Irreducible Representations of Finite Groups of Lie Type: On the Irreducible Restriction Problem and Some Local-Global Conjectures

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Statement by Author

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Abstract

In this thesis, we investigate various problems in the representation theory of finite groups of Lie type. In Chapter 2, we hope to make sense of the last statement - we will introduce some background and notation that will be useful for the remainder of the thesis. In Chapter 3, we find bounds for the largest irreducible representation degree of a finite unitary group, along the lines of [42]. In Chapter 4, we describe the block distribution and Brauer characters in cross characteristic for $Sp_6(2^a)$ in terms of the irreducible ordinary characters. This will be useful in Chapter 5 and Chapter 7, which focus primarily on the group $Sp_6(2^a)$ and contain the main results of this thesis, which we now summarize.

Given a subgroup $H \leq G$ and a representation V for G, we obtain the *restriction* $V|_H$ of V to H by viewing V as an $\mathbb{F}H$-module. However, even if V is an irreducible representation of G, the restriction $V|_H$ may (and usually does) fail to remain irreducible as a representation of H. In Chapter 5, we classify all pairs (V, H), where H is a proper subgroup of $G = Sp_6(q)$ or $Sp_4(q)$ with q even, and V is an ℓ-modular representation of G for $\ell \neq 2$ which is absolutely irreducible as a representation of H. This problem is motivated by the Aschbacher-Scott program on classifying maximal subgroups of finite classical groups.

The local-global philosophy plays an important role in many areas of mathematics. In the representation theory of finite groups, the so-called "local-global" conjectures would relate the representation theory of G to that of certain proper subgroups, such as the normalizer $N_G(P)$ of a Sylow subgroup. One might hope that these conjectures could be proven by showing that they are true for all simple groups. Though this turns out not quite to be the case, some of these conjectures have been reduced to showing that a finite set of stronger conditions hold for all finite simple groups. In Chapter 7, we show that $Sp_6(q)$ and $Sp_4(q)$, q even, are "good" for these reductions.

Chapter 1
Introduction

The study of group theory was motivated by the desire to understand the symmetry of an object, whether it be in nature, art, communication networks, or any other place that symmetry might play a role. Representation theory is a tool used to better understand the structure of a group and the symmetries it represents. Roughly speaking, representations provide a way to view, in some sense, an abstract group as a group of matrices whose structure is often easier to understand. In particular, we are interested in irreducible representations, which are in a sense the building blocks of all representations.

It is well-known that any finite group G has a composition series, i.e. a subnormal series $1 \leq N_1 \lhd N_2 \lhd .. \lhd N_k \lhd G$ in which each factor N_i/N_{i-1} is simple. The factors of this series are called the composition factors, and the Jordan-Hölder Theorem states that any finite group has a unique set of composition factors (up to isomorphism and reordering). This suggests that many questions about finite groups can be reduced to questions about finite simple groups or groups closely related to simple groups, such as *almost simple* groups (groups satisfying $G_0 \lhd G \leq \mathrm{Aut}(G_0)$ for a finite simple group G_0) and *quasisimple* groups (perfect groups for which $G/Z(G)$ is simple). Hence, for several decades, the main goal of many group theorists was to completely classify all finite simple groups.

In 2004, the Classification of Finite Simple Groups was completed, and is seen by many as the most important result in finite group theory. The completion of the Classification has opened the door to many interesting questions in group theory by giving us a hope of reducing to the case of simple groups and using the Classification to finish the proof. This is precisely the idea behind recent reductions to various local-global conjectures in representation theory, which will be of interest in Chapter 7. The Classification states that every nonabelian finite simple group falls into one of the following three categories:

- the alternating groups A_n, $n \geq 5$
- the finite groups of Lie type
- one of 26 sporadic groups

The finite groups of Lie type are sometimes called finite reductive groups, and are analogues of Lie groups over finite fields. The Classification of Finite Simple Groups tells us that "most" finite non-abelian simple groups are groups of Lie type, which is why they are of such interest, and the focus of this thesis. This class of groups can be further divided into the finite classical groups and the so-called exceptional groups. In the remainder of this thesis, we look at various problems concerning the cross-characteristic representations of finite classical groups, with a particular emphasis on the symplectic group $Sp_6(2^a)$ and the unitary groups $GU_n(q)$. In Chapter 2, we introduce some background and notation that will be useful for the remainder of the thesis.

One problem of interest in the representation theory of finite groups is to find the largest dimension of an irreducible representation of a given group. Though an explicit formula is often very difficult to achieve, it is sometimes possible to find bounds for this number, and the question often turns into finding the "correct" asymptotic. In Chapter 3, we find bounds for the largest dimension of an irreducible representation for a finite unitary group defined over a field $\mathbb{F}_q$ of characteristic p. Our bounds show that this number divided by the p-part of the group order grows like a polynomial in $\log_q$ of the rank, as we vary the rank of the group.

In Chapter 4, we describe the block distribution and Brauer characters in cross characteristic for $Sp_6(2^a)$ in terms of the irreducible ordinary characters. In particular, we classify the low-dimensional irreducible ℓ-modular representations of this group. This information will be crucial in Chapter 5 and Chapter 7, which focus primarily on the group $Sp_6(2^a)$ and contain the main results of this thesis.

The first of the main problems dealt with in this thesis is concerned with the restrictions of representations to proper subgroups. Given a subgroup $H \leq G$ and an $\mathbb{F}G$-module V, we obtain the *restriction* $V|_H$ of V to H by viewing V as an $\mathbb{F}H$-module. However, even if V is an irreducible representation of G, the restriction $V|_H$ may (and usually does) fail to remain irreducible as a representation of H. In Chapter 5, we classify all pairs (V, H), where H is a proper subgroup of $G = Sp_6(q)$ or $Sp_4(q)$ with q even, and V is an ℓ-modular representation of G for $\ell \neq 2$ which is absolutely irreducible as a representation of H. We note that these results can also be found in a more concise form in [62], which is available on the ArXiv (arXiv:1204.5514v1). In Chapter 6, we also discuss this problem for complex representations of the unitary groups $GU_n(q)$ with $n < 10$. This "restriction problem" is motivated by the Aschbacher-Scott program on classifying maximal subgroups of finite classical groups. In Section 1.1, we describe in more detail the motivation behind this problem and state our main results regarding the problem.

The other main problem we are concerned with in this thesis involves local-global conjectures in representation theory. The local-global philosophy plays an important role in many areas of mathematics, and in the representation theory of finite groups, the so-called "local-global" conjectures relate the representation theory of G to that of certain proper subgroups, such as the normalizer $N_G(P)$ of a Sylow p-subgroup. One might hope that these conjectures could be proven by showing that they are true for all simple groups. Though this turns out not quite to be the case, some of these conjectures have been reduced to showing that a finite set of stronger conditions hold for all finite simple groups. In Chapter 7, we show that $G = Sp_6(q)$ and $Sp_4(q)$, q even, are "good" for these reductions. That is, we show that these groups satisfy each condition in these lists. A more concise version of these results was submitted for publication in 2012 and is available on ArXiv (arXiv:1212.5622v1). In Section 1.2, we discuss the specific conjectures that we will be concerned with and state our main result.

1.1 The Aschbacher-Scott Program and the Irreducible Restriction Problem

In this section, we provide a brief overview and motivation for the Aschbacher-Scott program and the restriction problem discussed in Chapters 5 and 6.

The main motivation for the Aschbacher-Scott program and the classification of maximal subgroups is to understand the finite primitive permutation groups, which have been a topic of interest going back to the time of Galois and have applications to many areas of mathematics, including number theory, algebraic geometry, graph theory, and combinatorics. Since a transitive permutation group $X \leq \mathrm{Sym}(\Omega)$ is primitive if and only if any point stabilizer $H = \mathrm{stab}_X(\alpha)$, for $\alpha \in \Omega$, is a maximal subgroup, we can view the study of primitive permutation groups as equivalent to studying maximal subgroups. Thanks to the Aschbacher-O'Nan-Scott Theorem [7], the problem can be reduced to the case of *almost quasi-simple* groups (that is, central extensions of almost simple groups), and the results of Liebeck-Praeger-Saxl [44] and Liebeck-Seitz [45] allow us to further reduce to the case that X is a classical group.

In this case, Aschbacher has described all possible choices for the maximal subgroup H (see [8]). Namely, he has described 8 collections $\mathcal{C}_1, ..., \mathcal{C}_8$ of subgroups obtained in natural ways (for example, stabilizers of certain subspaces of the natural module for X), and has shown that if H is not contained in one of these subgroups, then H lies in a collection $\mathcal{S}$ of almost quasi-simple groups which act absolutely irreducibly on the natural module, V, for X. The question of whether a subgroup H in $\bigcup_{i=1}^{8} \mathcal{C}_i$ is in fact maximal has been answered by Kleidman and Liebeck (see [37]) in the case that $\dim V \geq 13$. The case that V has smaller dimension is considered in [10], as well as in the early work on the problem in [19] and [51] for the cases of $SL_2(q)$ and $SL_3(q)$. When $H \in \mathcal{S}$, we want to decide whether there is some maximal subgroup G such that $H < G < X$, that is, if H is not maximal. The most challenging case is when G also lies in the collection $\mathcal{S}$. This suggests the following problem, which is the motivation for Chapters 5 and 6.

Problem 1. Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell \geq 0$. Classify all triples (G, V, H) where G is a finite group with $G/Z(G)$ almost simple, V is an $\mathbb{F}G-$module of dimension greater than 1, and H is a proper subgroup of G such that the restriction $V|_H$ is irreducible.

Although the motivation may suggest that we fix the group H and try to find all pairs (G, V) which create a triple as in Problem 1, in practice it is more practical to fix G and find all pairs (V, H). In [13], [38], and [39], Brundan, Kleshchev, Sheth, and Tiep have solved Problem 1 for $\ell > 3$ when $G/Z(G)$ is an alternating or symmetric group. Liebeck, Seitz, and Testerman have obtained results for groups of Lie type in defining characteristic in [43], [63], and [65].

Assume now that G is a finite group of Lie type defined in characteristic $p \neq \ell$, with q a power of p. In [57], Nguyen and Tiep show that when $G = {}^3D_4(q)$, the restrictions of irreducible representations are reducible over every proper subgroup, and in [28], Himstedt, Nguyen, and Tiep prove that this is the case for $G = {}^2F_4(q)$ as well. Nguyen shows in [55] that when $G = G_2(q), {}^2G_2(q)$, or ${}^2B_2(q)$, there are examples of triples as in Problem 1 and finds all such examples.

Gary Seitz [64] has made a huge breakthrough in the Aschbacher-Scott program by providing a list of possibilities for (H, G) as in Problem 1 in the case that H is a finite group of Lie type and G is a finite classical group, both defined in the same characteristic. In Seitz's main theorem restated below, the notation $H = \underline{H}(p^a)$ means a group of Lie type defined over the field $\mathbb{F}_{p^a}$, with corresponding simple algebraic group $\underline{H}$ over an algebraically closed field of characteristic p.

Theorem (Seitz: Main Theorem of [64]). *Let X be a classical group with natural module $V = \mathbb{F}_{\ell^c}$ where $\ell \neq p$. Assume (i) $H = \underline{H}(p^a)$ with $p^a > 3$, $G = \underline{G}(p^b)$, and G is of classical type; (ii) $H < G < X$; and (iii) H is not a subgroup of a group in $\mathcal{C}(X)$. Then there is a quasisimple group A with $H \leq A \leq G$ such that the pair $(A/Z(A), G/Z(G))$ is one of the following:*

(i) $(PSp_{2n}(q), PSL^{\pm}_{2n}(q))$;

(ii) $(P\Omega_{n-1}(q), P\Omega_n(q))$;

(iii) $(PSp_{2n}(q^s), PSp_{2ns}(q))$;

(iv) $(G_2(q), P\Omega_7(q))$ *or* $(G_2(q), PSp_6(q))$ *(if* $p = 2$*)*;

(v) $(PSU_{n-1}(q), PSU_n(q))$, *with* $(q+1)|n$.

Another important breakthrough to this problem in the case that G is a finite group of Lie type was achieved in [40], where Kleshchev and Tiep solve Problem 1 in the case that $SL_n(q) \leq G \leq GL_n(q)$, which resolves the pair $(PSp_{2n}(q), PSL_{2n}(q))$ of (i) in Seitz' list.

For the remaining pairs in Seitz' list, the question of whether (H, G) indeed gives rise to triples as in Problem 1 remains open, and for some of these pairs, the resolution of the question would require major advancements in the cross-characteristic representation theory of finite groups of Lie type. Our goal here is more modest - we solve Problem 1 in the case that $G = Sp_{2n}(q)$ for $n = 2, 3$ with q even, and H is a proper subgroup. This will resolve one of the cases in Seitz' list, namely the pair $(G_2(2^a), Sp_6(2^a))$.

Note that in order to restrict irreducibly to a proper subgroup, a representation must have sufficiently small degree. Hence, in considering this problem, it will be useful to understand the low-dimensional ℓ-modular representations of $Sp_6(q)$. In Section 4.2 of Chapter 4 below, we prove the following theorem, which describes these representations. In the theorem, let

$$\delta_1 := \begin{cases} 1_G, & \ell|(q^2+q+1), \\ 0, & \text{otherwise,} \end{cases} \quad \delta_2 := \begin{cases} 1_G, & \ell|(q+1), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\delta_3 := \begin{cases} 1_G, & \ell|(q^3+1), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $\alpha_3, \beta_3, \rho_3^1, \rho_3^2, \tau_3^i$, and ζ_3^i denote the complex Weil characters of $Sp_6(q)$, as in [27] (see Table 4.2), and let χ_j, $1 \le j \le 35$ be as in the notation of [76].

Theorem 1.1.1. *Let $G = Sp_6(q)$, with $q \ge 4$ even, and let $\ell \ne 2$ be a prime dividing $|G|$. Suppose $\chi \in \mathrm{IBr}_\ell(G)$. Then:*

A) If χ lies in a unipotent ℓ-block, then either

1. $\chi \in \left\{1_G, \widehat{\alpha}_3, \widehat{\rho}_3^1 - \delta_1, \widehat{\beta}_3 - \delta_2, \widehat{\rho}_3^2 - \delta_3\right\}$,

2. χ is as in the following table:

Condition on ℓ	χ	*Degree $\chi(1)$*
$\ell\|(q^3-1)$ *or* $3 \ne \ell\|(q^2-q+1)$	$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\ell\|(q^2+1)$	$\widehat{\chi}_6 - 1_G$	$q^2(q^4+q^2+1)-1$
$\ell\|(q+1)$	$\widehat{\chi}_{28}$ $= \widehat{\chi}_6 - \widehat{\chi}_3 - \widehat{\chi}_2 + 1_G$	$(q^2+q+1)(q-1)^2(q^2+1)$

,

3. χ is as in the following table:

Condition on ℓ	χ	*Degree $\chi(1)$*
$\ell\|(q^3-1)$ *or* $3 \ne \ell\|(q^2-q+1)$	$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\ell\|(q^2+1)$	$\widehat{\chi}_7 - \widehat{\chi}_4$	$q^3(q^4+q^2+1) - q(q+1)(q^3+1)/2$
$\ell\|(q+1)$	$\widehat{\chi}_{35} - \widehat{\chi}_5$ $= \widehat{\chi}_7 - \widehat{\chi}_6 + \widehat{\chi}_3 - \widehat{\chi}_1$	$(q-1)(q^2+1)(q^4+q^2+1) - q(q-1)(q^3-1)/2$

,

or

4. $\chi(1) \ge D$, where D is as in the table:

Condition on ℓ	D
$\ell\|(q^3-1)(q^2+1)$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$
$\ell\|(q+1)$, $(q+1)_\ell \ne 3$	$\frac{1}{2}q(q^3-2)(q^2+1)(q^2-q+1) - \frac{1}{2}q(q-1)(q^3-1)+1$
$\ell\|(q+1)$, $(q+1)_\ell = 3$	$\frac{1}{2}q(q^3-2)(q^2+1)(q^2-q+1)+1$
$3 \ne \ell\|(q^2-q+1)$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1) - \frac{1}{2}q(q-1)^2(q^2+q+1) = \frac{1}{2}q(q^3-1)^2(q-1)$

B) If χ does not lie in a unipotent block, then either

1. $\chi \in \{\widehat{\tau}_3^i, \widehat{\zeta}_3^j\}_{1 \le i \le ((q-1)_{\ell'}-1)/2, 1 \le j \le ((q+1)_{\ell'}-1)/2}$,

2. $\chi(1) = (q^2+1)(q-1)^2(q^2+q+1)$ *or* $(q^2+1)(q+1)^2(q^2-q+1)$ *(here* χ *is the restriction to* ℓ*-regular elements of the semisimple character indexed by a semisimple* ℓ' *- class in the family* $c_{6,0}$ *or* $c_{5,0}$ *respectively, in the notation of [47] - see Table 4.1),*

3. $\chi(1) = (q-1)(q^2+1)(q^4+q^2+1)$ *or* $(q+1)(q^2+1)(q^4+q^2+1)$ *(here* χ *is the restriction to* ℓ*-regular elements of the semisimple character indexed by a semisimple* ℓ' *- class in the family* $c_{10,0}$ *or* $c_{8,0}$ *respectively, in the notation of [47] - see Table 4.1), or*

4. $\chi(1) \geq q(q^4+q^2+1)(q-1)^3/2$.

