since PN/N is a 2-group and $\varphi(1)$ is odd. Then $\varphi|_N = \psi \in \mathrm{Irr}_{2'}(N)$. Moreover, φ is linear since N is abelian.

Now, $g \in PN$ can be written $g = xy$ with $x \in P$, $y \in N$, so $\varphi(g) = \varphi(xy) = \varphi(x)\varphi(y) = \varphi(x)\psi(y)$. Then $\varphi^\sigma(g) = \varphi(g)^\sigma = \varphi(x)^\sigma\psi(y)^\sigma = \varphi(x)\psi^\sigma(y)$ since $\varphi(x)$ must be some 2-root of unity, and hence is fixed by σ. Moreover, by Lemma 3.6, ψ is σ-fixed, and hence φ is also σ-fixed.

We therefore see that any $\varphi \in \mathrm{Irr}_{2'}(PN)$ is fixed by σ, so by the minimality of G, either $N_{NP}(P) = P$ or $PN = G$. In the latter case, G is solvable, so Navarro's conjecture [22, Conjecture A] holds, and hence $N_G(P) = P$ by [22, Theorem 5.2]. (Note that the proof of [22, Theorem 5.2] for a given group G requires only that [22, Conjecture A] holds for G.) Then we may assume $N_{NP}(P) = P$. Since $R = N \cap P = 1$, we therefore have $N_N(P) = P$, and hence $1 = C_N(P) = C_{N_N(R)/R}(P)$. Therefore, by [25, Lemma 2.1(i)] (with $M := N, N := 1 =: Q$), we have $C_{N_G(P)/P}(1) = 1$, i.e. $N_G(P) = P$.

We may therefore assume N is a direct product of nonabelian simple groups. First, we claim that any P-invariant $\psi \in \mathrm{Irr}_{2'}(N)$ is fixed by σ.

Indeed, let $\psi \in \mathrm{Irr}_{2'}(N)$ be P-invariant. Note that PN/N is a 2-group (and hence solvable), ψ is PN-invariant, and $(\psi(1), [PN : N]) = 1$. Let $\lambda := \det(\psi)$ (where $\det(\psi)$ is as in [15, Problem (2.3)]). Then by [15, Theorem 6.25], ψ extends to PN if and only if λ extends to PN. However, as λ is a linear character for the product N of nonabelian simple groups, we see that $\lambda = 1_N$, so is certainly extendible to PN. Hence ψ extends to PN and by Lemma 3.6, $\psi^\sigma = \psi$.

Then by Lemma 3.1 (with $Q := P, G := N, P := R = P \cap N$), we see that $C_{N_N(R)/R}(P) = 1$. Moreover, since $N_G(PN) = PN$, [25, Lemma 2.1(i)] (with $M := N, N := 1 =: Q$), we see that $C_{N_G(P)/P}(1) = 1$, i.e. $N_G(P) = P$, completing the proof. □

4 Some Simple Groups That are SN2S-Good

Theorem 4.1. *The alternating groups A_n for $n \geq 3$ are SN2S-Good.*

Proof. For $3 \leq n \leq 7$, the statements can be verified readily using the GAP Character Table Library [2], [11]. We may therefore assume that $n \geq 8$, so that the Schur covering group of A_n is a double cover. For the second statement of the definition of SN2S-Goodness, it suffices to assume that $G = \widehat{A_n}$ is the double cover of A_n.

Characters of $\widehat{A_n}$ which are nontrivial on the center are known as spin characters. Each of these characters have even degree (see, for example, [30]), and hence $\mathrm{Irr}_{2'}(\widehat{A_n}) = \mathrm{Irr}_{2'}(A_n)$.

Moreover, note that A_n and $S_n = \mathrm{Aut}(A_n)$ have self-normalizing Sylow 2-subgroups. To prove the theorem, we are therefore left to see that every $\chi \in \mathrm{Irr}_{2'}(A_n)$ is σ-fixed, which follows according to the remarks after [22, Theorem (5.2)].

□

Theorem 4.2. *The sporadic simple groups are SN2S-Good.*

Proof. Let S be sporadic simple group. By [17], it follows that S has a self-normalizing Sylow 2-subgroup except in the case $S = J_1$, where $N_G(S) \cong 2^3.7.3$ or $S = J_2, J_3, Suz$, or HN, in which cases $[N_G(S) : S] = 3$.

Note J_1 has a trivial Schur multiplier and outer automorphism group, and hence it suffices in this case to note that the characters X.7 and X.8 of degree 77 (in the notation of GAP [2]) are interchanged by the action of σ. If $S = J_2, J_3, Suz$, or HN, then $|\mathrm{Out}(S)| = 2$. In each of these cases, $\mathrm{Aut}(S)$ has a self-normalizing Sylow 2-subgroup (see [29]). It is clear from inspection of the character tables in GAP [2] that in each case, S has odd-degree irreducible characters which are not fixed by σ. However, we also see that these come in pairs which are interchanged by the nontrivial outer automorphism. That is, $\chi \in \mathrm{Irr}_{2'}(S)$ is invariant under the outer automorphism exactly when χ is fixed by σ. This proves the claim for the groups HN and J_3, since in the case HN, the Schur multiplier is trivial, and in the case J_3, the Schur multiplier is size 3, and hence no nontrivial character of the multiplier can be fixed by σ. Further, for J_2, the multiplier is size 2, and for Suz it is size 6. Since no nontrivial character of a cyclic group of odd order can be fixed by σ, it suffices in either case to show that odd characters of $2.S$ which are invariant under the nontrivial outer automorphism are fixed by σ. Here $\chi \in \mathrm{Irr}_{2'}(2.S)$ is invariant under the outer automorphism exactly when χ extends to an irreducible character of $2.S.2$. Observing the character tables of these two groups in GAP, we see that every member of $\mathrm{Irr}_{2'}(2.S.2)$ is fixed by σ, which completes the proof in these cases.

Hence we may assume S is one of the 21 sporadic simple groups which has a self-normalizing Sylow 2-subgroup. In the cases that S has a Schur multiplier of 2-power order, let G be the covering group for S. If $S = M_{22}$, let $G = 4.M_{22}$, in the cases $S = McL$, $O'N$, or Fi_{24}', let $G = S$, and in the case $S = Fi_{22}$, let $G = 2.Fi_{22}$. Then it suffices to show that every $\chi \in \mathrm{Irr}_{2'}(G)$ is fixed by σ. The character tables for each of these is again available in GAP, and (tedious) inspection of the values on odd characters verifies the statement.

□

Theorem 4.3. *The simple groups $G_2(3), B_3(3)$, ${}^2G_2(3)', {}^2G_2(3^{2n+1})$ for $n \geq 1$, $G_2(q)$ for $q > 3$ odd, and ${}^3D_4(q)$ for q odd are each SN2S-Good.*

Proof. With the help of GAP and the GAP Character Table Library [2], [11] we see that both $G_2(3)$ and $B_3(3) = O_7(3)$ have self-normalizing Sylow 2-subgroups and that the members of $\mathrm{Irr}_{2'}(G_2(3))$ and $\mathrm{Irr}_{2'}(2.O_7(3))$ are integer-valued, proving the statement in these cases.