Note that in the case $n = 3$, Theorem 1.1.1 generalizes [27, Theorem 6.1], which gives the corresponding bounds for ordinary representations of $Sp_{2n}(q)$ with q even.

Our main result in Chapter 5 is the following complete classification of triples (G, V, H) as in Problem 1 in the case $G = Sp_6(q)$ with $q \geq 4$ even.

Theorem 1.1.2. *Let* q *be a power of* 2 *larger than* 2*, and let* (G, V, H) *be a triple as in Problem 1, with* $\ell \neq 2$*,* $G = Sp_6(q)$*, and* $H < G$ *a proper subgroup. Then:*

1. $P_3' \leq H \leq P_3$*, the stabilizer of a totally singular* 3*-dimensional subspace of the natural module* $\mathbb{F}_q^6$*, and the Brauer character afforded by* V *is the Weil character* $\widehat{\alpha_3}$*; or*

2. $H = G_2(q)$*, and the Brauer character afforded by* V *is one of the Weil characters*
   - $\widehat{\rho}_3^1 - \begin{cases} 1, & \ell \mid \frac{q^3-1}{q-1}, \\ 0, & \textit{otherwise}, \end{cases}$ *degree* $q(q+1)(q^3+1)/2 - \begin{cases} 1 \\ 0 \end{cases}$
   - $\widehat{\tau}_3^i$, $1 \leq i \leq ((q-1)_{\ell'} - 1)/2$, *degree* $(q^6-1)/(q-1)$
   - $\widehat{\alpha}_3$, *degree* $q(q-1)(q^3-1)/2$
   - $\widehat{\zeta}_3^i$, $1 \leq i \leq ((q+1)_{\ell'} - 1)/2$, *degree* $(q^6-1)/(q+1)$.

as in the notation of [27] (see Table 4.2).

Moreover, each of the above situations indeed gives rise to such a triple (G, V, H).

Note that Theorem 1.1.2 tells us that pair (ii) in the main theorem of [64] does not occur for the case $n = 7$, q even, and that pair (iv) does occur.

We also prove the following complete classifications of triples as in Problem 1 when H is a maximal subgroup of $G = Sp_4(q)$, $q \geq 4$ even, $G = Sp_6(2)$, and $G = Sp_4(2)$.

Theorem 1.1.3. *Let* q *be a power of* 2 *larger than* 2, $\ell \neq 2$, $G = Sp_4(q)$, *and* $H < G$ *a maximal subgroup. Then* (G, V, H) *is a triple as in Problem 1 if and only if* $H = P_2$, *the stabilizer of a totally singular* 2*-dimensional subspace of the natural module* $\mathbb{F}_q^4$, *and the Brauer character afforded by* V *is the Weil character* $\widehat{\alpha_2}$.

Theorem 1.1.4. *Let* (G, V, H) *be a triple as in Problem 1, with* $\ell \neq 2$, $G = Sp_4(2) \cong S_6$, *and* $H < G$ *a maximal subgroup. Then one of the following situations holds:*

1. $H = A_6$,

2. $H = A_5.2 = S_5$,

3. $H = O_4^-(2) \cong S_5 = A_6.2_1M3$ *in the notation of [11].*

Moreover, each of the above situations indeed gives rise to such a triple (G, V, H).

Theorem 1.1.5. *Let* (G, V, H) *be a triple as in Problem 1, with* $\ell \neq 2$, $G = Sp_6(2)$, *and* $H < G$ *a maximal subgroup. Then one of the following situations holds:*

1. $H = G_2(2) = U_3(3).2$, *and*
   - $\ell = 0, 5, 7$ *and* V *affords the Brauer character* $\widehat{\alpha}_3$, $\widehat{\zeta}_3^1$, $\widehat{\rho}_3^1 - \begin{cases} 1, & \ell = 7 \\ 0, & \text{otherwise}, \end{cases}$ *or* $\widehat{\chi}_9$, *where* χ_9 *is the unique irreducible complex character of* $Sp_6(2)$ *of degree* 56.
   - $\ell = 3$ *and* V *affords the Brauer character* $\widehat{\alpha}_3$ *or* $\widehat{\rho}_3{}^1$.

2. $H = O_6^-(2) \cong U_4(2).2$, *and the Brauer character afforded by* V *is the Weil character* $\widehat{\beta_3}$.

3. $H = O_6^+(2) \cong L_4(2).2 \cong A_8.2$, *and the Brauer character afforded by* V *is either the Weil character* $\widehat{\alpha_3}$, *the character* $\widehat{\chi_7}$ *where* χ_7 *is the unique irreducible character of degree* 35 *which is not equal to* ρ_3^2, *or the character* $\widehat{\chi_4}$ *where* χ_4 *is the unique irreducible character of degree* 21 *which is not equal to* ζ_3^1.

4. $H = 2^6 : L_3(2)$, *and the Brauer character afforded by* V *is* $\widehat{\alpha}_3$ *or* $\widehat{\chi_4}$ *where* χ_4 *is the unique irreducible character of* G *of degree* 21 *which is not equal to* ζ_3^1.

5. $H = L_2(8).3$, *and* V *affords one of the Brauer characters:*

- $\widehat{\alpha}_3$,
- $\widehat{\zeta}_3^1$, $\quad \ell \neq 3$,
- $\widehat{\rho}_3^1$, $\quad \ell \neq 7$, *or*
- $\widehat{\chi_4}$ *where* χ_4 *is the unique irreducible complex character of* $Sp_6(2)$ *of degree* 21 *which is not equal to* ζ_3^1, $\quad \ell \neq 3$.

Moreover, each of the above situations indeed gives rise to such a triple (G, V, H).

We note that unlike the case $q \geq 4$, we do not discuss the descent to non-maximal proper subgroups of $Sp_6(2)$ in Theorem 1.1.5, as there are many examples of such triples in this case.

In Chapter 6, we also begin a discussion of pair (v) of Seitz' list as stated above. There, we show that this pair yields no triples for $n < 8$ in the case that V is defined in characteristic 0.

1.2 Local-Global Conjectures

Much of the representation theory of finite groups is dedicated to showing the validity of various conjectures which relate certain invariants of a finite group with those of certain subgroups. Often, these have to do with the number of characters of the group of a given type. One of the first of these "local-global" or "counting" conjectures was proposed by McKay [50] in 1972. Though the original conjecture was more restricted, the following is the McKay conjecture as it is known today.

McKay Conjecture. Let G be a finite group, $\ell||G|$ a prime, and $P \in \mathrm{Syl}_\ell(G)$. Then $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))|$.

Here, $\mathrm{Irr}_{\ell'}(G)$ represents the set of irreducible characters of G with degree prime to ℓ. Though there is much evidence for the validity of the McKay conjecture, it is still open, and the question of *why* it should be true remains unclear. Many refinements to the conjecture have been proposed, and a reduction theorem has been proved, with the hope of providing not only a method by which to prove it, but also a better understanding of the deeper underlying reason behind it. For example, Alperin [2] later extended the McKay conjecture to include the role of blocks. The new conjecture, known as the Alperin-McKay conjecture, uses Brauer's First Main Theorem, which says that block induction $b \mapsto b^G$ gives a bijection between blocks B of G with defect group D and blocks b of $N_G(D)$ with defect group D. Recall that a character χ in the block B has height zero if its degree satisfies $\chi(1)_\ell = |G|_\ell/|D|_\ell$.

Alperin-McKay Conjecture. Let G be a finite group, B an ℓ-block of G with defect group D, and b the block of $N_G(D)$ with $b^G = B$. Then the number of height zero characters of B and b coincide.

In [34], Isaacs, Malle, and Navarro prove a reduction theorem for the McKay conjecture. They describe a list of conditions that a simple group must satisfy in order to be "good" for the McKay conjecture for a prime ℓ. The reduction says that if every finite simple group is "good" for the McKay conjecture for ℓ, then *every* finite group satisfies the McKay conjecture for the prime ℓ. In [69], Späth provides a reduction for the Alperin-McKay conjecture along the same lines, providing a list of conditions for a group to be "good", and proving that if every finite simple group is good for a prime ℓ, then all finite groups satisfy the Alperin-McKay conjecture for that prime.

The other conjectures that we will be concerned with in this thesis involve ℓ-*weights* of a finite group. An ℓ-weight of G is a pair (Q, μ), where Q is an ℓ-radical subgroup (i.e. an ℓ-subgroup with $Q = \mathbf{O}_\ell(N_G(Q))$) and μ is a defect-zero character of $N_G(Q)/Q$, that is, an irreducible character with $\mu(1)_\ell = |N_G(Q)/Q|_\ell$. In 1986, Alperin [3] made the following conjecture:

Alperin Weight Conjecture (AWC). Let G be a finite group and ℓ a prime. Then the number of irreducible Brauer characters of G equals the number of G-conjugacy classes of ℓ-weights of G.

The AWC aims to provide an analogue for finite groups to the situation of ℓ-modular representations of algebraic groups, where the representations are in bijection with the dominant weights of the algebraic group. This is the motivation for referring to the collection of such pairs (Q, μ) as weights for the finite group G.

More generally, a weight for a block B of G is a weight (Q, μ) as before, where μ lies in a block b of $N_G(Q)$ for which the induced block b^G is B. Again, we have an extension of this conjecture to one which involves the role of blocks of the group, giving the conjecture more structure:

Blockwise Alperin Weight Conjecture (BAWC). Let G be a finite group, ℓ a prime, and B an ℓ-block of G. Then the number of irreducible Brauer characters belonging to B equals the number of G-conjugacy classes of ℓ-weights of B.

In [53], Navarro and Tiep prove a reduction for the Alperin weight conjecture in the same spirit as those for the McKay and Alperin-McKay conjectures, and in [70], Späth extends this reduction to the blockwise version of the Alperin weight conjecture.

The reductions for these conjectures give us hope of proving them by appealing to the classification of finite simple groups. It has been shown that under certain conditions, a simple group of Lie type is "good" for the various conjectures, but it still needs to be shown in general. For example, it is known (see [53], [70]) that a simple group of Lie type defined in characteristic p is "good" for the Alperin weight and blockwise Alperin weight conjectures for the prime $\ell = p$, but the question is still open when $\ell \neq p$. Similarly, for $\ell = p \geq 5$, Späth has shown in [69, Proposition 8.4] that a simple group of Lie type defined in characteristic p is "good" for the Alperin-McKay conjecture (and therefore also the McKay conjecture) for the prime $\ell = p$. It is worth noting that for $q \geq 4$ a power of 2, the same argument shows that indeed, $Sp_6(q)$ and $Sp_4(q)$ are good for the Alperin-McKay conjecture for the prime 2, as the Schur multiplier is non-exceptional in these cases.

In [14], Cabanes shows that $Sp_4(2^a)$ is "good" for the McKay conjecture for all primes $\ell \neq 2$. According to the discussion preceding [70, Theorem A], G. Malle has shown that alternating groups, and therefore $Sp_4(2)' \cong A_6$, are "good" for the blockwise Alperin weight conjecture. The main theorem of Chapter 7 is the following statement, which therefore implies that $Sp_6(2^a)$ and $Sp_4(2^a)$ are "good" for each of these conjectures for every prime:

Theorem 1.2.1. *The simple groups $Sp_6(q)$ with q even and $Sp_4(q)$ with $q \geq 4$ even are "good" for the McKay, Alperin-McKay, Alperin weight, and blockwise Alperin weight conjectures for all primes $\ell \neq 2$. Moreover, the simple group $Sp_4(2)'$ is "good" for the Alperin-McKay conjecture for all primes ℓ (including $\ell = 2$) and $Sp_6(2)$ is "good" for the Alperin-McKay conjecture for the prime $\ell = 2$.*

Though our proof is rather specialized, we hope that it will lead us to find a more general underlying pattern which will give us an idea of how to extend these results to higher rank symplectic groups.

Chapter 2
Preliminaries

In this chapter, we introduce some notation and background that will be useful in the subsequent chapters, in an effort to make this thesis somewhat self-contained, though some of the notation will be reiterated in later chapters. As our focus will be on the finite classical groups, we describe a number of ways to construct them in Section 2.2. In Section 2.3, we discuss some basic information on representations before specializing to the representations of finite classical groups. In later chapters, we will be paying particular attention to the groups $GU_n(q)$ and $Sp_{2n}(2^a)$, so in Section 2.4, we discuss the structure of centralizers of semisimple elements in these groups.

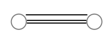
2.1 Groups of Lie Type

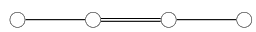
A group of Lie type G can be identified with the fixed points of a connected reductive algebraic group $\underline{G}$ under a Frobenius map F. (For this reason, groups of Lie type are sometimes referred to as *finite reductive groups*.) If G is a finite group of Lie type over $\mathbb{F}_q$, let $k = \overline{\mathbb{F}_q}$ and let $\underline{G}$ be regarded as a subgroup of some $GL_n(k)$. A map $F\colon (a_{ij}) \to (a_{ij}^q)$ which maps $\underline{G} \to \underline{G}$ is called a standard Frobenius map. More generally, a Frobenius map from $\underline{G}$ to $\underline{G}$ is a morphism such that some power is a standard Frobenius map. Given a Frobenius map F, the group $G = \underline{G}^F = \{g \in \underline{G} | F(g) = g\}$ is the set of fixed points under F. The untwisted groups are obtained from standard Frobenius maps, whereas the twisted groups arise from other Frobenius maps.

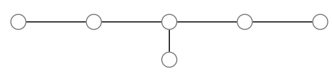
Simple groups of Lie type can also be thought of as the Chevalley groups $\mathfrak{L}(q)$ for simple Lie algebras $\mathfrak{L}$ over the field $\mathbb{F}_q$ and their twisted counterparts resulting from symmetries of the Dynkin diagram, as in the notation of [15]. $\mathfrak{L}(q)$ is the group of automorphisms of $\mathfrak{L}$ generated by the elements $x_r(t) := \exp(t\operatorname{ad} e_r)$, following the notation of Carter [15], where $t \in \mathbb{F}_q$, r is a root, and e_r is a root vector in a Chevalley basis for $\mathfrak{L}$.

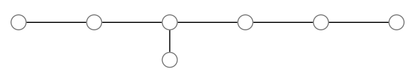
2.1.1 Exceptional Groups of Lie Type

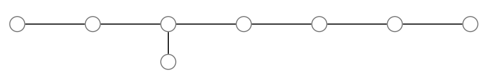
By an exceptional group of Lie type, we mean the fixed points under a Frobenius morphism of a simple algebraic group or the finite analogue of the Lie group (and their twisted counterparts) corresponding to one of the exceptional simple Lie algebras $\mathfrak{L} = G_2, F_4, E_6, E_7$, or E_8, whose root systems are shown below.

G_2 : 

F_4 : 

E_6 : 

E_7 : 

E_8 : 

These can come in one of two forms. First, G could be the Chevalley group $\mathfrak{L}(q)$ corresponding to the simple Lie algebras $\mathfrak{L}$ over a finite field $\mathbb{F}_q$, omitting the case $q = 2$ when $\mathfrak{L} = B_2, G_2$. (See [15].)

Second, G can be a twisted group obtained by a nontrivial graph automorphism (an automorphism induced from a nontrivial symmetry of the Dynkin diagram) of the Chevalley group $\mathfrak{L}(q)$ in the cases where $\mathfrak{L}$ is E_6 (for q a square), B_2 (for $q = 2^{2m+1} > 2$), D_4 (an order-3 symmetry, for q a cube), F_4 (for $q = 2^{2m+1} > 2$), or G_2 (for

$q = 3^{2m+1} > 3$). These twisted exceptional groups are denoted ${}^2E_6(q^2)$, ${}^2B_2(2^{2m+1}) := Sz(q)$, ${}^3D_4(q^3)$, ${}^2G_2(3^{2m+1})$, and ${}^2F_4(2^{2m+1})$. The latter two are sometimes called the Ree groups, $Sz(q)$ is called a Suzuki group, and ${}^3D_4(q^3)$ are sometimes called Steinberg triality groups. While the groups ${}^2B_2(2), {}^2F_4(2)$, and ${}^2G_2(3)$ are not simple, the commutator subgroups ${}^2F_4(2)'$ and ${}^2G_2(3)'$ are simple.

2.2 The Finite Classical Groups

There are various ways to construct the groups of Lie type, but unlike the exceptional groups, the classical groups can be realized nicely as certain groups of matrices over finite fields. They essentially fall into the classes of finite linear, unitary, symplectic, and orthogonal groups. There are many ways in which to view these groups. For example, we can view them as matrix groups or groups of transformations. We can also view them as certain groups of Lie type, or as a set of fixed points of a simple algebraic group with respect to a Frobenius endomorphism.

2.2.1 The Classical Groups as Matrix Groups

Let $V = \mathbb{F}_q^n$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$. The general linear group $GL(V)$ of V is the set of non-singular $\mathbb{F}_q$-linear transformations of V. Choosing a basis, we get $GL(V) \cong GL_n(q)$, the set of $n \times n$ matrices over $\mathbb{F}_q$ with nonzero determinant. Taking the subgroup of $GL_n(q)$ of elements with determinant 1, we obtain the special linear group $SL_n(q) \cong SL(V)$. Since $SL_n(q)$ is the kernel of the determinant map onto $\mathbb{F}_q^\times$, it is clear that $SL_n(q) \lhd GL_n(q)$ with index $q-1$. The set $\{\lambda \cdot I | \lambda \in \mathbb{F}_q^\times\}$ forms the center $Z(GL_n(q))$ of $GL_n(q)$, and taking the intersection $Z(GL_n(q)) \cap SL_n(q)$ gives the center of $SL_n(q)$. We obtain the projective general linear group and projective special linear group as the quotients $PGL_n(q) := GL_n(q)/Z(GL_n(q))$ and $PSL_n(q) := SL_n(q)/Z(SL_n(q))$, respectively. The groups $PSL_n(q)$ are simple as long as $(n, q) \notin \{(2, 2), (2, 3)\}$. Hence, when referring to the finite simple groups, $PSL_n(q)$ is the one we mean. Note also that $PGL_n(q)$ is almost simple, and $SL_n(q)$ is quasi-simple, aside from the above exceptions.