Now, note that ${}^2G_2(3)'$ and ${}^2G_2(3^{2n+1})$ have trivial Schur multipliers and odd outer automorphism groups. Hence it suffices to prove the statement of Problem 1 for these groups, and since each of these has a self-normalizing Sylow 2-subgroup, it therefore suffices to show that there are odd irreducible characters which are not fixed by σ. The character table for ${}^2G_2(3)' \cong PSL_2(8)$ is available in GAP [2], and we see that (among other examples) the characters X.7, X.8, and X.9 of degree 9 are interchanged by σ. The character table for ${}^2G_2(3^{2n+1})$ is available in CHEVIE [12], and in this case the characters χ_4 and χ_5 are not fixed by σ, since $\sqrt{3} = -i(2\zeta_3 + 1)$ is not fixed by σ.

Now consider $S := G_2(q)$ for $q > 3$ odd or ${}^3D_4(q)$ with q odd, so that S has a self-normalizing Sylow 2-subgroup by [17]. Again, the Schur multiplier is trivial, so it suffices to show that every $\chi \in \mathrm{Irr}_{2'}(S)$ is fixed by σ. Inspection of the character tables available in CHEVIE for $G_2(q)$ and for ${}^3D_4(q)$ yields that the odd characters are integer valued, so this is indeed the case.

□

Theorem 4.4. *Let S be the group $S := PSL_2(q)$ with $q \geq 5$ odd. Then S is SN2S-Good.*

Proof. Note that $PSL_2(5) \cong A_5$, and $PSL_2(9) \cong A_6$, so the statement is true for $q = 5, 9$ by Theorem 4.1.

So assume $q > 5$ and write $q = p^a$ for an odd prime p. Then $\mathrm{Aut}(S) = \langle S, \delta, \varphi \rangle$, where δ has order 2 and satisfies $\langle S, \delta \rangle = PGL_2(q)$, and φ is a field automorphism induced from the map $x \mapsto x^p$ on $\mathbb{F}_q$. (See, for example, [31].)

Let $P \in \mathrm{Syl}_2(S)$. Then $P < N_S(P)$ if and only if $q \equiv \pm 3 \mod 8$, and in these cases, $N_S(P) \cong A_4$, so that $|N_S(P)/P| = 3$, by [17]. Let $S \leq A \leq \mathrm{Aut}(S)$ be obtained from S by adjoining a 2-group of automorphisms Q. Then A has a self-normalizing Sylow 2-subgroup if $q \equiv \pm 1 \mod 8$. Moreover, if $q \equiv \pm 3 \mod 8$, then A has a self-normalizing Sylow 2-subgroup if and only if $PGL_2(q)$ is contained in A. (See [29].)

Now, in the notation of the generic character table available in CHEVIE [12], $\mathrm{Irr}_{2'}(S)$ is the set $\{\chi_1, \chi_2, \chi_3, \chi_4\}$. (Note that χ_3 and χ_4 are actually families of characters.) We see that χ_1, χ_2 are σ-fixed, since they are rational-valued. However, χ_3 and χ_4 take values of the form $\frac{\pm 1 \pm \sqrt{q}}{2}$, which are fixed by σ if and only if $q \equiv \pm 1 \mod 8$, which can be seen using Gauss sums. Hence if $q \equiv \pm 1 \mod 8$, every odd character is σ-fixed, as desired. If $q \equiv \pm 3 \mod 8$, we need to show that χ_3 and χ_4 are not A-invariant when A has a self-normalizing Sylow 2-subgroup.

By [31], χ_1 and χ_2 are fixed by $\mathrm{Aut}(S)$, while the families χ_3 and χ_4 are fixed by φ but interchanged by δ. Hence we see that if $\delta \in A$, then every A-invariant odd character is fixed by σ, and if $\delta \notin A$, then there exist A-invariant odd characters not fixed by σ. Hence S satisfies Condition 2.2.

Finally, applying similar arguments to $SL_2(q)$ shows that as long as S does not have an exceptional Schur multiplier (i.e. $q \neq 9$), if $G/Z(G) \cong S$, then G satisfies Condition 2.1, completing the proof.

□

Theorem 4.5. *The simple groups $F_4(q)$ and $E_7(q)$ with q odd are SN2S-Good.*

Proof. From [17], we see that these groups have self normalizing Sylow 2-subgroups. Moreover, $F_4(q)$ is its own covering group and $E_7(q)_{sc}$ is the cover for $E_7(q)$. Hence it suffices to show that every odd-degree irreducible character of $F_4(q)$ and $E_7(q)_{sc}$ are fixed by σ. However, from [18], we see that the only characters of these groups of odd degree have multiplicity one, completing the proof.

□

We remark that the above argument just barely fails for $E_8(q)$, as in this case there are 4 odd character degrees which have multiplicity 2.

4.1 Groups of Lie Type in Characteristic 2

In this section, let G be a group of Lie type defined over $\mathbb{F}_q$, where $q = 2^a$. That is, $G = \mathbb{G}^F$ is the group of fixed points of a connected reductive algebraic group $\mathbb{G}$ over $\overline{\mathbb{F}}_q$ under a Frobenius morphism F. Let $(\mathbb{G}^*, F^*)$ be dual to $(\mathbb{G}, F)$ and set $G^* := (\mathbb{G}^*)^{F^*}$.

Note that in this case, the unipotent radical U is a Sylow 2-subgroup, and the Borel subgroup B is its normalizer $N_G(U)$. Here G has a self-normalizing Sylow 2-subgroup if and only if $q = 2$ and G is untwisted (with the exception of the Tits group ${}^2F_4(2)'$, which has a self-normalizing Sylow 2-subgroup). (See, for example, [6, Proposition 3.6.7]).

For $s \in G^*$ semisimple, we denote by $[s]$ and (s) the G^* and $\mathbb{G}^*$ conjugacy class of s, respectively. Further, we denote by $\mathcal{E}(G, [s])$ the rational Lusztig series indexed by the G^*-semisimple conjugacy class $[s]$ and by $\mathcal{E}(G, (s))$ the geometric Lusztig series indexed by the $\mathbb{G}^*$-semisimple conjugacy class (s) (see [7, the discussion after 14.40 and Definition 13.16]). When $C_{\mathbb{G}^*}(s)$ is connected, the two series are the same, and we simply write $\mathcal{E}(G, s)$. In this case, there is a character $\chi_s \in \mathcal{E}(G, s)$ of degree $[G^* : C_{G^*}(s)]_{2'}$, called the semisimple character. (See [7, Chapter 14] for an explicit description of χ_s.)