The other classical groups are subgroups of the linear groups, and can be defined as certain isometry groups for particular forms. A bilinear form $f\colon V \times V \to \mathbb{F}_q$ is symmetric if $f(v,w) = f(w,v)$ for all $w, v \in V$ and is skew-symmetric if $f(v,w) = -f(w,v)$ for all $v, w \in V$. A non-degenerate skew-symmetric bilinear form satisfying $f(v,v) = 0$ for all $v \in V$ is called a symplectic form. If $\mathbb{F}_q$ has a field automorphism $\lambda \mapsto \overline{\lambda}$ of order two, then a non-degenerate left-linear form satisfying $f(v,w) = \overline{f(w,v)}$ is called a Hermitian or unitary form. Notice that in this case, q must be a square.

The symplectic groups $Sp(V,f)$ and general unitary groups $GU(V,f)$ are the subgroups of $GL(V)$ preserving a symplectic form or unitary form, f, respectively. That is, elements g of these subgroups are those satisfying $f(gv,gv) = f(v,v)$ for all $v \in V$. We note that n-dimensional symplectic forms (resp. unitary forms) over $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$) are unique up to similarity, and hence the corresponding groups $Sp(V,f)$ (resp. $GU(V,f)$) are unique up to isomorphism. With a proper choice of basis, we can identify these groups with the matrix groups

$$Sp_n(q) = \{g \in GL_n(q) | ^T gJg = J\}$$

,

$$GU_n(q) = \{g \in GL_n(q^2) | ^T\overline{g}g = I\}$$

where J is the $n \times n$ matrix

$$J := \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$$

and $\overline{g}$ is the matrix obtained from g by taking the qth power of each entry. Note that for the symplectic groups, n must be even and it turns out that $Sp_n(q) \leq SL_n(q)$. We obtain the special unitary group $SU_n(q)$ as the subgroup of $GU_n(q)$ of matrices with determinant 1.

From these groups, we obtain the projective symplectic, projective general unitary, and projective special unitary groups by taking the quotient of the corresponding group by its center:

$$PSp_n(q) := Sp_n(q)/Z(Sp_n(q)), \quad PGU_n(q) := GU_n(q)/Z(GU_n(q)),$$

$$PSU_n(q) := SU_n(q)/Z(SU_n(q)).$$

For most (n, q), the groups $PSp_n(q)$ and $PSU_n(q)$ are simple.

The last type of finite classical group consists of the orthogonal groups. These are the transformations preserving a non-degenerate quadratic form. A quadratic form is a map $Q\colon V \to V$ such that $Q(\lambda v) = \lambda^2 Q(v)$ for $\lambda \in \mathbb{F}_q$, $v \in V$ and the map $f_Q\colon V \times V \to V$ given by $f_Q(v, w) = Q(v + w) - Q(v) - Q(w)$ is a bilinear form. Q is called non-degenerate if f_Q is non-degenerate. If $n = 2m + 1$ is odd, there is one non-degenerate quadratic form in dimension n over $\mathbb{F}_q$, up to similarity. When q is odd, the group $O(V, Q)$ of isometries of the form Q is unique up to isomorphism and can be identified, after choosing a suitable basis, with the matrix group

$$O_{2m+1}(q) = \{g \in GL_{2m+1}(q) | {}^T gMg = M\},$$

where

$$M := \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When q is even, $O_{2m+1}(q) \cong Sp_{2m}(q)$, so we usually assume that if q is even, then so is n.

If $n = 2m$ is even, there are two isometry classes of non-degenerate quadratic forms. Choosing a standard basis, we see that the isometry groups $O(V, Q)$ of these two forms are isomorphic to a subgroup of one of the following matrix groups:

$$O^+_{2m}(q) \leq \{g \in GL_{2m}(q) | {}^T gKg = K\}$$

$$O^-_{2m}(q) \leq \{g \in GL_{2m}(q) | {}^T gLg = L\}$$

with

$$K := \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

and

$$L := \begin{pmatrix} 0 & I_{m-1} & 0 \\ I_{m-1} & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 2 & 1 \\ 1 & 2\zeta \end{pmatrix} \end{pmatrix},$$

where $x^2 + x + \zeta$ is irreducible over $\mathbb{F}_q$. When q is odd, the quadratic form Q is determined by its associated bilinear form f_Q, and hence the above are actually equalities. When q is even, the bilinear form is symplectic, so $O^{\pm}_{2m}(q) \le Sp_{2m}(q)$ is the subgroup of $g \in Sp_{2m}(q)$ such that $Q(gv) = Q(v)$ for all v in the chosen basis.

As before, we obtain the special orthogonal groups $SO^{\epsilon}_n(q)$ (for $\epsilon = \pm$, or null) as the subgroup of $O^{\epsilon}_n(q)$ of matrices with determinant 1. Taking the quotient by the center, we obtain the projective special orthogonal groups $PSO^{\epsilon}_n(q)$ as before. However, in the case of orthogonal groups, this group is not in general simple. Except for the case $SO^+_4(2)$, $SO^{\epsilon}_n(q)$ contains a unique subgroup of index 2, which is denoted by $\Omega^{\epsilon}_n(q)$. The projective group $P\Omega^{\epsilon}_n(q)$, which is the quotient of $\Omega^{\epsilon}_n(q)$ by scalar matrices, is the group which is usually simple in the orthogonal group case.

We will sometimes denote by $I(V, f)$ or $I(V, Q)$ the group of isometries of a bilinear or Hermitian form f or quadratic form Q, making the convention that f is the zero map in the case of linear groups.

Example 1. Consider the group $G = PSp_{2n}(q)$ when q is even. (This group, particularly when $n = 3$, will be the group we are primarily interested in in Chapter 5 and Chapter 7.) In general, we have $Z(Sp_{2n}(q)) = \{\pm I\}$, so since the characteristic of our field is 2, we have $Z(Sp_{2n}(q)) = I$. Hence $PSp_{2n}(q) = Sp_{2n}(q)$ when q is even, and this group is simple for $(n, q) \neq (2, 2), (1, 2)$. We note that although $Sp_4(2)$ is not simple, the commutator $Sp_4(2)'$ is simple and is isomorphic to the alternating group A_6.

If f is the skew-symmetric bilinear form defining G (that is, $G = I(V, f)$), we have $f(v, w) = -f(w, v) = f(w, v)$ since the characteristic is 2. Thus f is actually a symmetric bilinear form, and G contains $O_{2n}^{\pm}(q)$ as a subgroup.

We can find a standard symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ for $V = \mathbb{F}_q^{2n}$ such that $f(e_i, e_j) = 0 = f(f_i, f_j)$ for all i, j and $f(e_i, f_j) = \delta_{ij}$. Therefore, the Gram matrix of f is the matrix

$$J = K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

defined above.

(Note that in the case q is odd, the standard symplectic basis is defined the same way, but then the Gram matrix has $-I_n$ in the lower left.)

2.2.2 The Classical Groups as Groups of Lie Type

We can also identify the finite classical group G with the fixed points of a connected reductive algebraic group $\underline{G}$ under a Frobenius map F. For example, for $G = GL_n(q)$, we can take $\underline{G} = GL_n(k)$ and F to be the standard Frobenius map

$$F\colon (a_{ij}) \mapsto (a_{ij}^q)$$

mapping each matrix entry to its qth power. Similarly,

$$SL_n(k)^F = SL_n(q), \quad Sp_{2n}(k)^F = Sp_{2n}(q),$$

$$O_{2n+1}(k)^F = O_{2n+1}(q), \quad SO_{2n+1}(k)^F = SO_{2n+1}(q),$$

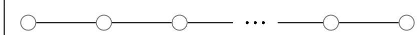
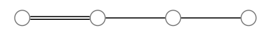
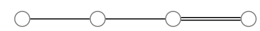
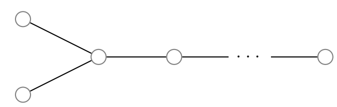
$$O_{2n}(k)^F = O_{2n}^+(q), \quad \text{and} \quad SO_{2n}(k)^F = SO_{2n}^+(q),$$

with the Gram matrices of the forms as above.

If instead we take F' to be the inverse-transpose map composed with F, i.e. $F'\colon (a_{ij}) \mapsto ({}^T(a_{ij}^q))^{-1}$, then

$$GL_n(k)^{F'} = GU_n(q) \quad \text{and} \quad SL_n(k)^{F'} = SU_n(q).$$

Table 2.1: Identifications of Finite Chevalley Groups with Finite Classical Groups

Chevalley Group $\mathfrak{L}(q)$	Simple Classical Group	Dynkin diagram for $\mathfrak{L}$
$A_\ell(q)$ ${}^2A_\ell(q^2)$	$PSL_{\ell+1}(q)$ $PSU_{\ell+1}(q)$	
$B_\ell(q)$	$P\Omega_{2\ell+1}(q)$, q odd	
$C_\ell(q)$	$PSp_{2\ell}(q)$	
$D_\ell(q)$ ${}^2D_\ell(q^2)$	$P\Omega^+_{2\ell}(q)$ $P\Omega^-_{2\ell}(q)$	

Note here that $(F')^2\colon (a_{ij}) \mapsto (a_{ij}^{q^2})$ is the standard Frobenius with respect to $\mathbb{F}_{q^2}$.

To obtain $SO^-_{2n}(q)$ in this way, we let F'' be the composition of F with conjugation by the matrix

$$t := \left(\begin{array}{c|cc} I_{2n-2} & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right).$$

Since $t \in O_{2n}(k)$ normalizes $SO_{2n}(k)$, the map $F''\colon (a_{ij}) \mapsto t^{-1}(a_{ij}^q)t$ sends $SO_{2n}(k)$ to itself. Then

$$SO_{2n}(k)^{F''} = SO^-_{2n}(q).$$

Note that the map F'' also squares to the standard Frobenius with respect to $\mathbb{F}_{q^2}$.

The finite simple classical groups can also be thought of as the Chevalley groups $\mathfrak{L}(q)$ for the simple Lie algebras $\mathfrak{L} = A_\ell, B_\ell, C_\ell, D_\ell$ over the field $\mathbb{F}_q$ and their twisted counterparts resulting from symmetries of the Dynkin diagram, as in the notation of [15]. The Dynkin diagrams for the root systems and the identifications of the groups $\mathfrak{L}(q)$ as finite classical groups for each of these Lie algebras are shown in Table 2.1. For the diagrams of B_n and C_n, we have made the convention that the long root is on the right.

2.3 Representations

Given a finite group G and a field $\mathbb{F}$, an $\mathbb{F}$-representation of G is a homomorphism $\mathfrak{X}\colon G \to GL(V)$ for some $\mathbb{F}$-vectorspace V. Equivalently, if $\dim V = n$, we can fix a basis for V to obtain $\mathfrak{X}\colon G \to GL_n(\mathbb{F})$. Given any $\mathbb{F}G$-module V which has dimension n as a vector space, we obtain a representation defined by $\mathfrak{X}(g)v := g \cdot v$ for $g \in G, v \in V$. Conversely, given a representation $\mathfrak{X}\colon G \to GL_n(\mathbb{F})$, we obtain an $\mathbb{F}G$-module by taking V to be the column space $\mathbb{F}^n$ and defining the action of $g \in G$ by $g \cdot v := \mathfrak{X}(g)v$. Hence $\mathbb{F}$-representations of G can be identified with the $\mathbb{F}G$-modules which afford them, and we call a representation *irreducible* if it is afforded by an irreducible $\mathbb{F}G$-module V, and *reducible* otherwise. If a representation $\mathfrak{X}$ remains irreducible when viewed over any extension field $\mathbb{E}$ of $\mathbb{F}$, we say that $\mathfrak{X}$ is *absolutely irreducible*. In particular, if $\mathbb{F}$ is algebraically closed, then any irreducible $\mathbb{F}$-representation is also absolutely irreducible.

Any $\mathbb{F}G$-module V has a composition series

$$0 = V_0 \leq V_1 \leq \ldots \leq V_{k-1} \leq V_k = V$$

with each factor V_i/V_{i-1} irreducible. In this sense, the irreducible representations of G form the building blocks of all representations.

Now, when $\mathbb{F}$ is an algebraically closed field of characteristic ℓ relatively prime to $|G|$, we say that $\mathfrak{X}$ (or V) is an *ordinary* representation, and the representation theory is the same as in the case $\mathbb{F} = \mathbb{C}$. In this case, Maschke's theorem tells us that in fact V is a direct sum of its irreducible composition factors. When ℓ divides $|G|$, the situation is more complicated, and we call V an *ℓ-modular* representation.

We will denote by $\mathfrak{d}_\ell(G)$ the smallest degree of an absolutely irreducible representation of G of degree larger than 1 in characteristic ℓ. Similarly, $\mathfrak{m}_\ell(G)$ denotes the largest such degree. When $\ell = 0$, we write $b(G) = \mathfrak{m}_0(G) =: \mathfrak{m}_\mathbb{C}(G)$, or sometimes simply $\mathfrak{m}(G)$. In particular, we have $\mathfrak{m}_\ell(G) \leq \mathfrak{m}(G) = b(G)$ for all $\ell \geq 0$.

2.3.1 Characters and Blocks of Finite Groups

Given a representation $\mathfrak{X}\colon G \to GL_n(\mathbb{F})$, we obtain the *character* $\chi := \mathrm{Tr} \circ \mathfrak{X}\colon G \to \mathbb{F}$ afforded by $\mathfrak{X}$ by taking the traces of the images $\mathfrak{X}(g)$ for $g \in G$. The character χ is called irreducible if the representation $\mathfrak{X}$ is irreducible. We denote the set of irreducible ordinary characters of G by $\mathrm{Irr}(G)$. Given an arbitrary ordinary character ψ of G, we can write ψ uniquely as a linear combination

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi,$$

where $0 \leq a_\chi \in \mathbb{Z}$.

If $\mathfrak{X}$ is an ℓ-modular representation over an algebraically closed field $\mathbb{F}$, we generalize the notion of characters as follows. Let $G^\circ := \{g \in G\colon \ell \not| |g|\}$ denote the set of *ℓ-regular* elements of G (that is, ℓ'- elements). If $g \in G^\circ$, then the eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{F}^\times$ of $\mathfrak{X}(g)$ are $|g|$th roots of unity in $\mathbb{F}^\times$. Fixing an isomorphism $*$ between $|G|_{\ell'}$th roots of unity in $\mathbb{F}^\times$ and $|G|_{\ell'}$th roots of unity in $\mathbb{C}$, we obtain the *ℓ-Brauer character* $\varphi\colon G^\circ \to \mathbb{C}$ by taking $\varphi(g) := \sum_{i=1}^n \lambda_i^*$. (Here we use $n_{\ell'}$ to denote the ℓ'-part of the integer n. We will also use n_ℓ to denote the ℓ-part of n.) We again call φ irreducible if $\mathfrak{X}$ is afforded by an irreducible $\mathbb{F}G$-module, and the set of irreducible ℓ- Brauer characters of G is denoted $\mathrm{IBr}_\ell(G)$. As in the case of the ordinary characters, we can write an arbitrary ℓ-Brauer character θ as a unique sum

$$\theta = \sum_{\varphi \in \mathrm{IBr}_\ell(G)} b_\varphi \varphi,$$

where $0 \leq b_\varphi \in \mathbb{Z}$.

In particular, taking the restriction of $\chi \in \mathrm{Irr}(G)$ to G° yields a (possibly reducible) Brauer character, which we will denote $\widehat{\chi}$, and we can write

$$\widehat{\chi} = \sum_{\varphi \in \mathrm{IBr}_\ell(G)} d_{\chi,\varphi} \varphi$$

for nonnegative integers $d_{\chi,\varphi}$. The numbers $d_{\chi,\varphi}$ are called the *decomposition numbers*, and the matrix $(d_{\chi,\varphi})$ is called the *decomposition matrix* for G.

The set $\mathrm{IBr}_\ell(G) \cup \mathrm{Irr}(G)$ is partitioned into sets called *ℓ-blocks* of G. The blocks satisfy that $\chi, \chi' \in \mathrm{Irr}(G)$ are in the same block if and only if there exist $\chi = \chi_1, \chi_2, ..., \chi_m = \chi' \in \mathrm{Irr}(G)$ and $\varphi_1, ..., \varphi_{m-1} \in \mathrm{IBr}_\ell(G)$ such that d_{χ_i, φ_i} and $d_{\chi_{i+1}, \varphi_i}$ are both nonzero for each i. Moreover, $d_{\chi,\varphi} \neq 0$ if and only if χ, φ are in the same block. Hence, by reordering $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$, the decomposition matrix can be put into block-diagonal form, with the blocks of the matrix corresponding to the ℓ-blocks of G. (It is worthwhile to note that the decomposition matrix is dependent on the choice of isomorphism $*$ fixed above.) When the prime ℓ is fixed, we will denote by $\mathrm{Bl}(G)$ the set of ℓ - blocks of G. Further, if $\chi \in \mathrm{Irr}(G) \cup \mathrm{IBr}_\ell(G)$, $\mathrm{Bl}(G|\chi)$ will denote the block of the group G containing χ. We will use $\mathrm{IBr}_\ell(B) := B \cap \mathrm{IBr}_\ell(G)$ to denote the irreducible Brauer characters in the block B and $\mathrm{Irr}(B) := B \cap \mathrm{Irr}(G)$ for the irreducible ordinary characters in B.