We continue to denote by σ the Galois automorphism in Problem 1. The following lemma will be useful:

Lemma 4.6. *Let G be a group of Lie type defined in characteristic 2, and keep the notation above. Let $s \in G^*$ with $C_{\mathbb{G}^*}(s)$ connected. Then $\mathcal{E}(G, s)^\sigma = \mathcal{E}(G, s^2)$. In particular, $\chi_s^\sigma = \chi_{s^2}$. Hence $\chi_s = \chi_s^\sigma$ if and only if s is conjugate in G^* to s^2.*

Proof. Note that $n := |s|$ is odd. Let $\gamma \in \mathbb{Q}_n$ be an n'th root of unity. Then $\sigma(\gamma) = \gamma^2$, by definition. Then the proof of [23, Lemma 9.1] shows that $\chi_s^\sigma = \chi_{s^2}$ and $\mathcal{E}(G, s)^\sigma = \mathcal{E}(G, s^2)$. Certainly, then, if $(s) = (s^2)$, then $\chi_s = \chi_s^\sigma$. Conversely, if $\chi_s = \chi_s^\sigma$, then $\mathcal{E}(G, s) = \mathcal{E}(G, s^2)$, so $(s) = (s^2)$. $\square$

If α is an automorphism of G which is the restriction of a bijective morphism $\alpha_1 \colon \mathbb{G} \to \mathbb{G}$ of algebraic groups which commutes with F (in particular this is the case when $\mathbb{G}$ is a simple simply connected algebraic group and α is an automorphism of G), then α induces an automorphism $\alpha^* \colon G^* \to G^*$, and by [26, Corollaries 2.4, 2.5], we have

$$\mathcal{E}(G, (s))^\alpha = \mathcal{E}(G, (s^{\alpha^*})), \quad \mathcal{E}(G, [s])^\alpha = \mathcal{E}(G, [s^{\alpha^*}])),$$

and if $C_{\mathbb{G}^*}(s)$ is connected,

$$\chi_s^\alpha = \chi_{s^{\alpha^*}}. \tag{1}$$

Let $q = 2^a$ and let $\varphi \colon G \to G$ denote the field automorphism of G induced from the map $x \mapsto x^2$ in $\mathbb{F}_q$.

Keeping our notation from above, let $\chi \in \mathrm{Irr}_{2'}(G)$. Then $\chi \in \mathcal{E}(G, [s])$ for some semisimple $s \in G^*$. By Lusztig correspondence, there is some unipotent character ψ of $C_{G^*}(s)$ of odd degree with

$$\chi(1) = [G^* : C_{G^*}(s)]_{2'} \psi(1). \tag{2}$$

Hence it is useful to note that by [19, Theorem 6.8], if $C_{\mathbb{G}^*}(s)$ is connected, then $C_{G^*}(s)$ has exactly one unipotent character of odd degree (namely, $1_{C_{G^*}(s)}$), unless $C_{G^*}(s)$ contains a component of type $C_n(2), G_2(2)$, or $F_4(2)$. We deal with these situations separately:

Theorem 4.7. *Let $S = G_2(2)'$ or $F_4(2)$. Then S is SN2S-Good.*

Proof. Let $S = G_2(2)'$ or $F_4(2)$. From computation in GAP and the GAP Character Table Library ([11], [2]), we see that S has a self-normalizing Sylow 2-subgroup and that every odd character of S (and in the case of $F_4(2)$, every odd character of the covering group $2.F_4(2)$) is fixed by σ. Hence the statement is true in either case.

□

Theorem 4.8. *Let $S = Sp_{2n}(2)$ with $n \geq 3$ or $S = \Omega_{2n}^+(2)$ with $n \geq 4$. Then S is SN2S-Good.*

Proof. First, let $S = Sp_6(2)$, so that the covering group of S is $2.Sp_6(2)$ or let $S = \Omega_8^+(2)$, so that the covering group is $2^2.\Omega_8(2)$. Either choice of S and their covering groups have self-normalizing Sylow 2-subgroups. By a direct check in GAP [11], [2], we see every odd character of either covering group is fixed by σ, and hence the statement is true in these cases.

We may therefore assume that $n > 3$ in the case $S = Sp_{2n}(2)$ or $n > 4$ in the case $S = \Omega_{2n}^+(2)$, so $G := S$ is its own covering group. Note that $G \cong G^*$, $Z(\mathbb{G}^*) = 1$ is connected, and that G has a self-normalizing Sylow 2-subgroup. Therefore, it suffices to show that every member of $\mathrm{Irr}_{2'}(G)$ is fixed by σ.

Let $\chi \in \mathrm{Irr}_{2'}(G)$ and let s and ψ be as in (2). If $G = Sp_{2n}(2)$, we have

$$C_{G^*}(s) \cong Sp_{2m_0}(2) \times X$$

where X is a direct product of groups of the form $GL_k(2^d)$ and $GU_k(2^d)$ and $2m_0 = \dim_{\mathbb{F}_2} \ker(s-1)$. Note that every nontrivial unipotent character of $GL_k(2^d)$ and $GU_k(2^d)$ has even degree. Hence $\psi = \psi_1 \times 1_X$, where ψ_1 is a unipotent character of $Sp_{2m_0}(2)$ of odd degree. If $G = \Omega_{2n}^+(2)$, then $G \cong G^*$ is the commutator subgroup of $O_{2n}^+(2)$, and $C_{G^*}(s)$ is index at most 2 in $C_{O_{2n}^+(2)}(s)$. Moreover,

$$C_{O_{2n}^+(2)}(s) \cong O_{2m_0}^+(2) \times X,$$

where X is of the same form as above. Then since every nontrivial unipotent character of $O_{2m_0}^+(2)$ also has even degree, it follows that ψ is the trivial character for $C_{G^*}(s)$.

Hence, if we are in the case $\Omega_{2n}^+(2)$ or s has no fixed points on $\mathbb{F}_2^{2n}$, then $\chi = \chi_s$ and $\chi^\sigma = \chi$ since $[s^2] = [s]$. (See, for example, [13, Lemma 2.4].)

Then assume $G = Sp_{2n}(2)$ and let $2m_0 \geq 4$. Then $Y := Sp_{2m_0}(2)$ has 5 unipotent characters of odd degree, namely 1_Y, $\rho_{m_0}^1$, $\rho_{m_0}^2$, α_{m_0}, and β_{m_0}, in the notation of [14]. If $m_0 > 2$, then no two of these characters have the same degree. Hence if $m_0 > 2$, then since $\mathcal{E}(G, s)^\sigma = \mathcal{E}(G, s)$ by Lemma 4.6, we see that $\chi^\sigma = \chi$.

Now assume that $m_0 = 1$ or 2. Note that $C_{\mathbb{G}^*}(s)$ is some rational Levi subgroup L^* of $\mathbb{G}^*$, and hence by [7, Theorem 13.25], the bijection $\mathcal{E}(C_{G^*}(s), 1) \to \mathcal{E}(G, s)$ is given by Deligne-Lusztig induction R_L^G. Now, from a direct check using GAP [2], we see that $\psi_1^\sigma = \psi_1$, and hence ψ is σ-fixed. Moreover, since the Green functions are rational-valued, by [7, Proposition 12.2], $\chi = R_L^G \psi$ is also σ-fixed, which completes the proof.

□

We remark that $Sp_4(2)' \cong A_6$ is covered by Theorem 4.1.

Theorem 4.9. *Let S be one of the simple groups ${}^2E_6(2), E_6(2), E_7(2)$ or $E_8(2)$. Then S is SN2S-Good.*

Proof. Let S be one of $E_6(2), E_7(2)$ or $E_8(2)$. Then S is its own covering group and has a self-normalizing Sylow 2-subgroup, so it suffices to show that every $\chi \in \mathrm{Irr}_{2'}(S)$ is fixed by σ. Note $S \cong S^*$ is self-dual, we are in the situation of Lemma 4.6, and that the odd characters of S are exactly χ_s for $s \in S^*$ semisimple. Every semisimple element $s \in S^*$ is conjugate to s^2 (see [13, Lemma 2.4]), so $\chi_s^\sigma = \chi_{s^2} = \chi_s$ for every odd character χ_s of S.