Usually, blocks are defined using *central characters*. (For a more complete discussion, we refer the reader to [33, Chapter 15].) Given $\chi \in \mathrm{Irr}(G)$, we will denote the central character associated to χ by $\omega_\chi \colon Z(\mathbb{C}G) \to \mathbb{C}$. This function is defined by $\omega_\chi(\mathcal{K}^+) = \frac{|\mathcal{K}|\chi(g)}{\chi(1)}$, where $\mathcal{K}$ is the conjugacy class of G containing g and given a set $\mathfrak{S}$, we define $\mathfrak{S}^+$ to be the sum $\sum_{x \in \mathfrak{S}} x$. We set $\lambda_B := \omega_\chi^* \colon Z(\mathbb{F}G) \to \mathbb{F}$ for $B = \mathrm{Bl}(G|\chi)$, as in [33, Chapter 15]. If $Y \leq G$ is a subgroup, and $b \in \mathrm{Bl}(Y)$, then the induced block b^G is the unique block B so that $\lambda_b^G(\mathcal{K}^+) = \lambda_B(\mathcal{K}^+)$ for all conjugacy classes $\mathcal{K}$ of G, if such a B exists. (In this situation, b^G is said to be *defined*.) Recall that $\lambda_b^G(\mathcal{K}^+)$ is given by $\lambda_b\left((\mathcal{K} \cap Y)^+\right)$.

To each ℓ - block B of G, there is associated an ℓ-radical subgroup D called the *defect group* of the block, which is unique up to G-conjugacy. If $P \in \mathrm{Syl}_\ell(G)$, then the *defect* of the block (or of any character in the block) is $d(B)$, where $|D| = \ell^{d(B)}$. If $|P| = \ell^r$, then $\ell^{r-d(B)}$ is the largest power of ℓ which divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(B)$, and the *height* of $\chi \in \mathrm{Irr}(B)$ is the integer h such that $\chi(1)_\ell = \ell^{r-d(B)+h}$. In particular, if $\chi(1)_\ell = |G|_\ell$, then χ is said to have defect zero (in this case $\mathrm{Bl}(G|\chi)$ is comprised exactly of χ and $\widehat{\chi}$) and if $D \in \mathrm{Syl}_\ell(G)$, then B is said to have maximal defect. We will denote by $\mathrm{Irr}_0(G|D)$ the set of height-zero characters of G which lie in any block with defect group D and by $\mathrm{dz}(G)$ the set of defect-zero characters of G.

2.3.2 Representations of Finite Classical Groups

Suppose now that G is a finite classical group in characteristic p. Then the p-modular representation theory differs significantly from the ℓ-modular representation theory for $\ell \neq p$. For this reason, we distinguish between the two cases, and say that $\mathfrak{X}$ is a *natural-* (or *defining-*) characteristic representation when $\ell = p$ and a *cross-* (or *non-defining-*) characteristic representation when $\ell \neq p$.

The representations of the finite classical groups in natural characteristic are closely related to the representations of the corresponding simple algebraic group. In particular, if G is a finite classical group and $G = \underline{G}^F$ where $\underline{G}$ is a simply connected, simple algebraic group over the algebraic closure $\mathbb{F} = \overline{\mathbb{F}_q}$ and F is the corresponding Frobenius endomorphism, then the irreducible $\mathbb{F}G$-modules are exactly the restrictions of a particular set of irreducible $\mathbb{F}\underline{G}$-modules. To be more precise, these irreducible $\mathbb{F}\underline{G}$-modules are of the form $M(\lambda)$, the unique irreducible $\mathbb{F}\underline{G}$-module with highest weight λ (under the ordering $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots), where $\lambda = \sum_i c_i\lambda_i$, $0 \leq c_i \leq q-1$, $\{\lambda_i\}$ is the basis of $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ dual to $\{\alpha_i^*\} = \{2\alpha_i/(\alpha_i, \alpha_i)\}$ where $\Pi = \{\alpha_i\}$ is the set of fundamental roots. Here $X(T)$ is the set of characters of a maximal torus T of $\underline{G}$. The set of roots Φ is obtained as the set of weights of the Lie algebra of $\underline{G}$ under the natural action. (Weights are those $\mu \in X(T)$ such that there is some nonzero v in the module (here the Lie algebra) such that $tv = \mu(t)v$ for all $t \in T$.)

However, these representations can still be quite difficult to find. In [48], Lübeck finds low-dimensional irreducible representations in natural characteristic for Chevalley groups of small rank. However, for each type, he designates a degree bound and is only able to find all such representations up to this degree. (Lübeck's results improve upon the results of Liebeck (see [37, Theorem 5.4.11]) which do this for classical groups, up to a smaller bound on the degree.)

For the purposes of this thesis, we will focus primarily on cross-characteristic representations.

Example 2. Consider the group $Sp_6(q)$ with q even. Although this group is of small rank, it is still actually quite difficult to find the character tables. In [76], Donald White makes significant progress in the cross characteristic case by finding the decomposition numbers of the unipotent blocks of the cross characteristic ℓ-modular characters, but even here there were, until recently, a few unknowns in the case $\ell|(q+1)$. These unknowns have been found in recent work by Olivier Dudas using the ℓ-adic cohomology of Deligne-Lusztig varieties.

2.3.3 Some Deligne-Lusztig Theory

We now present a short overview of some Deligne-Lusztig theory. Deligne-Lusztig theory can be thought of as a way to define a "Jordan decomposition" for irreducible characters into a "unipotent part" and a "semisimple part", in analogue to the Jordan decompositions of Lie groups and Lie algebras.

Let $G = \underline{G}^F$ for a connected reductive algebraic group $\underline{G}$, defined in characteristic $p \neq \ell$, and Frobenius map F, and write $G^* = (\underline{G}^*)^{F^*}$, where $(\underline{G}^*, F^*)$ is dual to $(\underline{G}, F)$. We can write $\mathrm{Irr}(G)$ as a disjoint union $\bigsqcup \mathcal{E}(G,(s))$ of Lusztig series corresponding to G^*- conjugacy classes of semisimple (i.e. p'-) elements $s \in G^*$. In the case that the centralizer $C_{\underline{G}^*}(s)$ is connected (in particular, this is the case if $Z(\underline{G})$ is connected), apart from a few exceptional cases, the Lusztig series $\mathcal{E}(G,(s))$ contains a unique character with p'-degree, and this character is called a *semisimple* character. Characters in the series $\mathcal{E}(G,(1))$ are called the *unipotent* characters, and there is a bijection $\mathcal{E}(G,(s)) \leftrightarrow \mathcal{E}(C_{G^*}(s),(1))$ such that if $\chi \mapsto \psi$, then $\chi(1) = [G^* : C_{G^*}(s)]_{p'}\psi(1)$. Note that the semisimple character in $\mathcal{E}(G,(s))$ has degree $[G^* : C_{G^*}(s)]_{p'}$.

Let $\chi \in \mathrm{Irr}(G)$ and assume χ belongs to the Lusztig series $\mathcal{E}(G,(s))$ and that t is the ℓ'-part of the semisimple element $s \in G^*$. Then $\chi \in \mathcal{E}_\ell(G,(t)) := \bigcup \mathcal{E}(G,(ut))$, where the union is taken over all ℓ-elements u in $C_{G^*}(t)$. By a fundamental result of Broué and Michel [12], $\mathcal{E}_\ell(G,(t))$ is actually a union of ℓ-blocks. Hence, we may view $\mathcal{E}_\ell(G,(t))$ as a collection of ℓ-Brauer characters as well as a set of ordinary characters.

Moreover, it follows (see, for example [31, Proposition 1]) that the degree of any irreducible Brauer character $\theta \in \mathcal{E}_\ell(G,(t))$ is divisible by $[G^* : C_{G^*}(t)]_{p'}$. Hence, if $\chi \in \mathcal{E}_\ell(G,(t)) \cap \mathrm{Irr}(G)$ and $\chi(1) = [G^* : C_{G^*}(t)]_{p'}$, then $\widehat{\chi}$ is irreducible. Furthermore, if H is a subgroup of G such that the restriction $\theta|_H$ to H is irreducible, and $[G^* : C_{G^*}(t)]_{p'} > \mathfrak{m}_\ell(H)$, then θ cannot be a member of $\mathcal{E}_\ell(G,(t))$. Also, any irreducible Brauer character in $\mathcal{E}_\ell(G,(t))$ appears as a constituent of the restriction $\widehat{\chi}$ to G° for some ordinary character χ in $\mathcal{E}(G,(t))$ (see [30, Theorem 3.1]), so $\mathcal{E}_\ell(G,(1))$ is a union of unipotent blocks. In particular, if $\theta|_H$ is irreducible and $[G^* : C_{G^*}(t)]_{p'} > \mathfrak{m}_\ell(H)$ for all nonidentity semisimple ℓ'- elements t of G^*, then θ must belong to a unipotent block.

In [9], Bonnafé and Rouquier show that when $C_{\underline{G}^*}(t)$ is contained in an F^*-stable Levi subgroup, $\underline{L}^*$, of $\underline{G}^*$, then Deligne-Lusztig induction R_L^G yields a Morita equivalence between $\mathcal{E}_\ell(G,(t))$ and $\mathcal{E}_\ell(L,(t))$, where $L = (\underline{L})^F$ and $(\underline{L}, F)$ is dual to $(\underline{L}^*, F^*)$. This fact will be very important in what follows.

Remark. As this thesis is primarily concerned with the groups $GU_n(q)$ and $Sp_{2n}(q)$, we make a few remarks about these cases. We note that in the case of $\underline{G} = GL_n(\overline{\mathbb{F}_q})$, the dual group $\underline{G}^*$ is actually isomorphic to $\underline{G}$, and $G = \underline{G}^F$ is also isomorphic to the dual $G^* = (\underline{G}^*)^{F^*}$. Therefore, in this situation we can simplify things by making the substitution $G^* = G$ in the above discussion. In this case, we also have that the center of $\underline{G}$ is connected, and therefore so is $C_{\underline{G}^*}(s)$ for any semisimple $s \in G^*$.

Also, for $G = Sp_{2n}(q)$ with q even, the dual G^* is isomorphic to G. This follows from the fact that the duality switches the root systems of types B_n and C_n, but these are the same when our field has characteristic 2. Hence in this case as well, we can make the substitution $G^* = G$ in the above discussion. Moreover, when $G = Sp_6(q)$, q even, with $G = \underline{G}^F$ and $(\underline{G}^*, F^*)$ in duality with $(\underline{G}, F)$, each semisimple conjugacy class (s) of $G^* = (\underline{G}^*)^{F^*}$ satisfies that $|s|$ is odd. Hence by [20, Lemma 13.14(iii)], the centralizer $C_{\underline{G}^*}(s)$ is connected.

2.4 Centralizers of Semisimple Elements of Unitary and Symplectic Groups

We present here some well-known results pertaining to the structure of centralizers of semisimple elements of the finite unitary groups (over fields with arbitrary order) and finite symplectic groups over fields with even order, as these will be useful in later chapters. Similar calculations can be found in various papers. For example, in [56], Nguyen uses analogous calculations to describe the centralizers of semisimple elements of the orthogonal groups.

We begin by introducing some notation. Let f be an irreducible polynomial of degree d over a field of size q. If α is one root of f in some extension field of $\mathbb{F}_q$, then the set of all roots of f is $\{\alpha, \alpha^q, \alpha^{q^2}, ..., \alpha^{q^{(d-1)}}\}$. Note that when the field is size q^2, the roots are $\alpha, \alpha^{q^2}, ..., \alpha^{q^{2(d-1)}}$, and the map $J_1 \colon \alpha \mapsto \alpha^{-q}$ may or may not keep the set of roots fixed. Thus J_1 induces an action on the set of irreducible polynomials, and when viewed as such is an involution. Similarly, the map $J_2 \colon \alpha \mapsto \alpha^{-1}$ may or may not keep the set of roots fixed when the field is of size q, and J_2 induces an involutory action on the set of irreducible polynomials.

We define $J := J_1$ in the case of the unitary groups and $J := J_2$ in the case of the symplectic groups. Let $f^{\checkmark} := J(f)$ denote the irreducible polynomial of degree d with roots $\alpha^{-q}, \alpha^{-q^3}, ..., \alpha^{-q^{2(d-1)+1}}$ or $\alpha^{-1}, \alpha^{-q}, ..., \alpha^{-q^{d-1}}$ in the case $J = J_1$ or J_2, respectively. We say that a polynomial f over $\mathbb{F}_{q^2}$ (respectively, $\mathbb{F}_q$) is *self-check* if $f = f^{\checkmark}$ and is *not self-check* otherwise.

2.4.1 Centralizers of Semisimple Elements of $GU_n(q)$

Note that for an irreducible polynomial f over $\mathbb{F}_{q^2}$, to be self-check, the degree d of f must necessarily be odd, and the roots must satisfy $\alpha^{q^d+1} = 1$. Moreover, these two conditions are sufficient to ensure that f is self-check.

Theorem 2.4.1. *Let $G = GU_n(q)$, with natural module $V = \mathbb{F}_{q^2}^n$, and let s be a semisimple element of G. Decompose the characteristic polynomial $P(t) \in \mathbb{F}_{q^2}[t]$ of s acting on V in the form*

$$P(t) = \prod_{i=1}^{\ell} f_i(t)^{k_i} \prod_{j=\ell+1}^{m} (g_j(t)g_j(t)^{\checkmark})^{r_j}$$

where each f_i, g_j is an irreducible polynomial over $\mathbb{F}_{q^2}$ and

- $f_i = f_i^{\checkmark}$*, and* $\deg f_i = d_i$
- $g_j \neq g_j^{\checkmark}$ *and* $\deg g_j = \deg g_j^{\checkmark} = d_j$
- $\sum_{j=1}^{\ell} k_j d_j + 2\sum_{j=\ell+1}^{m} r_j d_j = n.$

Then

$$C_G(s) \cong \prod_{j=1}^{\ell} GU_{k_j}(q^{d_j}) \times \prod_{j=\ell+1}^{m} GL_{r_j}(q^{2d_j}).$$

Proof. Let

$$V_i = \left[\frac{P(t)}{f_i(t)^{k_i}}\right]_{t=s} (V),$$

and

$$W_j = \left[\frac{P(t)}{(g_j(t)g_j^{\checkmark}(t))^{r_j}}\right]_{t=s} (V).$$

Then $V = \bigoplus_i V_i \oplus \bigoplus_j W_j$ is an orthogonal decomposition of nondegenerate subspaces and $s = \prod_i s|_{V_i} \cdot \prod_j s|_{W_j}$ is a product of semisimple elements acting on these subspaces. Note that since $C := C_G(s)$ commutes with s, it fixes each of V_i, W_j for all i, j. Then letting $C_W := C_{Sp(W)}(s|_W)$ for $W = V_i$ or W_j, we have $C = \prod_i C_{V_i} \times \prod_j C_{W_j}$.

Claim 1. $C_{V_i} \cong GU_{k_i}(q^{d_i})$.

Write $W := V_i$, $d := d_i$, $k := k_i$, and $f := f_i$. Then the characteristic polynomial of s on W is $f(t)^k$. Let $\lambda \in \mathbb{F}_{q^{2d}}$ be an eigenvalue of the action of s on W. That is, λ is a root of f in $\overline{\mathbb{F}_q}$. If $d = 1$, then the claim is clear, as $W = \ker(s - \lambda)$. So let $d \geq 2$. In this case, the claim follows from the argument in part (B1) of [73, Proof of Theorem 4.1].

Claim 2. $C_{W_j} \cong GL_{r_j}(q^{2d_j})$.

Now let $W := W_j$, $m := r_j$, $d := d_j$, and $g := g_j$. The characteristic polynomial of s acting on W is $(g(t)g^{\checkmark}(t))^m$. Let $\lambda \in \mathbb{F}_{q^{2d}}$ be a root of $g(t)$, so λ is an eigenvalue of s acting on W. Then $\lambda, \lambda^{q^2}, \lambda^{q^4}, ..., \lambda^{q^{2(d-1)}}$ are all of the roots of $g(t)$ and $\lambda^{-q}, \lambda^{-q^3}, ..., \lambda^{-q^{2(d-1)+1}}$ are the roots of $g^{\checkmark}(t)$. Let $\widetilde{W} := W \otimes_{q^2} \mathbb{F}_{q^{2d}}$, choose a basis $e_1, ..., e_{2md}$ for W, and define σ to be a Frobenius endomorphism on $\widetilde{W}$ given by $\sigma\colon \sum x_i e_i \mapsto \sum x_i^{q^2} e_i$, where $x_i \in \mathbb{F}_{q^{2d}}$.