Now let S be ${}^2E_6(2)$. The following observations can be verified using GAP and the GAP Character Table Library [2], [11], along with the calculations of [32]. If $S \leq A \leq \mathrm{Aut}(S)$ is obtained by adjoining a 2-group of automorphisms, then A has a self-normalizing Sylow 2-subgroup exactly when A contains a field automorphism of order 2 (i.e. $A = S.2$). Hence $\chi \in \mathrm{Irr}_{2'}(S)$ is invariant under A if and only if χ extends to an irreducible character of A. Moreover, every irreducible character of $2^2.{}^2E_6(2).2$ with odd degree is fixed by σ. Hence we see Condition 2.1 is satisfied. Moreover, there exist odd-degree irreducible characters of S which are not fixed by σ, so we see that Condition 2.2 is satisfied.

□

Theorem 4.10. *Let S be simple of type $E_7(2^a), E_8(2^a), F_4(2^a)$, or $G_2(2^a)$ with $a > 1$, or let S be $E_6(2^a)$ with $2^a \equiv 2 \mod 3$ and $a > 1$. Then if G is quasisimple with $G/Z(G) = S$, then G satisfies Condition 2.1.*

Proof. (1) First, if $S = G_2(4)$, this is readily verified using GAP and observations as in the preceding theorem. So we further assume that $a > 2$ for type G_2.

(2) We see that the Schur multiplier of S is trivial, so that it suffices to prove the converse of Condition 2.2 for $G = S$. Since G is of untwisted type with nontrivial torus, we see that G does not have a self-normalizing Sylow 2-subgroup. Moreover, by [29], we see that A with $G \leq A \leq \mathrm{Aut}(G)$ has a self-normalizing Sylow 2-subgroup exactly when A contains a field automorphism φ with $|\langle G, \varphi\rangle/G| = a$. Hence such an A can only be obtained by adjoining a 2-group of automorphisms Q in the case that $a = 2^t$ is a 2-power.

Suppose A has a self-normalizing Sylow 2-subgroup, so that A contains φ and $a = 2^t$ for some $t \geq 1$, or $t \geq 2$ in the case of $G_2(2^a)$. Then it suffices to show that every $\chi \in \mathrm{Irr}_{2'}(G)$ which is fixed by φ is also fixed by σ. Note that we are in the situation of Lemma 4.6 and (1), since $Z(\mathbb{G}) = 1$, and that the odd characters of G are exactly χ_s for $s \in G^*$ semisimple. We note also that G is self-dual. That is, $G \cong G^*$.

(3) Let $s \in G^*$ be a semisimple element. Note that we may view G^* as a Chevalley group, as in [5]. By [5, Section 7.1] and the fact that $C_{\mathbb{G}^*}(s)$ is connected, we see that s is conjugate in G^* to a product $h = h_{p_1}(\lambda_1)h_{p_2}(\lambda_2)\cdots h_{p_n}(\lambda_n)$, where n is the rank of G^* as a group of Lie type, $\{p_1, ..., p_n\}$ are a set of fundamental roots for the root system corresponding to G^*, $\lambda_i \in \mathbb{F}_q^\times$, and $h_{p_i}(\lambda_i)$, as defined in [5, Proposition 6.4.1], is the image of $\mathrm{diag}(\lambda_i, \lambda_i^{-1})$ under the homomorphism $SL_2(\mathbb{F}_q) \to \langle X_{p_i}, X_{-p_i}\rangle$ given in [5, Theorem 6.3.1]. (Here X_{p_i} is the root subgroup corresponding to the root p_i.)

In particular, from [5, Section 12.2], we see that $h_{p_i}(\lambda_i)^\varphi = h_{p_i}(\lambda_i^2)$, so that $h^\varphi = \prod_{i=1}^n h_{p_i}(\lambda_i^2)$. Then by [5, Proposition 6.4.1] h^φ acts on the Chevalley basis (see [5, Theorem 4.2.1] for the definition of the Chevalley basis) as follows:

$$h^\varphi \cdot e_x = \left(\prod_{i=1}^n \lambda_i^{2A_{p_i x}}\right) e_x \quad \text{and} \quad h^\varphi \cdot h_r = h_r$$

for x a root and r a fundamental root. Here $A_{p_i x}$ is the Cartan integer $\frac{2(p_i,x)}{p_i,p_i}$. (See [5, Section 3.3].) Since h^2 acts on the Chevalley basis the same way, we see that s^φ is conjugate in G^* to s^2, and hence $\chi_s^\sigma = \chi_{s^2} = \chi_s^\varphi$ by Lemma 4.6 and (1). Thus χ is fixed under σ whenever it is φ-invariant.

(4) Now suppose that A does not have a self-normalizing Sylow 2-subgroup. In particular, $\varphi \notin A$. In this case, we need only show that there is some $\chi \in \mathrm{Irr}_{2'}(G)$ which is A-invariant but

for which $\chi^\sigma \neq \chi$. Write $q = 2^a$ with $a = 2^t m > 1$, where $t \geq 0$ and $m \geq 1$ is odd. Consider the field automorphism ψ, where ψ is defined to be φ^m if $m \neq 1$ and $\psi := \varphi^2$ if $m = 1$. Then ψ has the largest order among field automorphisms which may be contained in A. (Note that if $t = 0$ or $t = 1 = m$, then $\psi = 1$.) Our goal is to illustrate a semisimple character χ_s such that χ_s is fixed by ψ, as well as any graph automorphisms (recall that G has no diagonal automorphisms), but such that $\chi_s^\sigma \neq \chi_s$. We therefore wish to exhibit a semisimple element s such that $(s^\psi) = (s)$ but $(s^2) \neq (s)$.

If $m \neq 1$, note that $\mathbb{F}_q^\times$ contains an element with multiplicative order dividing $2^m - 1 = \sqrt[2^t]{q} - 1$ but larger than 3, since $q \geq 8$. When $m = 1$ and $q > 4$, note that $\mathbb{F}_q^\times$ contains an element of order 5, since $q = 2^{2^t}$ for $t \geq 2$, and hence $q - 1$ is divisible by 5. For $G_2(q)$, we see that the element $s = h_\gamma(i, -i, 0)$ in the notation of [8] satisfies our conditions when γ^i has multiplicative order dividing $2^m - 1$ but larger than 3 if $m \neq 1$ and multiplicative order 5 if $m = 1$. For $F_4(q)$, the element $s = h_4 = (1, z, z, z)$ in the notation of [28] works when $q \geq 4$, where z is chosen the same way as γ^i, and the element $h_{25} = (z_1, 1, z, z^4)$ with $z^5 = 1, z_1^3 = 1$ works with $q = 4$. Finally, in the case of $E_6(q), E_7(q)$, and $E_8(q)$, we may choose elements s similarly, where s is of type D_5, D_6, and D_7, respectively. (See [21], [9], and [10] for a discussion of the semisimple conjugacy classes of these groups.)