Now we can decompose $\widetilde{W}$ as

$$\widetilde{W} \cong \widetilde{W}_1 \oplus ... \oplus \widetilde{W}_k \oplus \widetilde{W}'_1 \oplus ... \oplus \widetilde{W}'_k$$

where $\widetilde{W}_j := \ker(s - \lambda^{q^{2(j-1)}})$ and $\widetilde{W}'_j := \ker(s - \lambda^{-q^{2(j-1)+1}})$. Since s is semisimple, we know that each $\widetilde{W}_j$ and $\widetilde{W}'_j$ has dimension m. Also, σ permutes the $\widetilde{W}_j$ and $\widetilde{W}'_j$ cyclically: $\sigma(\widetilde{W}_j) = \widetilde{W}_{j+1}; \sigma(\widetilde{W}'_j) = \widetilde{W}'_{j+1}$. Further, C_W fixes each $\widetilde{W}_j$ and $\widetilde{W}'_j$, and $h \in C_W$ commutes with σ. Thus the action of h on $\widetilde{W}$ is completely determined by its action on $\widetilde{W}_1 \oplus \widetilde{W}'_1$. Hence, $C_W \hookrightarrow GL_{2m}(q^{2d})$.

Let $(\cdot, \cdot)$ denote the nondegenerate Hermitian form on V, which we may view as a form on W and extend to a nondegenerate Hermitian form on $\widetilde{W}$. If $w \in \widetilde{W}_j$ and $u \in \widetilde{W}_1$ for $1 \leq j \leq k$, then

$$(w, u) = (sw, su) = (\lambda^{q^{2(j-1)}} w, \lambda u) = \lambda^{q^{2(j-1)}} \cdot \lambda^q (w, u)$$

so that either $(w, u) = 0$ or $\lambda^{q^{2(j-1)}} = \lambda^{-q}$. Since $g \neq g^{\checkmark}$, the latter gives a contradiction, yielding

$$\widetilde{W}_1^{\perp} \supset \bigoplus_j \widetilde{W}_j.$$

If $w' \in \widetilde{W}_j'$, then

$$(w', u) = (sw', su) = (\lambda^{-q^{2(j-1)+1}} w', \lambda u) = \lambda^{-q^{2(j-1)+1}} \cdot \lambda^q (w', u)$$

so that either $(w', u) = 0$ or $\lambda^{-q} = \lambda^{-q^{2(j-1)+1}}$. The latter case would mean that $j = 1$, so we have

$$\widetilde{W}_1^{\perp} \supset \bigoplus_{j \geq 2} \widetilde{W}_j'.$$

Then we have

$$\widetilde{W}_1^{\perp} \supset \bigoplus_i \widetilde{W}_i \oplus \bigoplus_{j \geq 2} \widetilde{W}_j'. \tag{2.4.1}$$

Choose a basis $w_1, ..., w_m$ for $\widetilde{W}_1$ and a basis $w_1', ..., w_m'$ for $\widetilde{W}_1'$ and suppose that with respect to these bases, $h \in GL(\widetilde{W}_1 \oplus \widetilde{W}_1')$ acts via the matrix $A = (a_{k\ell})$ on $\widetilde{W}_1$ and via $B = (b_{k\ell})$ on $\widetilde{W}_1'$. From (2.4.1), we see that $h \in GL(\widetilde{W}_1 \oplus \widetilde{W}_1')$ is in C_W if and only if

$$(w_j, w_i') = (hw_j, hw_i') = \left(\sum_k a_{kj} w_k, \sum_\ell b_{\ell i} w_\ell' \right) = \sum_{k,\ell} a_{ki} b_{\ell j}^q (w_k, w_\ell')$$

for all i, j. Since $(\cdot, \cdot)$ is nondegenerate, we see that B is completely determined by A, so $C_W \cong GL_m(q^k)$, as claimed.

□

2.4.2 Centralizers of Semisimple Elements of $Sp_{2n}(q)$, q even

Recall that in this case, an irreducible polynomial f of degree d over $\mathbb{F}_q$ is self-check if and only if for every root α of f in the extension field $\mathbb{F}_{q^d}$, the element α^{-1} is also a root of f. Let $V = \mathbb{F}_q^{2n}$ be the usual module for $G = Sp_{2n}(q)$ with nondegenerate symplectic form $(\cdot,\cdot)$. Fix a basis for V and let $\mathbf{J}$ denote the corresponding Gram matrix for $(\cdot,\cdot)$.

Theorem 2.4.2. *Let $G = Sp_{2n}(q)$ with q even, and let s be a semisimple element of G. Decompose the characteristic polynomial $P(t) \in \mathbb{F}_q[t]$ of s acting on V in the form*

$$P(t) = (t-1)^{m_0} \cdot \prod_i f_i(t)^{m_i} \cdot \prod_j g_j(t)^{n_j} g_j^{\checkmark}(t)^{n_j}$$

where each f_i, g_j is an irreducible polynomial over $\mathbb{F}_q$ and

- $f_i = f_i^{\checkmark}$, 1 *is not a root of* f_i, *and* $\deg f_i = d_i$
- $g_j \neq g_j^{\checkmark}$ *and* $\deg g_j = \deg g_j^{\checkmark} = k_j$
- $2n = m_0 + \sum_i d_i m_i + 2\sum_j k_j n_j$.

Then

$$C_G(s) \cong Sp_{m_0}(q) \oplus \bigoplus_i GU_{m_i}(q^{d_i/2}) \oplus \bigoplus_j GL_{n_j}(q^{k_j}).$$

Proof. If we let

$$U = \ker(s-1),$$
$$V_i = \left[\frac{P(t)}{f_i(t)^{m_i}}\right]_{t=s}(V),$$

and

$$W_j = \left[\frac{P(t)}{(g_j(t)g_j^{\checkmark}(t))^{n_j}}\right]_{t=s}(V),$$

then $V = U \oplus \bigoplus_i V_i \oplus \bigoplus_j W_j$ is an orthogonal decomposition of nondegenerate subspaces. Note that since $C := C_G(s)$ commutes with s, it fixes each of U, V_i, W_j for all i, j. As in the case of unitary groups above, let $C_W := C_{Sp(W)}(s|_W)$ for $W = U, V_i$, or W_j, so that $C = C_U \times \prod_i C_{V_i} \times \prod_j C_{W_j}$.

Claim 3. $C_U \cong Sp_{m_0}(q)$.

First note that m_0 is even, since each d_i must be even (as 1 is not a root of f_i, so the roots come in pairs λ, λ^{-1}) and $2n = m_0 + \sum_i d_i m_i + 2\sum_j k_j n_j$. Now, U is the set of fixed points in V under s and has dimension m_0. That is, $C_U = C_{Sp(U)}(s|_U) = C_{Sp(U)}(1_U)$. Therefore, we see that $C_U \cong Sp(U) = Sp_{m_0}(q)$.

Claim 4. $C_{V_i} \cong GU_{m_i}(q^{d_i/2})$.

Write $W := V_i$, $d := d_i$, $m := m_i$, and $f := f_i$. Then the characteristic polynomial of s on W is $f(t)^m$. Let $\lambda \in \mathbb{F}_{q^d}$ be an eigenvalue of the action of s on W. That is, λ is a root of f in $\overline{\mathbb{F}_q}$. Then the roots of f are $\lambda, \lambda^q, \lambda^{q^2}, ..., \lambda^{q^{d-1}}$ and since λ^{-1} is a root of f and 1 is not, we have $\lambda^{-1} = \lambda^{q^r}$ for some $1 \le r \le d-1$. But this means $\lambda^{q^d} = \lambda = \lambda^{-q^r} = \lambda^{q^{2r}}$ and therefore $r = d/2$. So $\lambda^{-1} = \lambda^{q^{d/2}}$.

Now define $\widetilde{W} := W \otimes_q \mathbb{F}_{q^d}$ and fix a basis $e_1, ..., e_{md}$ of W. Define a Frobenius endomorphism on $\widetilde{W}$ by

$$\sigma\colon \sum_{i=1}^{md} x_i e_i \mapsto \sum_{i=1}^{md} x_i^q e_i$$

where $x_i \in \mathbb{F}_{q^d}$. Because s is semisimple, we know that its minimal polynomial has distinct roots, so $(t - \lambda^{q^{j-1}})$ is the largest elementary divisor which is divisible by $(t - \lambda^{q^{j-1}})$, and the eigenspace $\ker(s - \lambda^{q^{j-1}})$ has dimension m. Thus we can decompose $\widetilde{W}$:

$$\widetilde{W} = \widetilde{W}_1 \oplus ... \oplus \widetilde{W}_d$$

where $\widetilde{W}_j = \ker(s - \lambda^{q^{j-1}})$ and $\dim \widetilde{W}_j = m$ for $j = 1, ..., d$. Note that σ permutes the $\tilde{W}_j$s cyclically: $\sigma(\widetilde{W}_j) = \widetilde{W}_{j+1}$, where we define $\widetilde{W}_{d+1} := \widetilde{W}_1$.

Further, σ commutes with $g \in C_W$, since

$$g\sigma\left(\sum x_i e_i\right) = g\left(\sum x_i^q e_i\right) = \sum x_i^q g(e_i) = \sigma\left(\sum x_i g(e_i)\right) = \sigma g\left(\sum x_i e_i\right)$$

and g fixes each $\widetilde{W}_j$, since

$$(s-\lambda^{q^{j-1}})(gw) = sg(w) - \lambda^{q^{j-1}} g(w) = gs(w) - g\lambda^{q^{j-1}}(w) = g(s-\lambda^{q^{j-1}})(w) = g(0) = 0$$

for $w \in \widetilde{W}_j$.

Thus the action of g on $\widetilde{W}$ is determined by the action of g on $\widetilde{W}_1$, as

$$g(\sigma^j w) = \sigma^j(gw)$$

for $w \in \widetilde{W}_1$. Therefore, $C_W \hookrightarrow GL(\widetilde{W}_1) = GL_m(q^d)$.

Now, if $u \in \widetilde{W}_i$ and $v \in \widetilde{W}_j$ with $1 \leq i, j \leq d$, then $(u, v) = (su, sv) = \lambda^{q^{i-1}+q^{j-1}}(u, v)$, which means either $(u, v) = 0$ or $\lambda^{q^{i-1}+q^{j-1}} = 1$. Letting $i = 1$, the latter case would imply that $\lambda^{-1} = \lambda^{q^{j-1}}$, so $j = d/2+1$. Similarly, if $i = 1+d/2$, then either $(u, v) = 0$ or $j = 1$. This and the nondegeneracy of $(\cdot, \cdot)$ implies that

$$\widetilde{W}_1^{\perp} = \bigoplus_{j \neq d/2+1} \widetilde{W}_j \quad \text{and} \quad \widetilde{W}_{1+d/2} \cap \widetilde{W}_1^{\perp} = 0. \tag{2.4.2}$$

Choosing a basis $\{w_1, ..., w_m\}$ for $\widetilde{W}_1$ gives a basis $\{v_i = \sigma^{d/2}(w_i)\}$ for $\widetilde{W}_{1+d/2}$, and we have

$$(\sigma w_i, \sigma v_j) = (w_i, v_j)^q,$$

so

$$(v_i, w_j) = (\sigma^{d/2} w_i, \sigma^{d/2} v_j) = (w_i, v_j)^{q^{d/2}}.$$

Because of this and (2.4.2), we see that $(\cdot, \cdot)$ determines a nondegenerate hermitian space of dimension m over $\mathbb{F}_{q^{2(d/2)}} = \mathbb{F}_{q^d}$.

Then $h \in C_W$ if and only if $(w_i, v_j) = (hw_i, hv_j) = (\sum_k a_{ki} w_k, \sum_\ell a_{\ell j}^{q^{d/2}} v_\ell) = \sum_{a_{ki}} \sum a_{\ell j}^{q^{d/2}} (w_k, v_\ell)$ where h acts by the matrix $A = (a_{ij})$ with respect to the basis (w_i) and as $(a_{ij}^{q^{d/2}})$ with respect to the basis (v_i). That is, $h \in C_W$ if and only if $h \in GU_m(q^{d/2})$, which proves the claim.

Claim 5. $C_{W_j} \cong GL_{n_j}(q^{k_j})$.

Now let $W := W_j$, $m := n_j$, $k := k_j$, and $g := g_j$. The characteristic polynomial of s acting on W is $(g(t)g^{\checkmark}(t))^m$. Let $\lambda \in \mathbb{F}_{q^k}$ be a root of $g(t)$, so λ is an eigenvalue of s acting on W. Then $\lambda, \lambda^q, \lambda^{q^2}, ..., \lambda^{q^{k-1}}$ are all of the roots of $g(t)$ and

$\lambda^{-1}, \lambda^{-q}, ..., \lambda^{-q^{k-1}}$ are the roots of $g^{\checkmark}(t)$. As before, let $\widetilde{W} := W \otimes_q \mathbb{F}_{q^k}$, choose a basis $e_1, ..., e_{2mk}$ for W, and define σ to be a Frobenius endomorphism on $\widetilde{W}$ given by $\sigma\colon \sum x_i e_i \mapsto \sum x_i^q e_i$, where $x_i \in \mathbb{F}_{q^k}$.

Now we can decompose $\widetilde{W}$ as

$$\widetilde{W} \cong \widetilde{W}_1 \oplus ... \oplus \widetilde{W}_k \oplus \widetilde{W}'_1 \oplus ... \oplus \widetilde{W}'_k$$

where $\widetilde{W}_j := \ker(s - \lambda^{q^{j-1}})$ and $\widetilde{W}'_j := \ker(s - \lambda^{-q^{j-1}})$. As before, since s is semisimple, we know that each $\widetilde{W}_j$ and $\widetilde{W}'_j$ has dimension m. Also, σ permutes the $\widetilde{W}_j$ and $\widetilde{W}'_j$ cyclically: $\sigma(\widetilde{W}_j) = \widetilde{W}_{j+1}; \sigma(\widetilde{W}'_j) = \widetilde{W}'_{j+1}$. Again, C_W fixes each $\widetilde{W}_j$ and $\widetilde{W}'_j$, and $h \in C_W$ commutes with σ. Thus in this case, the action of h on $\widetilde{W}$ is completely determined by its action on $\widetilde{W}_1 \oplus \widetilde{W}'_1$. Hence, $C_W \hookrightarrow GL_{2m}(q^k)$.

Moreover, if $w \in \widetilde{W}_1$ and $u \in \widetilde{W}_j$ for $1 \leq j \leq k$, then

$$(w, u) = (sw, su) = (\lambda w, \lambda^{q^{j-1}} u) = \lambda \cdot \lambda^{q^{j-1}} (w, u)$$

so that either $(w, u) = 0$ or $\lambda^{q^{j-1}} = \lambda^{-1}$. Since $g \neq g^{\checkmark}$, the latter gives a contradiction, so we have

$$\widetilde{W}_1^{\perp} \supset \bigoplus_j \widetilde{W}_j.$$

If $u' \in \widetilde{W}'_j$, then

$$(w, u') = (sw, su') = (\lambda w, \lambda^{-q^{j-1}} u') = \lambda \cdot \lambda^{-q^{j-1}} (w, u')$$

so that either $(w, u') = 0$ or $\lambda = \lambda^{q^{j-1}}$. The latter case would mean that $j = 1$, so we have

$$\widetilde{W}_1^{\perp} \supset \bigoplus_{j \geq 2} \widetilde{W}'_j.$$

Then since $(\cdot, \cdot)$ is nondegenerate, we have

$$\widetilde{W}_1^{\perp} \cap \widetilde{W}'_1 = 0 \quad \text{and} \quad \widetilde{W}_1^{\perp} = \bigoplus_i \widetilde{W}_i \oplus \bigoplus_{j \geq 2} \widetilde{W}'_j. \tag{2.4.3}$$

Now we choose a basis $w_1, ..., w_m$ for $\widetilde{W}_1$ and a basis $w'_1, ..., w'_m$ for $\widetilde{W}'_1$. Suppose h acts by the matrix $A = (a_{k\ell})$ on $\widetilde{W}_1$ and by the matrix $B = (b_{k\ell})$ on $\widetilde{W}'_1$ with respect to these bases. From (2.4.3), we see that $h \in GL(\widetilde{W}_1 \oplus \widetilde{W}'_1)$ is in C_W if and only if

$$(w_j, w'_i) = (hw_j, hw'_i) = \left(\sum_k a_{kj} w_k, \sum_\ell b_{\ell i} w'_\ell\right) = \sum_{k,\ell} a_{ki} b_{\ell j} (w_k, w'_\ell)$$

for all i, j. Since $(\cdot, \cdot)$ is nondegenerate, this means that B is completely determined by A, so $C_W \cong GL_m(q^k)$, which completes the proof of the claim, and therefore of the theorem.

$\square$

Chapter 3
Bounds for Character Degrees of Unitary Groups

Given a finite group G, let $b(G) := \max\{\chi(1)|\chi \in \mathrm{Irr}(G)\}$ denote the largest irreducible complex character degree of G. (Note that $b(G)$ is also an upper bound for the largest absolutely irreducible degree in characteristic $\ell \neq 0$.) One problem of interest in the representation theory of finite groups is to determine information about $b(G)$. We may ask whether we can find an explicit formula for this number, or if we can bound it somehow. Of course, nothing can be said in general, so we restrict our attention to the case of simple groups and groups closely related to simple groups. It remains an open question to determine $b(G)$ explicitly for many nonabelian simple groups.

Certainly we know that $b(G) \leq \sqrt{|G|}$ and that $b(C_n) = 1$ for a cyclic group C_n, and we can use the Atlas [18] to obtain information about the character degrees for sporadic groups. It is well-known that the irreducible complex representations of S_n are labeled by partitions $\lambda \vdash n$ of n, and that the degree of the character χ_λ corresponding to the partition λ is given by the hook-length formula. However, it is still a difficult question to determine from this formula which partition actually yields the largest degree. The best result regarding this problem is due to Vershik-Kerov [74] and Logan-Shepp [46], and says that there are universal constants $0 < A < B$ so that

$$\exp(-B\sqrt{n})\sqrt{n!} < b(S_n) < \exp(-A\sqrt{n})\sqrt{n!}.$$

The question we are concerned with is how $b(G)$ can be bounded for G a simple group of Lie type. For exceptional groups of Lie type, F. Lübeck [49] has computed all character degrees, so we are interested in the case of finite classical groups. G.

Seitz [64] has shown that for groups of Lie type defined over a field of characteristic p with q elements, $b(G) \leq |G|_{p'}/|T_0|$, where T_0 is a maximal torus of minimal order. Moreover, he has shown that for q sufficiently large, this is actually an equality. For this reason, we are particularly interested in the case that q is small. For example, if $G = SL_n(2)$, then $T_0 = 1$, so in this case, Seitz' bound gives us no more information than the trivial bound $\sqrt{|G|}$.

Now, viewed as polynomials in q, Seitz' bound has the same degree as $|G|_p = \mathrm{St}(1)$, where St is the Steinberg character for G, which suggests that we consider the ratio $\frac{b(G)}{|G|_p} = \frac{b(G)}{\mathrm{St}(1)}$. Note that this ratio is always at least 1. Fixing q, we may ask whether there is some universal constant C such that $\frac{b(G)}{\mathrm{St}(1)} < C$ for any n. It turns out that the answer to this question is no. In fact, the main goal of this chapter is to prove the following theorem:

Theorem 3.0.3. *Let G be a finite unitary group (i.e. $G = GU_n(q), PGU_n(q), SU_n(q)$, or $PSU_n(q)$). Then*

$$\max\left\{1, \frac{1}{4}\left(\log_q((n-1)(1-q^{-2})+q^4)\right)^{2/5}\right\} < \frac{b(G)}{q^{n(n-1)/2}} < 2\left(\log_q(n(q^2-1)+q^2)\right)^{1.27}.$$

Note that in the case of finite unitary groups, $\mathrm{St}(1) = q^{(n(n-1)/2}$. We also note that similar bounds are found for the other groups of Lie type in [42]. This shows that if we fix q, then as n grows infinitely large, so does the ratio $\frac{b(G)}{\mathrm{St}(1)}$.

In the remainder of the chapter, we prove Theorem 3.0.3, beginning by showing that the degree of the Steinberg character of G is larger than that of any other unipotent character.

3.1 The Largest Degree of a Unipotent Character in Finite Unitary Groups

Let q be a power of p and let G be $GU_n(q), SU_n(q), PU_n(q)$, or $PSU_n(q)$. The unipotent characters of G are in one-to-one correspondence with partitions α of n of the form $\alpha = (\alpha_1, ..., \alpha_m)$ with $1 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m$. Denote $\lambda_i := \alpha_i + i - 1$ for

$1 \leq i \leq m$. Then the unipotent character corresponding to α, which we denote by χ^α, has degree

$$\chi^\alpha(1) = \frac{(q+1)(q^2-1)...(q^n-(-1)^n)\prod_{i'<i}(q^{\lambda_i}-(-1)^{\lambda_i+\lambda_{i'}}q^{\lambda_{i'}})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\dots}\prod_i\prod_{k=1}^{\lambda_i}(q^k-(-1)^k)}. \tag{3.1.1}$$

(See, for example, [16]).

The Steinberg character is the unipotent character St corresponding to the partition $(1, ..., 1)$, which has degree $\mathrm{St}(1) = q^{n(n-1)/2} = |G|_p$. In this section, we show that $\mathrm{St}(1) > \chi^\alpha(1)$ for any unipotent character $\chi^\alpha \neq \mathrm{St}$ of G. (We note that this is also shown in [42] using a different approach.)

The following inequalities, which can be found in [73], will be useful in what follows.

Lemma 3.1.1. *Let $2 \leq a_1 < a_2 < ... < a_\ell$ be integers and $\varepsilon_1, ..., \varepsilon_\ell \in \{1, -1\}$. Then*

$$\frac{1}{2} < \frac{(q^{a_1}+\varepsilon_1)\cdot ... \cdot(q^{a_\ell}+\varepsilon_\ell)}{q^{a_1+a_2+...+a_\ell}} < 2.$$

Theorem 3.1.2. *Let $G = GU_n(q), SU_n(q), PU_n(q)$, or $PSU_n(q)$. Let $\chi \in \mathrm{Irr}(G)$ be a unipotent character of G which is not the Steinberg character,* St*. Then $\chi(1) < \mathrm{St}(1)$.*

Proof. We want to show that if α is a partition of n of the form above and $\alpha \neq (1, ..., 1)$, then $\chi^\alpha(1) < \chi^{(1,...,1)}(1)$. We proceed by induction on n. For $n = 2, 3$, the statement can be verified by direct calculation. So suppose that the statement holds for unitary groups of degree smaller than n. We will use St_n to denote the Steinberg character for a unitary group of degree n. Note that if $\alpha = (n)$, then $\chi^\alpha(1) = 1$, so assume that $\alpha = (\alpha_1, ..., \alpha_m)$ with $m > 1$ and $\alpha_m > 1$. Let β be the partition of $n - \alpha_m$ given by $\beta = (\alpha_1, ..., \alpha_{m-1})$. Then by induction, we can assume that $\chi^\beta(1) \leq \mathrm{St}_{n-\alpha_m}(1) = q^{(n-\alpha_m)(n-\alpha_m-1)/2}$. Thus, we have

$$\chi^\alpha(1) = \frac{\chi^\alpha(1)}{\chi^\beta(1)}\chi^\beta(1) \leq \frac{\chi^\alpha(1)}{\chi^\beta(1)}q^{(n-\alpha_m)(n-\alpha_m-1)/2}.$$

We will show that $f(\alpha) := \frac{\chi^\alpha(1)}{\chi^\beta(1)} \frac{q^{(n-\alpha_m)(n-\alpha_m-1)/2}}{q^{n(n-1)/2}}$ is smaller than 1.

We have

$$\chi^\alpha(1) = \frac{\prod_{k=1}^n (q^k - (-1)^k) \prod_{i=2}^m \prod_{k=1}^{i-1} (q^{\lambda_i} - (-1)^{\lambda_i+\lambda_k} q^{\lambda_k})}{q^{\binom{m-1}{2} + \binom{m-2}{2} + \ldots} \prod_{i=1}^m \prod_{k=1}^{\lambda_i} (q^k - (-1)^k)}$$

and

$$\chi^\beta(1) = \frac{\prod_{k=1}^{n-\alpha_m} (q^k - (-1)^k) \prod_{i=2}^{m-1} \prod_{k=1}^{i-1} (q^{\lambda_i} - (-1)^{\lambda_i+\lambda_k} q^{\lambda_k})}{q^{\binom{m-2}{2} + \ldots} \prod_{i=1}^{m-1} \prod_{k=1}^{\lambda_i} (q^k - (-1)^k)}$$

so that

$$\begin{aligned} \frac{\chi^\alpha(1)}{\chi^\beta(1)} &= \frac{\prod_{k=n-\alpha_m+1}^n (q^k - (-1)^k) \prod_{k=1}^{m-1} (q^{\lambda_m} - (-1)^{\lambda_m+\lambda_k} q^{\lambda_k})}{q^{\binom{m-1}{2}} \prod_{k=1}^{\lambda_m} (q^k - (-1)^k)} \\ &= \frac{\prod_{k=n-\alpha_m+1}^n (q^k - (-1)^k) \prod_{k=1}^{m-1} q^{\lambda_k} \prod_{k=1}^{m-1} (q^{\lambda_m-\lambda_k} - (-1)^{\lambda_m-\lambda_k})}{q^{\binom{m-1}{2}} \prod_{k=1}^{\lambda_m} (q^k - (-1)^k)} \end{aligned}$$

Now,

$$\sum_{k=1}^{m-1} \lambda_k = \sum_{k=1}^{m-1} (\alpha_k + k - 1) = \sum_{k=1}^{m-1} \alpha_k + \sum_{k=1}^{m-1} k - (m-1) = (n - \alpha_m) + \sum_{k=1}^{m-2} k$$

and

$$\binom{m-1}{2} = (m-1)(m-2)/2 = \sum_{k=1}^{m-2} k$$

so that

$$\frac{\chi^\alpha(1)}{\chi^\beta(1)} = \frac{q^{n-\alpha_m} \prod_{k=n-\alpha_m+1}^n (q^k - (-1)^k) \prod_{k=1}^{m-1} (q^{\lambda_m-\lambda_k} - (-1)^{\lambda_m-\lambda_k})}{\prod_{k=1}^{\lambda_m} (q^k - (-1)^k)}.$$

Note that $q^{n(n-1)/2} = \prod_{i=1}^{n-1} q^i$ and $q^{(n-\alpha_m)(n-\alpha_m-1)/2} = \prod_{i=1}^{n-\alpha_m-1} q^i$, so

$$\begin{aligned}
f(\alpha) &= \frac{q^{n-\alpha_m}\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{k=1}^{m-1}(q^{\lambda_m-\lambda_k}-(-1)^{\lambda_m-\lambda_k})}{\prod_{k=1}^{\lambda_m}(q^k-(-1)^k)\prod_{k=n-\alpha_m}^{n-1}q^k} \\
&= \frac{\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{k=1}^{m-1}(q^{\lambda_m-\lambda_k}-(-1)^{\lambda_m-\lambda_k})}{\prod_{k=1}^{\lambda_m}(q^k-(-1)^k)\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&= \frac{(q^{\lambda_m-\lambda_{m-1}}-(-1)^{\lambda_m-\lambda_{m-1}})\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{k=1}^{m-2}(q^{\lambda_m-\lambda_k}-(-1)^{\lambda_m-\lambda_k})}{(q+1)\prod_{k=2}^{\lambda_m}(q^k-(-1)^k)\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&< \frac{2^3\prod_{k=n-\alpha_m+1}^{n}q^k\prod_{k=1}^{m-1}q^{\lambda_m-\lambda_k}}{\prod_{k=1}^{\lambda_m}q^k\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&= 2^3q^R
\end{aligned}$$

where $R = n + \sum_{k=1}^{m-1}(\lambda_m - \lambda_k) - \sum_{k=1}^{\lambda_m} k$.

Note that the second-to-last inequality follows by Lemma 3.1.1 since $\lambda_m - \lambda_k = \alpha_m - \alpha_k + m - k \geq 2$ when $k < m-1$, $\lambda_m - \lambda_k > \lambda_m - \lambda_{k-1}$, and $n - \alpha_m + 1 \geq 2$ since we assume $m > 1$. Moreover, if $\lambda_m - \lambda_{m-1} = 1$, then we have that $(q^{\lambda_m-\lambda_{m-1}} - (-1)^{\lambda_m-\lambda_{m-1}})/(q+1) = 1 = q^{\lambda_m-\lambda_{m-1}}/q$ and otherwise, we have $\lambda_m - \lambda_k \geq 2$ for $k \leq m-1$, and clearly $1/(q+1) < 1/q$.

Now, we have

$$\begin{aligned}
\sum_{k=1}^{m-1}(\lambda_m - \lambda_k) &= \sum_{k=1}^{m-1}(\alpha_m - \alpha_k + m - k) \\
&= \sum_{k=1}^{m-1}\alpha_m - \sum_{k=1}^{m-1}\alpha_k + \sum_{\ell=1}^{m-1}\ell \\
&= (m-1)\alpha_m - (n-\alpha_m) + \sum_{\ell=1}^{m-1}\ell
\end{aligned}$$

so that

$$\begin{aligned}
R &= n + m\alpha_m - n + \sum_{\ell=1}^{m-1}\ell - \sum_{k=1}^{\lambda_m}k \\
&= m\alpha_m - \sum_{k=m}^{\alpha_m+m-1}k = m\alpha_m - (m + (m+1) + (m+2) + ... + (m+\alpha_m-1)) \\
&= m\alpha_m - \left(m\alpha_m + \sum_{k=1}^{\alpha_m-1}k\right) = -\sum_{k=1}^{\alpha_m-1}k = -(\alpha_m-1)\alpha_m/2
\end{aligned}$$

Thus we have $f(\alpha) < \frac{2^3}{q^{(\alpha_m-1)\alpha_m/2}} \leq 1$ for $\alpha_m \geq 3$.

Now suppose that $\alpha_m = 2$. This means that $\alpha = (1, 1, ..., 1, 2, ..., 2)$. Say j is the first position for which $\alpha_j = 2$. That is, $\alpha_\ell = 1$ for $\ell < j$ and $\alpha_k = 2$ for $k \geq j$. Thus, $\lambda_i = i$ for $i < j$ and $\lambda_i = i+1$ for $i \geq j$. Also, $n = 2m-j+1$ and $n-\alpha_m = 2m-j-1$. In this case, we have

$$\begin{aligned}
f(\alpha) &= \frac{\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{k=1}^{j-1}(q^{\lambda_m-k}-(-1)^{\lambda_m-k})\prod_{k=j}^{m-1}(q^{\lambda_m-\lambda_k}-(-1)^{\lambda_m-\lambda_k})}{\prod_{k=1}^{\lambda_m}(q^k-(-1)^k)\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&= \frac{\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{\ell=\lambda_m-j+1}^{\lambda_m-1}(q^\ell-(-1)^\ell)\prod_{k=j}^{m-1}(q^{m-k}-(-1)^{m-k})}{\prod_{k=1}^{\lambda_m}(q^k-(-1)^k)\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&= \frac{\prod_{k=n-\alpha_m+1}^{n}(q^k-(-1)^k)\prod_{\ell=1}^{m-j}(q^\ell-(-1)^\ell)}{(q^{\lambda_m}-(-1)^{\lambda_m})\prod_{k=1}^{\lambda_m-j}(q^k-(-1)^k)\prod_{k=n-\alpha_m+1}^{n-1}q^k} \\
&= \frac{\prod_{k=n-1}^{n}(q^k-(-1)^k)\prod_{\ell=1}^{m-j}(q^\ell-(-1)^\ell)}{(q^{\lambda_m}-(-1)^{\lambda_m})\prod_{k=1}^{\lambda_m-j}(q^k-(-1)^k)\prod_{k=n-1}^{n-1}q^k} \\
&= \frac{(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)\prod_{\ell=1}^{m-j}(q^\ell-(-1)^\ell)}{q^{n-1}(q^{\lambda_m}-(-1)^{\lambda_m})\prod_{k=1}^{\lambda_m-j}(q^k-(-1)^k)} \\
&= \frac{(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)\prod_{\ell=1}^{m-j}(q^\ell-(-1)^\ell)}{q^{n-1}(q^{m+1}-(-1)^{m+1})\prod_{k=1}^{m+1-j}(q^k-(-1)^k)} \\
&= \frac{(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)}{q^{n-1}(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})}
\end{aligned}$$

So we see that

$$f(\alpha) = \frac{(q^{2m-j}-(-1)^{2m-j})(q^{2m-j+1}-(-1)^{2m-j+1})}{q^{2m-j}(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})}.$$

If $2m-j$ is even, then we have

$$\begin{aligned}
f(\alpha) &= \frac{(q^{2m-j}-1)(q^{2m-j+1}+1)}{q^{2m-j}(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})} \\
&< \frac{q^{2(2m-j)+1}}{q^{2m-j}(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})} \\
&= \frac{q^{2m-j+1}}{(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})} \\
&< \frac{2q^{2m-j+1}}{q^{2m-j+2}} \quad \text{by Lemma 3.1.1} \\
&= 2/q \leq 1.
\end{aligned}$$

If $2m-j$ is odd then j is odd, and we get

$$f(\alpha) = \frac{(q^{2m-j}+1)(q^{2m-j+1}-1)}{q^{2m-j}(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})}.$$

Note that since j is odd, $m+1-j$ is the opposite parity of $m+1$. Consider $(q^{m+1}-(-1)^{m+1})(q^{m+1-j}-(-1)^{m+1-j})$. We have that this is either $(q^{m+1}-1)(q^{m+1-j}+1)$ or $(q^{m+1}+1)(q^{m+1-j}-1)$. In the first case, we have

$$(q^{m+1}-1)(q^{m+1-j}+1) = q^{2m-j+2} - q^{m-j+1} + q^{m+1} - 1 \geq q^{2m-j+2},$$

where the last inequality is because $m-j+1 < m+1$. In the second case, we have

$$(q^{m+1}+1)(q^{m+1-j}-1) > \frac{2}{3}q^{2m+2-j},$$

unless we are in the case $q=2$ and $j=m$. This inequality is because

$$\frac{(q^{m+1}+1)(q^{m+1-j}-1)}{q^{m+1}q^{m+1-j}} > \frac{q^{m+1}(q^{m+1-j}-1)}{q^{m+1}q^{m-j+1}} = \frac{q^{m+1-j}-1}{q^{m+1-j}},$$

which is minimized when q^{m+1-j} is minimized, so is at least $2/3$ unless $q^{m-j+1}=2$.