$\square$

Theorem 4.11. *Let $a \geq 1$ and let S be one of the simple groups $Sp_{2n}(2^a)$ with $n \geq 3$ or $\Omega_{2n}^\epsilon(2^a)$ with $n \geq 4$ and $\epsilon \in \{\pm\}$. Then S is SN2S-Good.*

Proof. If $S = Sp_{2n}(2)$ or $\Omega_{2n}^+(2)$, this is Theorem 4.8, so we further assume that $a \geq 2$ in the case of $Sp_{2n}(q)$ or $\Omega_{2n}^+(q)$.

Let $q := 2^a$. First, note that since q is even, $G := S$ is its own covering group and we may identify G with its dual group. Moreover, note that when $G = \Omega_{2n}^\epsilon(2^a)$, we have $O_{2n}^\epsilon(q) = G : 2$. Let $g \in O_{2n}^\epsilon(q)$ be a representative for the nontrivial coset of $O_{2n}^\epsilon(q)/G$ and let $\varphi \colon G \to G$ denote the field automorphism of G induced from the map $x \mapsto x^2$ in $\mathbb{F}_q$. Let $\overline{\varphi} := \varphi$ in the case of $Sp_{2n}(q)$ or $\epsilon = +$. In the case of $\epsilon = -$, let $\overline{\varphi}$ be the field automorphism induced by $x \mapsto x^2$ in $\mathbb{F}_{q^2}$, viewing $\Omega_{2n}^-(q)$ naturally inside $\Omega_{2n}^+(q^2)$. (See, for example, [5, Theorem 14.5.2] for one such embedding.) Further, write $\overline{q} := q^2$ in the case $\epsilon = -$ and $\overline{q} := q$ otherwise.

First assume $n > 4$ in the case of $\Omega_{2n}^+(q)$, so that the outer automorphism group of G has size $2a$ when $G = \Omega_{2n}^\epsilon(q)$, and size a in the case $G = Sp_{2n}(q)$. Moreover, in these cases, the outer automorphism group is generated by $\overline{\varphi}$ and the action of g in the case $\epsilon = +$ and by $\overline{\varphi}$ in the case $\epsilon = -$ or $G = Sp_{2n}(q)$. In the case $G = \Omega_8^+(2^a)$, notice that $\mathrm{Out}(G) = a.S_3$. However, if $A \leq \mathrm{Aut}(G)$ is a group obtained by adjoining a 2-group of automorphisms on G, then it must be that again $|A/G| \leq 2a$.

Now, in any case, it follows from [29] that if $A \leq \mathrm{Aut}(G)$ is a group obtained by adjoining a 2-group of automorphisms on G, then A has a self-normalizing Sylow 2-subgroup exactly when A contains a generating field automorphism, and without loss of generality, we consider this automorphism to be $\overline{\varphi}$. When this is the case, the same argument as in part (3) of the proof of Theorem 4.10 shows that every A-invariant odd-degree irreducible character of G is fixed by σ. (Note that in the case $\Omega_{2n}^-(q) = {}^2D_n(q)$, we are considering the element h as an element of $D_n(q^2)$ which is fixed by an appropriate automorphism induced by the nontrivial symmetry on the Dynkin diagram, as in [5, Section 13.4].)

Suppose A does not have a self-normalizing Sylow 2-subgroup, so that $\overline{\varphi} \notin A$. Then we must exhibit a character $\chi \in \mathrm{Irr}_{2'}(G)$ which is invariant under A but so that $\chi^\sigma \neq \chi$. Write $\overline{q} = 2^{\overline{a}}$ with $\overline{a} = 2^t m > 1$, where $t \geq 0$ and $m \geq 1$ is odd. Consider the field automorphism $\psi = \varphi^m$ if $m \neq 1$

and $\psi = \varphi^2$ if $m = 1$, so that ψ again has the largest order among field automorphisms which may be contained in A. Again note that if $t = 0$ (which cannot happen for $\epsilon = -$) or $t = 1 = m$, then $\psi = 1$. As before, we will illustrate a semisimple character χ_s such that χ_s is fixed by ψ, as well as any graph automorphisms, but such that $\chi_s^\sigma \neq \chi_s$. We view $G^* \cong G$ as a matrix group, corresponding to a form with Gram matrix:

$$\begin{pmatrix} & I_{n-1} & & \\ I_{n-1} & & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

in the case $G = Sp_{2n}(q)$. When $G = \Omega_{2n}^+(q)$, we can view G^* as a subgroup of $Sp_{2n}(q)$ defined with this Gram matrix, and when $G = \Omega_{2n}^-(q)$, we instead view G^* as the subgroup of $\Omega_{2n}^+(q^2)$ (defined as a subgroup of $Sp_{2n}(q^2)$ with this Gram matrix) comprised of elements y such that $S^{-1}yS$ has entries in $\mathbb{F}_q$, where S is an appropriate change of basis matrix $\mathrm{diag}\,(I_{2n-1}, S_0)$, as described in [5, Theorem 14.5.2]. We view $O_{2n}^\epsilon(q)$ similarly. Write $\underline{G} := G$ if $G = Sp_{2n}(q)$ and $\underline{G} := O_{2n}^\epsilon(q)$ if $G = \Omega_{2n}^\epsilon(q)$. Then we still have $\underline{G}^* \cong \underline{G}$.

First, assume $m \neq 1$. Let $\lambda \in {\mathbb{F}_q}^\times$ with $\lambda^3 \neq 1$ and the multiplicative order of λ dividing $2^m - 1 = \sqrt[2^t]{q} - 1$. (Note that such a λ exists in $\mathbb{F}_q^\times$ since $q > 4$ in this case.) Now let s be the semisimple element $\mathrm{diag}(\lambda \cdot I_{n-1}, \lambda^{-1} \cdot I_{n-1}, 1, 1)$ in $\underline{G}^* \cong \underline{G}$. Note that s^2 is not conjugate to s, since $\lambda^2 \neq \lambda$ or λ^{-1}. Further, $s^\psi = \mathrm{diag}(\lambda^{2^m} \cdot I_{n-1}, \lambda^{-2^m} \cdot I_{n-1}, 1, 1) = s$. Hence $\chi_s \in \mathrm{Irr}_{2'}(\underline{G})$ is ψ-invariant but not fixed by σ.

Now suppose that $m = 1$, so $\overline{q} = 2^{2^t}$ for $t \geq 1$. Let $s \in \underline{G}^*$ have order 5. Then certainly, by looking at the eigenvalues of s and s^2 on the action of the natural module (possibly in some extension field), we see that again s cannot be conjugate to s^2. However, we claim that s can be chosen so that s is conjugate to s^ψ. Indeed, note that the action of ψ on the eigenvalues of the action of s on the natural module will be $\lambda \mapsto \lambda^4 = \lambda^{-1}$. Moreover, the derived subgroup $G \cong G^*$ is of simply connected type, so since s can be chosen so that s and s^ψ are conjugate to the same diagonal matrix over the algebraic group $\underline{\mathbb{G}}^*$ corresponding to $\underline{G}^*$, we see that s is conjugate to s^ψ over $\underline{G}^*$ as well. Hence we see again that χ_s is invariant under ψ but not under σ.

Now, since $[\underline{G} : G] \leq 2$, we see by Clifford theory that χ_s must be irreducible when restricted to G, since χ_s has odd degree. Let $\chi := \chi_s|_G \in \mathrm{Irr}_{2'}(G)$. Then χ is also fixed by ψ, and by Lemma 3.4, it follows that $\chi^\sigma \neq \chi$, which completes the proof.