Now, this means that, unless $q=2$ and $m=j$ (a case in which explicit computation yields the result),

$$f(\alpha) < \frac{3(q^{2m-j}+1)(q^{2m-j+1}-1)}{2q^{2m-j}q^{2m-j+2}} < \frac{3(q^{2m-j}+1)(q^{2m-j+1})}{2q^{2m-j}q^{2m-j+2}} = \frac{3(q^{2m-j}+1)}{2q^{2m-j+1}} \leq 1$$

since for $x \geq 4$ and $q \geq 2$, $3x+3 \leq 2qx$. (Note that here we've used the fact that $2m-j = n-1 \geq 2$ and $q \geq 2$.)

Thus we have shown that in any case, $f(\alpha) < 1$, and therefore that the Steinberg representation has larger degree than any other unipotent representation for G a finite group of unitary type.

□

3.2 Proof of Theorem 3.0.3

In this section, we prove our bounds for the ratio $\frac{b(G)}{\mathrm{St}(1)}$ in the case that G is a finite unitary group. For more clarity, we may sometimes write St_G for the Steinberg character of the group G. Throughout this section, let q be a power of the prime p. Note that if for $G = GU_n(q)$, we have $b(G)/\mathrm{St_G}(1) < \mathrm{C}$ for some bound C, then the same is true for $G = SU_n(q), PSU_n(q)$, or $PGU_n(q)$. Therefore, to find an upper bound for $b(G)/\mathrm{St_G}(1)$ for G any of these finite unitary groups, it suffices to find one for $G = GU_n(q)$.

Lemma 3.2.1. *Let $G = GU_n(q)$, and denote by $b(G)$ the largest irreducible character degree. Then*

$$\frac{b(G)}{q^{n(n-1)/2}} = \max_{k_i, d_i, m} P$$

where P is

$$P = \frac{\prod_{i=1}^{n}(1-(-1)^i q^{-i})}{\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^i q^{-id_j})\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2id_j})}$$

and the maximum is taken over possible characteristic polynomials

$$\prod_{i=1}^{\ell} f_i^{k_i} \prod_{i=\ell+1}^{m} (g_i g_i^{\checkmark})^{r_i}$$

for semisimple elements. Here $f_i = f_i^{\checkmark}$ and $g_i \neq g_i^{\checkmark}$, in the sense of Section 2.4.1, and d_i is the degree of f_i or g_i, depending on the index i.

Proof. Since $G = G^*$ is self-dual, we know from Lusztig's correspondence that

$$\frac{b(G)}{q^{n(n-1)/2}} = \max_{s,\psi}\left\{\frac{[G : C_G(s)]_{p'}\psi(1)}{q^{n(n-1)/2}}\right\},$$

where s is a semisimple element of G and ψ is a unipotent character of $C_G(s)$. Moreover, from Section 2.4, we know that the centralizer of s is of the form

$$C_G(s) \cong \prod_{j=1}^{\ell} GU_{k_j}(q^{d_j}) \times \prod_{j=\ell+1}^{m} GL_{r_j}(q^{2d_j}),$$

where the characteristic polynomial of s acting on $\mathbb{F}_{q^2}^n$ is

$$f(t) = \prod_{i=1}^{\ell} f_i(t)^{k_i} \prod_{j=\ell+1}^{m} (g_j(t)g_j(t)^{\checkmark})^{r_j}$$

with $f_i = f_i^{\checkmark}$ and $g_i \neq g_i^{\checkmark}$, $\deg f_i = d_i$, $\deg g_j = \deg g_j^{\checkmark} = d_j$. Note that $\sum_{j=1}^{\ell} k_j d_j + 2\sum_{j=\ell+1}^{m} r_j d_j = n$.

Now, by Theorem 3.1.2 we know that this is maximized when $\psi = \mathrm{St}_{\mathrm{C_G(s)}}$, the product of the Steinberg characters for the factors $GU_{k_j}(q^{d_j})$ and $GL_{r_j}(q^{2d_j})$. So we see that

$$\frac{b(G)}{q^{n(n-1)/2}} = \max_s \left\{ \frac{|G|_{p'} \prod_{j=1}^{\ell} q^{d_j k_j (k_j-1)/2} \prod_{j=\ell+1}^{m} q^{2d_j r_j (r_j-1)/2}}{|C_G(s)|_{p'} q^{n(n-1)/2}} \right\}.$$

We have

$$\begin{aligned} |G|_{p'} &= \prod_{i=0}^{n-1} (q^{n-i} - (-1)^{i+n}) \\ &= \prod_{j=1}^{n} (q^j - (-1)^j) \\ &= \prod_{j=1}^{n} q^j \prod_{j=1}^{n} (1 - (-1)^j q^{-j}) \\ &= q^{n(n-1)/2+n} \prod_{j=1}^{n} (1 - (-1)^j q^{-j}) \\ &= q^n q^{n(n-1)/2} \prod_{j=1}^{n} (1 - (-1)^j q^{-j}) \end{aligned}$$

and

$$
\begin{aligned}
|C_G(s)|_{p'} &= \prod_{j=1}^{\ell}\prod_{i=0}^{k_j-1}(q^{d_j(k_j-i)}-(-1)^{i+k_j})\cdot\prod_{j=\ell+1}^{m}\prod_{i=0}^{r_j-1}(q^{2d_j(r_j-i)}-1)\\
&= \prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(q^{d_ji}-(-1)^i)\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(q^{2d_ji}-1)\\
&= \prod_{j=1}^{\ell}\prod_{i=1}^{k_j}q^{id_j}\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}q^{2id_j}\cdot\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^iq^{-d_ji})\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2d_ji})\\
&= \prod_{j=1}^{\ell}q^{d_jk_j(k_j-1)/2+d_jk_j}\prod_{j=\ell+1}^{m}q^{2d_jr_j(r_j-1)/2+2d_jr_j}\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^iq^{-d_ji})\\
&\quad\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2d_ji})\\
&= q^{\sum_{j=1}^{\ell}k_jd_j+2\sum_{j=\ell+1}^{m}r_jd_j}\prod_{j=1}^{\ell}q^{d_jk_j(k_j-1)/2}\prod_{j=\ell+1}^{m}q^{2d_jr_j(r_j-1)/2}\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^iq^{-d_ji})\\
&\quad\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2d_ji})\\
&= q^n\prod_{j=1}^{\ell}q^{d_jk_j(k_j-1)/2}\prod_{j=\ell+1}^{m}q^{2d_jr_j(r_j-1)/2}\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^iq^{-d_ji})\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2d_ji})
\end{aligned}
$$

Thus we see that

$$\frac{|G|_{p'}\prod_{j=1}^{\ell}q^{d_jk_j(k_j-1)/2}\prod_{j=\ell+1}^{m}q^{2d_jr_j(r_j-1)/2}}{|C_G(s)|_{p'}q^{n(n-1)/2}}=\frac{\prod_{j=1}^{n}(1-(-1)^jq^{-j})}{\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^iq^{-d_ji})\cdot\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2d_ji})},$$

which completes the proof.

□

The following bounds will be very useful in the remainder of this section and are proved in [42, Lemma 4.1].

Lemma 3.2.2. *Let $q \geq 2$. Then*

- $\prod_{i=1}^{\infty}(1-q^{-i}) > \exp(-\alpha/q)$, *where* $\alpha = 2\ln(32/9)$.
- $\prod_{i=2}^{\infty}(1-q^{-i}) > 9/16$.

- $\prod_{i=5}^{\infty}(1+q^{-i}) < 16/15$.
- $1 < \prod_{i=1}^{n}(1-(-1)^i q^{-i}) \leq 3/2$.

Lemma 3.2.3. *Let k be an odd positive integer, ℓ an integer with $\ell \geq k$, and q an integer with $q \geq 2$. Then $\prod_{i=k}^{\ell}(1-(-1)^i q^{-i}) \geq 1$.*

It will be useful to estimate the number of monic irreducible polynomials of a given degree over a certain finite field, which is the purpose of the next two lemmas.

Lemma 3.2.4. *Let $p_d(q)$ denote the number of monic irreducible polynomials of degree $d \geq 2$ over $\mathbb{F}_q$. Then*

$$\frac{q^d}{2d} < p_d(q) < \frac{q^d}{d}.$$

Proof. Write $p_d := p_d(q)$. Given a monic irreducible polynomial $f \in \mathbb{F}_q[t]$ of degree d, we know that f has d distinct roots, each of which are elements of $\mathbb{F}_{q^d}$. But the number of elements of $\mathbb{F}_{q^d}$ which are not contained in any proper subfield is at most $q^d - 1$. Thus $dp_d < q^d$. Moreover, the number of elements of $\mathbb{F}_{q^d}$ not contained in any proper subfield is at least $|\mathbb{F}_{q^d}| - \sum_{p|d} |\mathbb{F}_{q^{d/p}}|$ where the sum is taken over the distinct primes which divide d. But this means

$$dp_d \geq q^d - \sum_{p|d} q^{d/p} \geq q^d - \sum_{r=1}^{d-1} q^r = q^d - (q^d - q)/(q-1).$$

Now, we have that $2(q^d - q)/(q-1) \leq q^d - q < q^d$ for $q \geq 3$. Thus $(q^d - q)/(q-1) < q^d/2$. So $q^d - (q^d - q)/(q-1) > q^d - q^d/2 = q^d/2$ for $q \geq 3$.

Now let $q = 2$.

$$dp_d \geq q^d - \sum_{p|d} q^{d/p} \geq q^d - \sum_{r=1}^{d/2} q^r = q^d - (q^{d/2+1} - q)/(q-1)$$

We have $2(q^{d/2+1} - q)/(q-1) = 2(2^{d/2+1} - 2) = 4(2^{d/2} - 1) = 2^{d/2+2} - 4$. But $2^d - (2^{d/2+2} - 4)$ is positive for all d, so $(2^{d/2+1} - 2)/(2-1) < 2^d/2$, which means that $2^d - (2^{d/2+1} - 2)/(2-1) > 2^d - 2^d/2 = 2^d/2$, and this completes the proof.

$\square$

Lemma 3.2.5. *Let n_d denote the number of pairs $\{f, f^{\checkmark}\}$, where f is a monic irreducible polynomial in $\mathbb{F}_{q^2}[t]$ of degree d such that $f \neq f^{\checkmark}$, where $f^{\checkmark} = J_1(f)$ as in Section 2.4.1. Then if $d \geq 2$,*

$$\frac{2q^{2d}}{5d} < n_d < \frac{q^{2d}}{2d},$$

except in the case $(q, d) = (2, 2)$.

Proof. First note that the upper bound is clear from Lemma 3.2.4, since $2n_d \leq p_d(q^2)$. Moreover, a monic irreducible polynomial $f \in \mathbb{F}_{q^2}[t]$ satisfies $f = f^{\checkmark}$ if and only if $\deg(f)$ is odd and any root α satisfies $\alpha^{q^d+1} = 1$ (see, for example, [73, Part (B1) of the proof of Theorem 4.1]). Thus we see that for d even, $2n_d = p_d$.

First, consider the case that d is odd. If $d = 3$, then $\mathbb{F}_{q^6}/\mathbb{F}_{q^2}$ has no intermediate fields, and therefore

$$2dn_d = 6n_3 \geq q^{2d} - q^2 - (q^d + 1) = q^6 - q^3 - q^2 - 1,$$

which is larger than $4q^6/5$ for $q \geq 3$. Moreover, if $q = 2$, we know that every element of $\mathbb{F}_4^{\times}$ satisfies $\alpha^3 = 1$, so in particular $\alpha^{q^d+1} = \alpha^9 = 1$. Thus in this case, $2dn_d = 6n_3 \geq q^{2d} - (q^d + 1) - 1 = q^6 - q^3 - 2 = 54 > 4(2^6)/5 = 4q^{2d}/5$. Then for $d = 3$, the statement holds.

Now let $d > 3$ be odd. Subextensions of $\mathbb{F}_{q^{2d}}$ containing $\mathbb{F}_{q^2}$ are of the form $\mathbb{F}_{q^{2m}}$ where $m|d$. The number of $\alpha \in \mathbb{F}_{q^{2d}}$ which are roots of irreducible degree-d polynomials over $\mathbb{F}_{q^2}$ (so are not found in a proper subextension) and satisfy $\alpha^{q^d+1} = 1$ is therefore

$$\begin{aligned} 2dn_d \geq q^{2d} - \sum_{m|d} q^{2m} - (q^d + 1) &\geq q^{2d} - \left(q^d + 1 + \sum_{m=1}^{\lfloor d/3 \rfloor} q^{2m}\right) \\ &= q^{2d} - \left(q^d + 1 + \frac{q^{2(\lfloor d/3 \rfloor + 1)} - q^2}{q^2 - 1}\right) \end{aligned}$$

since we are assuming d is odd.

Thus it suffices to show that $\left(q^d + 1 + \frac{q^{2(\lfloor d/3\rfloor+1)}-q^2}{q^2-1}\right) \le q^{2d}/5$. We have that for $q \ge 3$,

$$\begin{aligned} 5\left(q^d + 1 + \frac{q^{2(\lfloor d/3\rfloor+1)}-q^2}{q^2-1}\right) &= 5q^d + 5 + \frac{5}{q^2-1}(q^{2(\lfloor d/3\rfloor+1)} - q^2) \\ &\le 5q^d + 5 + q^{2(\lfloor d/3\rfloor+1)} - q^2 \quad \text{since } q \ge 3 \\ &\le 5q^d + 5 + q^d - q^2 \quad \text{since } \lfloor d/3\rfloor + 1 \le d/2 \text{ for } d \ge 4 \\ &< 6q^d \text{ since } 5 < q^2 \text{ for } q \ge 3 \\ &\le q^{2d}. \end{aligned}$$

Now suppose $q = 2$. Then

$$5\left(q^d + 1 + \frac{q^{2(\lfloor d/3\rfloor+1)}-q^2}{q^2-1}\right) \le 5(2^d) + 5 + \frac{5}{3}2^d - 20/3 < \frac{10}{3}2^d < 2^{2d}$$

where the last inequality is because $10/3 < 4$ and $d + 2 \le 2d$ for $d \ge 2$.

Thus the claim holds in the case that d is odd.

Now suppose $d \ge 2$ is even. Then $2n_d = p_d$, so

$$2dn_d \ge q^{2d} - \sum_{m|d} q^{2m} \ge q^{2d} - \sum_{m=1}^{d/2} q^{2m} = q^{2d} - \frac{q^{2(d/2+1)} - q^2}{q^2-1} = q^{2d} - \frac{q^2}{q^2-1}(q^d - 1).$$

Thus it suffices to show that $\frac{q^2}{q^2-1}(q^d - 1) \le q^{2d}/5$. For $q \ge 3$, we have

$$5\left(\frac{q^2}{q^2-1}(q^d-1)\right) \le q^2(q^d - 1) < q^{d+2} \le q^{2d}$$

since $d \ge 2$. If $q = 2$, then

$$5\left(\frac{q^2}{q^2-1}(q^d-1)\right) = \frac{20}{3}(2^d - 1).$$

But $2^{2d} - \frac{20}{3}(2^d - 1)$ is positive when $d \ge 3$, so this completes the proof.

$\square$

Theorem 3.2.6. *Let G be of unitary type (i.e. $G = GU_n(q), PGU_n(q), SU_n(q)$, or $PSU_n(q)$). Then*

$$\frac{b(G)}{q^{n(n-1)/2}} < 2\left(\log_q(n(q^2-1) + q^2)\right)^{1.27}$$

Proof. We may assume that $G = GU_n(q)$, so from Lemma 3.2.1, we want to maximize

$$P = \frac{\prod_{i=1}^{n}(1-(-1)^i q^{-i})}{\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^i q^{-id_j})\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2id_j})}.$$

Now, the numerator is

$$(1+q^{-1})(1-q^{-2})(1+q^{-3})(1-q^{-4})\prod_{i=5}^{n}(1-(-1)^i q^{-i}).$$

It is clear that

$$\prod_{i=5}^{n}(1-(-1)^i q^{-i}) \le \prod_{i=5}^{n}(1+q^{-i}) \le \prod_{i=5}^{\infty}(1+q^{-i}) \le 16/15$$

by Lemma 3.2.2. The function $(1+x^{-1})(1-x^{-2})(1+x^{-3})(1-x^{-4})$ is decreasing on $[3,\infty)$, and is smaller at $x=2$ than $x=3$, so we have that $(1+q^{-1})(1-q^{-2})(1+q^{-3})(1-q^{-4}) \le (4/3)(8/9)(28/27)(80/81) < 1.214$ and therefore the numerator is no more than 1.3.

Now, $\prod_{j=1}^{\ell}\prod_{i=1}^{k_j}(1-(-1)^i q^{-id_j}) \ge 1$ by Lemma 3.2.2, so we have

$$P \le \frac{1.3}{\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2id_j})}.$$

Note that

$$\prod_{j=\ell+1}^{m}\prod_{i=1}^{r_j}(1-q^{-2id_j}) > \prod_{j=\ell+1}^{m}\prod_{i=1}^{\infty}(1-q^{-2id_j}) > \prod_{j=\ell+1}^{m}\exp(-\alpha q^{-2d_j})$$

by Lemma 3.2.2. Let b_d be the number of distinct irreducible polynomials g of degree d over $\mathbb{F}_{q^2}$ such that $g \neq g^{\checkmark}$ and g occurs in the decomposition of the characteristic polynomial of the semisimple element. Then $b_d < q^{2d}/d$. Note that $\sum_{d=1}^{n} b_d d \le n$ since the characteristic polynomial must have degree n. Letting L be the natural log of the denominator, we have

$$-L/\alpha < \sum_{j=\ell+1}^{m} q^{-2d_j} = \sum_{d=1}^{n}\frac{b_d}{2}q^{-2d}.$$

Note that the $b_d/2$ comes from the fact that each index j represents the product $g_j \cdot g_j^{\checkmark}$, so two distinct polynomials with $g \neq g^{\checkmark}$ occurring in the decomposition. Replacing $b_d/2$ with real numbers x_d, we wish to maximize $\sum_{d=1}^{\infty} x_d q^{-2d}$ subject to the conditions $\sum_{d=1}^{\infty} x_d d \leq n/2$ and $0 \leq x_d \leq q^{2d}/2d$.