□

Theorem 4.12. *Let S be one of the simple groups ${}^3D_4(q)$ with q even, ${}^2B_2(2^{2n+1})$ or ${}^2F_4(2^{2n+1})$ with $n \geq 1$, or ${}^2F_4(2)'$. Then S is SN2S-Good.*

Proof. First, we use GAP [11] and the GAP character table library for ${}^2B_2(8) = Sz(8)$ and ${}^2F_4(2)'$.

Let $S = Sz(8)$, so that $G = 2^2.S$ is the covering group for S. Note that neither S nor G have a self-normalizing Sylow 2-subgroup and that the outer automorphism group is odd. Hence it will be sufficient to show that there is $\chi \in \mathrm{Irr}_{2'}(S)$ satisfying $\chi^\sigma \neq \chi$. Now, inspecting the values of the characters χ_4, χ_5, and χ_6 of degree 35 in the notation of GAP, we see that they are permuted by σ, completing the proof for $Sz(8)$.

Now let $S = {}^2F_4(2)'$, so that S is its own covering group, the automorphism group is $A = {}^2F_4(2)$, and $|A/S| = 2$. Since S has a self-normalizing Sylow 2-subgroup, we need to show that every $\chi \in \mathrm{Irr}_{2'}(S)$ is fixed by σ. Indeed, by observing the character table in GAP, we see that this is the case, and therefore the statement is true for ${}^2F_4(2)'$.

So we may assume that $n \geq 2$ in the case $S = {}^2B_2(2^{2n+1})$ and $n \geq 1$ in the case $S = {}^2F_4(2^{2n+1})$, so that in all remaining cases, S is its own covering group and it suffices to show Condition 2.2 and its converse for $G := S$.

Moreover, note that $|\mathrm{Out}(G)| = 2n + 1$ is odd in the case of ${}^2B_2(2^{2n+1})$ and ${}^2F_4(2^{2n+1})$, so it suffices in either case to show that G satisfies Problem 1. Since G does not have a self-normalizing Sylow 2-subgroup, we need to illustrate some $\chi \in \mathrm{Irr}_{2'}(G)$ which is not fixed by σ. The generic character tables for each group is available in CHEVIE [12]. If $G = {}^2F_4(q^2)$, with $q^2 = 2^{2n+1}$, then inspection of the character table shows that $\chi_{22}(k)^\sigma = \chi_{22}(2k)$, where $\chi_{22}(k)$ is the family of characters in CHEVIE notation with degree $(q^4 - q^2 + 1)(q^4 + 1)(q^8 - q^4 + 1)(q^2 + 1)^2$, indexed by $1 \leq k \leq q^2 - 2$ with $\chi_{22}(k) = \chi_{22}(-k)$. If $G = {}^2B_2(q^2)$, where $q^2 = 2^{2n+1}$, then inspection of the character table shows that $\chi_6(k)^\sigma = \chi_6(2k)$, where $\chi_6(k)$ is the family of characters in CHEVIE notation with degree $(q^2 - \sqrt{2}q + 1)(q^2 - 1)$ indexed by $1 \leq k \leq q^2 + \sqrt{2}q$ with $\chi_6(k) = \chi_6(-k) = \chi_6(q^2k)$. Hence in either case, we see that there is an odd character which is moved by σ.

If $G = {}^3D_4(q)$, note that the outer automorphism group is generated by a field automorphism φ of order divisible by 3. Write $q = 2^a$ with $a = 2^t m$, where $t \geq 0$ and $m \geq 1$ is odd, so that φ has order $3a$. Let ψ be the field automorphism $\psi := \varphi^{3m}$. According to [29], an almost simple group obtained by adjoining outer automorphisms Q to ${}^3D_4(q)$ cannot have a self-normalizing nilpotent subgroup unless it contains a generating field automorphism, which means no such Q of 2-power order exists. Hence it suffices to show that there is some $\chi \in \mathrm{Irr}_{2'}(G)$ which is invariant under ψ (which generates the largest 2-group of field automorphisms) but for which $\chi^\sigma \neq \chi$. First, let $q = 2$. Then the characters X.7, X.9, and X.10 in the notation of the GAP Character Table Library are permuted by σ, so we are done in this case.

Now let $q \geq 4$. Again the generic character table is available in CHEVIE, and we see that $\chi_{11}(k)^\sigma = \chi_{11}(2k)$, where $\chi_{11}(k)$ is the family of characters in CHEVIE notion with degree $(q + 1)(q^2 - q + 1)^2(q^4 - q^2 + 1)$ indexed by $1 \leq k \leq q^2 + q$ with $\chi_{11}(k) = \chi_{11}(-k)$. Observing the character values on the classes $C_{11}(a)$, we see that $\chi_{11}(k)^\psi = \chi_{11}(2^{3m}k)$, so it is possible to choose k so that this character is fixed by ψ but not by σ, completing the proof.

□

4.1.1 Type A in Characteristic 2

We now turn our attention to groups of type A in characteristic 2. That is, we are interested in the simple groups $PSL_n^\pm(2^a)$, where we define $PSL_n^+(q) := PSL_n(q)$ and $PSL_n^-(q) := PSU_n(q)$. We will sometimes use the notation $PSL_n^\epsilon(q)$, where $\epsilon \in \{+, -\}$.

Write $G := SL_n^\pm(q)$, and note that in most cases, G is the covering group of $S = PSL_n^\pm(q)$, where $q = 2^a$, and $Z := Z(G)$. Further, write $\widetilde{G} := GL_n^\pm(q)$. Let $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ be the corresponding algebraic groups for G and $\widetilde{G}$, and let F be the Steinberg endomorphism so that $\mathbb{G}^F = G, \widetilde{\mathbb{G}}^F = \widetilde{G}$. Note that $\widetilde{\mathbb{G}}$ has connected center and $\widetilde{G}$ is self-dual, so that members of $\mathrm{Irr}_{2'}(\widetilde{G})$ are χ_s where s is a semisimple element, viewed in $\widetilde{G}$. Moreover, $\widetilde{G}/G$ is cyclic of order $q \mp 1$, which is odd, so $\mathrm{Irr}_{2'}(G)$ is exactly the set of irreducible constituents of members of $\mathrm{Irr}_{2'}(\widetilde{G})$ when restricted to G.

In this case, some results of Cabanes and Späth regarding generalized Gelfand-Graev Representations will be useful. We refer the reader to the notation developed in [3, Section 4.2], which we will adapt.

Lemma 4.13. *Let $q = 2^a$ for $a \geq 1$ and $G, \widetilde{G}, \mathbb{G}, \widetilde{\mathbb{G}}, F$ be as above. Let $\widetilde{\Gamma_\mathcal{C}} \in \mathbb{Z}_{\geq 0}\mathrm{Irr}(\widetilde{G})$ be as in [3, Section 4.2] and $\Gamma_u \in \mathbb{Z}_{\geq 0}\mathrm{Irr}(G)$ be as in [3, Theorem 4.6]. Then $\widetilde{\Gamma_\mathcal{C}}$ and Γ_u are fixed by σ.*

Proof. Let $\mathbb{U}$ be the unipotent radical of $\mathbb{G}$ and write $U := \mathbb{U}^F$. By [3, Section 4.2], there are subgroups $U_{2,\mathcal{C}} \leq U_{1,\mathcal{C}} \leq U$ so that $\widetilde{\Gamma_\mathcal{C}}$ and Γ_u are defined by inducing the character ψ_u' (in the notation of [3, Section 4.2]) from $U_{1,\mathcal{C}}$ to $\widetilde{G}$ and G, respectively and the character ψ'_u is an integer multiple of $(\psi_u)^{U_{1,\mathcal{C}}}$, where $\psi_u \in U_{2,\mathcal{C}}$ is as in the notation of [3, Section 4.2]. Hence it suffices to show that ψ_u is fixed by σ.

Now, the values of ψ_u are given by products of values of a certain linear character

$$\theta_0 \colon (\mathbb{F}_{q^2}, +) \to (\mathbb{C}^\times, \cdot).$$

But since $(\mathbb{F}_{q^2}, +)$ is an elementary abelian 2-group, we see that $\theta_0(\zeta)$ is a 2-root of unity for every $\zeta \in \mathbb{F}_{q^2}$. Hence $\theta_0(\zeta)^\sigma = \theta_0(\zeta)$ for every $\zeta \in \mathbb{F}_{q^2}$, so ψ_u is fixed by σ, as desired. □

The following lemma will also be useful, the proof of which can be found, for example, in [27, Lemma 3.2.7]:

Lemma 4.14. *Let X be a finite group with normal subgroup Y such that X/Y is cyclic and let $\chi \in \mathrm{Irr}(X)$. Then the number of irreducible constituents of $\chi|_Y$ is exactly the number of $\lambda \in \mathrm{Irr}(X/Y)$ so that $\lambda\chi = \chi$.*

Theorem 4.15. *Let $S = PSL_2(2^a)$ with $a \geq 2$, $PSL_3(2)$, $PSL_3^{\pm}(2^a)$ with $a \geq 2$, or $PSL_n^{\pm}(2^a)$ with $n \geq 4$ and $a \geq 1$. Then S is SN2S-Good.*

Proof. (1) First, note that $PSL_3(2)$ has a self-normalizing Sylow 2-subgroup. Inspection of the character table in GAP verifies that every odd irreducible character for this group is fixed by σ, completing the proof in this case. Further, Theorem 4.1 completes the proof for $PSL_2(4) \cong PSL_2(5) \cong A_5$ and $PSL_4(2) \cong A_8$.

Let S be $PSL_3(4)$, so that the covering group G has size 48. Note that to lie above a character $\lambda \in \mathrm{Irr}(Z(G))$ fixed by σ, χ must be a character of $4^2.S$. Also, from the GAP Character Table Library [2], we see that the two characters of $4^2.S$ of degree 63 are switched by σ, and that the remaining odd characters are fixed by σ. Moreover, calculation in GAP [11] using results of [32] yields that $C_{N_G(P)/P}(Q) = 1$ if and only if SQ contains the normal subgroup of $\mathrm{Aut}(S)$ of size 40320. Observing the character table of this group, we see that Q switches the two degree-63 characters, which completes the proof in this case.

Now let S be $PSU_6(2)$, so that the Schur multiplier is $2^2 \times 3$. Then for the same reasoning as above, we are only interested in characters of $2^2.S$. The pairs (in the notation of the GAP Character Table Library) X.29, X.30 (degree 10395) and X.42, X.43 (degree 25515) are switched by σ, but all other odd characters are fixed by σ. Moreover, observing the character table of $2^2.S:2$, we see that these pairs are fused in $2^2.S:2$, and since $S:2$ has a self-normalizing Sylow 2-subgroup and S does not, which can again be verified using GAP and the results of [32], the proof is finished in this case as well.

Finally, let $S = PSU_4(2)$, so that $2.S$ is the covering group. In this case, X.2 and X.3 (degree 5) are interchanged by σ, as are X.16, X.17 (degree 45), but all other odd characters of $2.S$ are fixed by σ. Moreover, we again see that $S:2$ has a self-normalizing Sylow 2-subgroup and S does not, and that these pairs of characters fuse in $2.S:2$.

(2) Hence we may assume that S does not have an exceptional Schur multiplier, and we keep the notation of $S, G, \widetilde{G}, \mathbb{G}, \widetilde{\mathbb{G}}$ from before.

From [29], it follows that $C_{N_G(P)/P}(Q) = 1$ exactly when Q contains a field automorphism of order a in the case $\epsilon = +$ (resp $2a$ in the case $\epsilon = -$). It suffices to consider the field automorphism φ induced by the map $F_2 \colon x \mapsto x^2$ on $\overline{\mathbb{F}}_q$. (Note that in the case $S = PSL_n(2)$, S has a self-normalizing Sylow 2-subgroup. In this case, φ is the identity automorphism.)

(3) First, we show that $G = SL_n^{\pm}(q)$ satisfies Condition 2.1. Assume that Q contains φ. Let $\chi \in \mathrm{Irr}_{2'}(G)$ be Q-invariant. In particular, this means $\chi^\varphi = \chi$. We will show that $\chi^\sigma = \chi$.

Let $\widetilde{\chi} \in \mathrm{Irr}_{2'}(\widetilde{G}|\chi)$. In particular, we may choose $\widetilde{\chi}$ to be fixed by φ. (Indeed, this follows from [15, Theorem 13.28], since Q is a 2-group and $\widetilde{G}/G$ is solvable, of odd order.) We may write $\widetilde{\chi} =: \chi_s$, where $s \in \widetilde{G}$ is semisimple. Then by (1) and Lemma 4.6, $\chi_s = \chi_s^{\varphi} = \chi_{s^2} = \chi_s^{\sigma}$, so $\widetilde{\chi}$ is fixed by σ.

Let $\mathcal{C}$ be a unipotent class of $\widetilde{\mathbb{G}}$ so that $\widetilde{\chi}$ is a constituent of $\widetilde{\Gamma_{\mathcal{C}}}$ with multiplicity one, as in [3, Theorem 4.4]. Letting $\Gamma_u \in \mathbb{Z}_{\geq 0}\mathrm{Irr}(G)$ be as in [3, Theorem 4.6] and noting that $\widetilde{\Gamma}_{\mathcal{C}} = (\Gamma_u)^{\widetilde{G}}$, we see there is a unique irreducible component χ_0 of $\widetilde{\chi}|_G$ (for example, by [3, Proposition 4.5 (a)]) which is multiplicity one as a constituent of Γ_u. Then χ_0^{σ} has multiplicity one as a constituent of Γ_u^{σ}, and hence of Γ_u by Lemma 4.13. Moreover, since $\widetilde{\chi}^{\sigma} = \widetilde{\chi}$, we also see that χ_0^{σ} is a constituent of $\widetilde{\chi}|_G$. Hence by uniqueness, $\chi_0^{\sigma} = \chi_0$.

Now by Clifford theory, we may write $\chi = \chi_0^g$ for some $g \in \widetilde{G}$, and hence for any $x \in G$,

$$\chi^{\sigma}(x) = \chi(x)^{\sigma} = (\chi_0^g(x))^{\sigma} = \big(\chi_0(g^{-1}xg)\big)^{\sigma} = \chi_0(g^{-1}xg) = \chi_0^g(x) = \chi(x)$$

and hence $\chi^{\sigma} = \chi$, as claimed, and we see that G satisfies Condition 2.1.

(4) Now, we show that S satisfies Condition 2.2. We show the contrapositive, namely, that if $S \leq A \leq \mathrm{Aut}(S)$ is obtained by adjoining a 2-group Q of automorphisms of S, then whenever A does not have a self-normalizing Sylow 2-subgroup (i.e. A does not contain φ), there exists a character $\chi \in \mathrm{Irr}_{2'}(S)$ which is A-invariant but not fixed by σ.

Let A be such a group. Note that a is necessarily at least 2 in the case $\epsilon = +$. We will exhibit $\chi \in \mathrm{Irr}_{2'}(\widetilde{G})$ which is A-invariant such that $\chi|_G$ is irreducible and $\chi|_Z$ is trivial, so that we may view $\chi \in \mathrm{Irr}_{2'}(S)$.

Recall that $\mathrm{Irr}_{2'}(\widetilde{G})$ is comprised of the characters χ_s for $s \in \widetilde{G}^* \cong \widetilde{G}$ semisimple. Moreover, $\mathrm{Irr}(\widetilde{G}/G)$ is given by characters χ_t for $t \in Z(\widetilde{G})$, and $\mathcal{E}(\widetilde{G}, s) \cdot \chi_t = \mathcal{E}(\widetilde{G}, (st))$ for such t. (See, for example [7, Proposition 13.30].) Hence by Lemma 4.14, we see that $\chi_s|_G \in \mathrm{Irr}(G)$ if and only if s is not $\widetilde{G}$-conjugate to st for any nontrivial $t \in Z(\widetilde{G})$. Further, if $s \in [\widetilde{G}, \widetilde{G}] = G$, then χ_s is trivial on $Z(\widetilde{G})$, and hence $Z(G)$ (see [24, Lemma 4.4(ii)]).

Recall that by Lemma 4.6 and (1), $\chi_s^{\sigma} = \chi_{s^2}$ and $\chi_s^{\omega} = \chi_{s^{\omega}}$ for any $\omega \in A$. Hence it suffices to find a semisimple element $s \in G$ so that s is not conjugate in $\widetilde{G}$ to s^2 or st for any $t \in Z(\widetilde{G})$ but the class (s) is invariant under the graph automorphism (in the case $\epsilon = +$) and the smallest power of φ (other than φ itself) which has 2-power order. As in the proof of Theorem 4.10 above, write $q = 2^a$ with $a = 2^t m > 1$, where $t \geq 0$ and $m \geq 1$ is odd, and let ψ be the field automorphism $\psi := \varphi^m$ if $m \neq 1$ and $\psi := \varphi^2$ if $m = 1$. Then again ψ has the largest order among field automorphisms which may be contained in A. (Note that if $\epsilon = +$ and $t = 1 = m$ or $t = 0$, then $\psi = 1$.)

Recall that the conjugacy class of s in $\widetilde{G}$ is determined by its eigenvalues. Let $\overline{q}$ denote q in the case $\epsilon = +$ and q^2 in the case of $\epsilon = -$.

First, assume $q \geq 8$. Consider an element $s \in G$ with eigenvalues $\{\lambda, \lambda^{-1}, 1, ..., 1\}$ for $\lambda \in \mathbb{F}_{\overline{q}}^{\times}$. If τ denotes the graph automorphism (given by inverse-transpose) in the case of $\epsilon = +$, then s^{τ} has eigenvalues $\{\lambda^{-1}, \lambda, 1, ..., 1\}$. Then s^{τ} is $\widetilde{G}$-conjugate to s. Also, for $1 \neq t \in Z(\widetilde{G})$, we may write $t = \mu \cdot I_n$ for $1 \neq \mu \in \mathbb{F}_{\overline{q}}^{\times}$, so that st has eigenvalues $\{\mu\lambda, \mu\lambda^{-1}, \mu, ..., \mu\}$. Then by comparing multiplicities of eigenvalues, we see that st cannot be conjugate to s in $\widetilde{G}$, except possibly when $\lambda^3 = 1$ and $n \leq 3$. Moreover, s^2 has eigenvalues $\{\lambda^2, \lambda^{-2}, 1, ..., 1\}$.

If $m \neq 1$, take λ such that $\lambda^{2^m-1} = 1$ but $\lambda^3 \neq 1$. (Note that such a λ exists because $2^m \geq 8$.) If $m = 1$, we have $a = 2^t$ with $t \geq 2$, so take λ with multiplicative order 5 in $\mathbb{F}_q^{\times}$ so that ψ maps λ to $\lambda^4 = \lambda^{-1}$. Then in either case $(s^2) \neq (s)$, but $(s^{\psi}) = (s)$. Hence $\chi := \chi_s \in \mathrm{Irr}(\widetilde{G})$ is irreducible on G, $\chi|_G$ can be viewed as an irreducible character of S, and χ is fixed by A but not by σ.

We claim that $\chi|_G$ is also moved by σ, which will complete the proof for $q \geq 8$. Suppose otherwise, so that $\chi_s|_G = \chi_{s^2}|_G$. By Gallagher's theorem, this means $\chi_{s^2} = \chi_s\chi_t = \chi_{st}$ for some

$1 \neq t \in Z(\widetilde{G})$. That is, s^2 is conjugate to st. But by again comparing eigenvalues, we see that this cannot occur for our choice of s.

Next, assume $q = 4$. Take s to have eigenvalues $\{\lambda, \lambda^4, 1, ..., 1\}$ for $\lambda \in \mathbb{F}_{4^2}^\times$ satisfying $\lambda^5 = 1$. Then the same argument as for $a = 2^t$ with $t \geq 2$ works.

Finally, let $q = 2$, so $\epsilon = -$ and $\overline{q} = 4$. Note that the two classes of $SU_4(2)$ containing elements of order 9 are fixed by ψ and interchanged by squaring. (The eigenvalues are $\{\lambda_4, \lambda_{64}^{14}, \lambda_{64}^{35}, \lambda_{64}^{56}\}$ and $\{\lambda_4^2, \lambda_{64}^7, \lambda_{64}^{28}, \lambda_{64}^{49}\}$, where λ_r denotes a generator of $\mathbb{F}_r^\times$.) Moreover, observing the eigenvalues shows that these classes are not fixed by multiplication by any $t \in Z(GU_4(2))$. Hence taking s to be an element embedded into $G = SU_n(2)$ from either of these classes in $SU_4(2)$, we see that $\chi := \chi_s$ satisfies our conditions.

□

5 Some Additional Remarks

We remark that to prove Problem 1, it remains to prove that most simple groups of Lie type in odd characteristic, as well as the groups $E_6^\pm(2^a)$ with nontrivial center, are SN2S-Good. One of the main difficulties that arises here is that for most of these groups, $Z(\mathbb{G})$ is disconnected. Hence when σ fixes a series $\mathcal{E}(G, s)$, the characters corresponding to a specific unipotent character of $C_{G^*}(s)^\circ$ may be permuted, so the strategy for much of Section 4.1 fails. However, we hope that an argument similar to the case of type A can be found.

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