Now there is some d_0 such that this is optimized when $x_d = q^{2d}/2d$ for $d \leq d_0$ and $x_d = 0$ for $d > d_0 + 1$. That is, d_0 is the largest integer such that

$$\sum_{d=1}^{d_0} (q^{2d}/2d)d \leq n/2,$$

or equivalently,

$$n \geq \sum_{d=1}^{d_0} q^{2d} = \frac{(q^{2(d_0+1)} - q^2)}{q^2 - 1},$$

and therefore

$$d_0 + 1 \leq \frac{\log_q(n(q^2-1) + q^2)}{2}.$$

Now, we have

$$\sum_{d=1}^{d_0+1} x_d q^{-2d} \leq \sum_{d=1}^{d_0+1} \frac{1}{2d} = \frac{1}{2} \sum_{d=1}^{d_0+1} 1/d < \frac{1}{2}(1 + \ln(d_0+1)),$$

so that

$$P \leq (1.3)e^{-L} < (1.3)e^{\frac{1}{2}\alpha(1+\ln(d_0+1))} = (1.3)e^{\alpha/2}(d_0+1)^{\alpha/2}.$$

Finally, this yields

$$P < (4.7)\left(\frac{\log_q(n(q^2-1)+q^2)}{2}\right)^{1.27} < 2\left(\log_q(n(q^2-1)+q^2)\right)^{1.27},$$

as desired.

$\square$

Let G be a finite group with normal subgroup N. Let $\chi \in \mathrm{Irr}(G)$ and define the constant $\kappa_N^G(\chi)$ to be the number of irreducible constituents of $\chi|_N$. The next lemma describes the value of $\kappa_N^G(\chi)$ in the case that G/N is cyclic.

Lemma 3.2.7. *Let G be a finite group with normal subgroup N such that G/N is cyclic. Then*

$$\kappa_N^G(\chi) = \#\{\lambda \in \mathrm{Irr}(G/N) \colon \lambda\chi = \chi\}.$$

Proof. Let φ be an irreducible constituent of χ_N and set $T = I_G(\varphi)$. Then by Clifford theory, we know that $\chi_N = e\sum_{i=1}^{[G:T]} \varphi^{g_i}$, so that

$$\kappa_N^G(\chi) = e[G:T]. \tag{3.2.1}$$

Moreover, since G/N is cyclic, we have that $e = 1$ (i.e. φ is extendable to T) (see, for example [33], corollary 11.22). Now, Gallagher's theorem implies that the $[T:N]$ distinct irreducible constituents of φ^T are of the form $\theta\beta$ where $\theta|_N = \varphi$ and $\beta \in \mathrm{Irr}(T/N)$. Then the irreducible constituents of φ^G are of the form $(\beta\theta)^G$ (see, for example, [33], theorem 6.11). But since G/N is cyclic, we know that any irreducible character of T/N is extendable to G/N. (To see this, note that the character values at the generator of T/N are $[T:N]$th roots of unity, obtained by taking powers of the $[G:N]$th roots of unity found as character values of the generator of G/N.) In particular, write $\beta = \lambda|_T$ for some $\lambda \in \mathrm{Irr}(G/N)$, so $(\beta\theta)^G = (\lambda_T\theta)^G = \lambda\theta^G$. But of course, χ is a constituent of φ^G, so that we can write any other irreducible constituent of φ^G as $\lambda'\chi$ for $\lambda' \in \mathrm{Irr}(G/N)$.

Thus we have that the number of irreducible constituents of φ^G is

$$\frac{|\mathrm{Irr}(G/N)|}{\#\{\lambda \in \mathrm{Irr}(G/N) \colon \lambda\chi = \chi\}} = \frac{[G:N]}{\#\{\lambda \in \mathrm{Irr}(G/N) \colon \lambda\chi = \chi\}}.$$

But also, the number of irreducible constituents of φ^G is $[T:N]$, since $\varphi^G(1) = \varphi(1)[G:N] = [G:N]\chi(1)/[G:T] = [T:N]\chi(1)$ and each irreducible constituent of $\varphi^G(1)$ has degree $\chi(1)$ (since it must be equal to χ when restricted to N, by Clifford's theorem). Thus we have shown that

$$\#\{\lambda \in \mathrm{Irr}(G/N) \colon \lambda\chi = \chi\} = \frac{[G:N]}{[T:N]} = [G:T],$$

which completes the proof by (3.2.1).

□

Now, we wish to find a lower bound for $b(G)/q^{n(n-1)/2}$ given that G is of unitary type. To do this, we will exhibit a character degree which occurs simultaneously for each of $GU_n(q), SU_n(q), PGU_n(q)$, and $PSU_n(q)$ by finding a semisimple element $s \in GU_n(q)$ which lies in $SU_n(q)$ such that the character $\chi \in \mathrm{Irr}(GU_n(q))$ corresponding to $((s), \mathrm{St}(C_{GU_n(q)}(s))$ is trivial at the center and remains irreducible when restricted to $SU_n(q)$. In the following discussion, let $G := GU_n(q)$ and $S := SU_n(q)$. From Lemma 3.2.7, we see that for $\chi \in \mathrm{Irr}(G)$, the restriction $\chi|_S$ is irreducible if any only if there is no nonidentity $\lambda \in \mathrm{Irr}(G/S)$ such that $\lambda\chi = \chi$.

Note that letting $\underline{G} = GL_n(\overline{\mathbb{F}_q})$ and F the Frobenius map $F\colon (a_{ij}) \mapsto {}^T(a_{ij}^q)^{-1}$, we can write $G = \underline{G}^F$. In this case, $\underline{G}^* = \underline{G}$ and $G = G^*$, and since $Z(\underline{G})$ is connected, we know $C_{\underline{G}}(s)$ is also connected for any semisimple $s \in G$. Hence it makes sense to discuss the semisimple character χ_s of G corresponding to s. We may think of χ_s as the character corresponding via Lusztig's correspondence to the product of principal characters in $C_G(s)$.

Now, characters in $\mathrm{Irr}(G/S)$ are precisely the characters χ_t for $t \in Z(G)$. But given a semisimple $s \in G$ and $t \in Z(G)$, the set of characters $\mathcal{E}(G,(s)) \cdot \chi_t$ is equal to the set of characters $\mathcal{E}(G,(st))$. (See, for example [20, Proposition 13.30].) Then since $\mathrm{Irr}(G)$ is the disjoint union,

$$\mathrm{Irr}(G) = \bigsqcup_{(s)\in G} \mathcal{E}(G,(s)),$$

over semisimple conjugacy classes (s) of G, to achieve our goal it suffices to choose a semisimple s lying in $SU_n(q)$ such that s and st are not conjugate for any $t \in Z(G)$.

Theorem 3.2.8. *Let $G = GU_n(q), SU_n(q), PGU_n(q)$, or $PSU_n(q)$. Then*

$$\frac{b(G)}{q^{n(n-1)/2}} > \frac{1}{4}\left(\log_q((n-1)(1-q^{-2})+q^4)\right)^{2/5}.$$

Proof. Let n_d be the number of pairs of monic irreducible polynomials $f \in \mathbb{F}_{q^2}[t]$ of degree d such that $f \neq f^{\checkmark}$. Let p_d be the total number of monic irreducible polynomials of degree d over $\mathbb{F}_{q^2}$. By Lemma 3.2.5, we have that

$$\frac{2q^{2d}}{5d} < n_d < \frac{q^{2d}}{2d}.$$

Suppose that $n-2 \leq 6n_3$. Then in particular, $n-2 < q^6$, so $n-1 \leq q^6$. In this case, the statement is evident, since

$$\begin{aligned}
\frac{1}{4}\left(\log_q((n-1)(1-q^{-2})+q^4)\right)^{2/5} &\leq \frac{1}{4}\left(\log_q(q^6(1-q^{-2})+q^4)\right)^{2/5} \\
&= \frac{1}{4}\left(\log_q(q^6-q^4+q^4)\right)^{2/5} \\
&= \frac{1}{4}(6)^{2/5} \\
&< 1 \leq \frac{b(G)}{\mathrm{St}(1)} = \frac{b(G)}{q^{n(n-1)/2}}.
\end{aligned}$$

Thus we may assume that $n-2 > 6n_3$. Let d_0 be the largest integer such that $n-2 > m := 2\sum_{d=3}^{d_0} dn_d$. Then in particular,

$$(q^{2(d_0+2)}-q^2)/(q^2-1) = \sum_{d=1}^{d_0+1} q^{2d} = \sum_{d=3}^{d_0+1} q^{2d}+q^2+q^4 > 2\sum_{d=3}^{d_0+1} dn_d+q^2+q^4 \geq n-2+q^2+q^4.$$

Since $2dn_d < q^{2d}$ is a strict inequality, we see that this implies

$$(q^{2(d_0+2)}-q^2)/(q^2-1) \geq n-1+q^2+q^4,$$

so

$$q^{2(d_0+2)} \geq (n-1+q^2+q^4)(q^2-1)+q^2 = (n-1)(q^2-1)+q^6,$$

and $q^{2(d_0+1)} > \frac{(n-1)(q^2-1)+q^6}{q^2}$. Thus we have

$$d_0+1 > \frac{\log_q((n-1)(1-q^{-2})+q^4)}{2}. \tag{3.2.2}$$

Consider the polynomial $h := \prod_{d=3}^{d_0}\prod_{i=1}^{n_d}(g_i g_i^{\checkmark}) \in \mathbb{F}_{q^2}[t]$, where the $g_i g_i^{\checkmark}$ for $i = 1, ..., n_d$ are all the pairs of non-self-check monic irreducible polynomials of degree

d. Let α be the product of the roots of h. Note that $\alpha^{q+1} = 1$, so $t - \alpha^{-1}$ is a self-check monic irreducible polynomial. (Indeed, we may write $\alpha = \lambda\lambda^{-q}$ where $\lambda \in \mathbb{F}_{q^2}$ is the product of the roots of the g_i's, so λ^{-q} is the product of roots of $g_i^{\checkmark}$'s. Hence since $\lambda^{q^2-1} = 1$, we see $\alpha^{q+1} = 1$.) Choose a semisimple element s of $GU_n(q)$ with characteristic polynomial $(t-1)^{n-m-1}(t-\alpha^{-1})h(t)$. Note that $\det(s) = 1$, so $s \in SU_n(q)$. Moreover, s is not conjugate to $s\gamma$ for any nontrivial $\gamma I \in Z(GU_n(q))$, so by Lemma 3.2.7 and the discussion following it, the character χ corresponding to $((s), \mathrm{St}_{C_{GU_n(q)}(s)})$ in Lusztig correspondence is irreducible when restricted to $SU_n(q)$. (Indeed, $\mathrm{spec}(\gamma s) = \gamma\mathrm{spec}(s)$, so if $\alpha \neq 1$, then γ has multiplicity $n-m-1 \geq 2$ in $\mathrm{spec}(s\gamma)$, but multiplicity at most 1 in $\mathrm{spec}(s)$. If $\alpha = 1$, then γ has multiplicity $n-m$ in $\mathrm{spec}(s\gamma)$, but multiplicity 0 in $\mathrm{spec}(s)$. In either case, s and $s\gamma$ have different eigenvalue multiplicities, so cannot be conjugate.) Moreover, since $s \in SU_n(q) = [GU_n(q), GU_n(q)]$, χ is trivial at $Z(GU_n(q))$ (see [54, Lemma 4.4(ii)]), so χ can be viewed as an irreducible character of G for $G = GU_n(q), SU_n(q), PGU_n(q)$, or $PSU_n(q)$.

In the case that $\alpha = 1$, the centralizer of s in $GU_n(q)$ is

$$C_{GU_n(q)}(s) \cong GU_{n-m}(q) \times \prod_{d=3}^{d_0} GL_1(q^{2d})^{n_d}$$

and if $\alpha \neq 1$, the centralizer is

$$C_{GU_n(q)}(s) \cong GU_{n-m-1}(q) \times GU_1(q) \times \prod_{d=3}^{d_0} GL_1(q^{2d})^{n_d}.$$

Thus in the first case,

$$\begin{aligned} \frac{b(G)}{q^{n(n-1)/2}} \geq \frac{\chi(1)}{q^{n(n-1)/2}} &= \frac{\prod_{i=1}^{n}(1-(-1)^i q^{-i})}{\prod_{i=1}^{n-m}(1-(-1)^i q^{-i})\prod_{d=3}^{d_0}(1-q^{-2d})^{n_d}} \\ &= \frac{\prod_{i=n-m+1}^{n}(1-(-1)^i q^{-i})}{\prod_{d=3}^{d_0}(1-q^{-2d})^{n_d}} \end{aligned}$$

In the second case, we get

$$\frac{b(G)}{q^{n(n-1)/2}} \geq \frac{\chi(1)}{q^{n(n-1)/2}} = \frac{\prod_{i=1}^{n}(1-(-1)^i q^{-i})}{(1+q^{-1})\prod_{i=1}^{n-m-1}(1-(-1)^i q^{-i})\prod_{d=3}^{d_0}(1-q^{-2d})^{n_d}}$$
$$= \frac{\prod_{i=n-m}^{n}(1-(-1)^i q^{-i})}{(1+q^{-1})\prod_{d=3}^{d_0}(1-q^{-2d})^{n_d}}$$

Thus since the second case gives a smaller bound, it suffices to consider only the second case.

Note that $\prod_{i=n-m}^{n}(1-(-1)^i q^{-i}) \geq 15/16$ by Lemma 3.2.3, since we know $n-m \geq 3$ and if $n-m$ is odd, then the product is at least 1, and if $n-m$ is even, then the product is at least $(1-q^{-(n-m)})$, which is at least $(1-2^{-4}) = 15/16$.

Now taking the natural log and noting that $1/(1-x) > e^x$ on $0 < x < 1$, we get

$$\ln\left(\prod_{d=3}^{d_0}\frac{1}{(1-q^{-2d})^{n_d}}\right) = \sum_{d=3}^{d_0} n_d \ln\left(\frac{1}{(1-q^{-2d})}\right) > \sum_{d=3}^{d_0} n_d q^{-2d} > \sum_{d=3}^{d_0}\frac{2q^{2d}}{5d}q^{-2d} = \frac{2}{5}\sum_{d=3}^{d_0}\frac{1}{d}$$
$$= \frac{2}{5}\left(\sum_{d=1}^{d_0}\frac{1}{d} - 3/2\right) > \frac{2}{5}\left(\ln(d_0+1) - \frac{3}{2}\right) = \ln\left(\left(\frac{d_0+1}{e^{3/2}}\right)^{2/5}\right)$$

This yields

$$\prod_{d=3}^{d_0}\frac{1}{(1-q^{-2d})^{n_d}} > \left(\frac{d_0+1}{e^{3/2}}\right)^{2/5}.$$

Then from (3.2.2), we have

$$\frac{b(G)}{q^{n(n-1)/2}} > \frac{15}{16(1+q^{-1})}\left(\frac{\log_q((n-1)(1-q^{-2})+q^4)}{2e^{3/2}}\right)^{2/5}$$
$$> \frac{5}{13}\frac{\left(\log_q((n-1)(1-q^{-2})+q^4)\right)^{2/5}}{1+q^{-1}}$$
$$\geq \frac{1}{4}\left(\log_q((n-1)(1-q^{-2})+q^4)\right)^{2/5},$$

which completes the proof.

□

Remark. Note that if $(r,\delta) \in \mathbb{Q}_{>0} \times \mathbb{Z}_{>0}$ is such that $n_d > \frac{rq^{2d}}{d}$ for $d \geq \delta$, then an argument analogous to the proof of Theorem 3.2.8 shows that for G finite of unitary type,

$$\frac{b(G)}{q^{n(n-1)/2}} > \frac{15}{16 \cdot C_r} \frac{\left(\log_q((n-1)(1-q^{-2}) + q^{2\delta-2})\right)^r}{1+q^{-1}},$$

with $C_r := 2^r \cdot e^{r \sum_{d=1}^{\delta-1} 1/d}$.

Recall that we are particularly interested in the case that q is small. For this reason, we note the following corollary.

Corollary 3.2.9.

$$\frac{1}{4} \left(\log_2(3n/4 + 19/2)\right)^{2/5} < \frac{b(PSU_n(2))}{2^{n(n-1)/2}} < 2 \left(\log_2(3n+4)\right)^{1.27}$$

Chapter 4

The Blocks and Brauer Characters of $Sp_6(2^a)$

In this chapter, we discuss some aspects of the blocks and Brauer characters of $Sp_6(2^a)$ that will be useful in Chapters 5 and 7. We begin in Section 4.1 by showing that Bonnafé and Rouquier's [9] results apply to the centralizers of nontrivial semisimple elements of $G = Sp_6(2^a)$, which yields a Morita equivalence between blocks of G and blocks of these centralizers. In Section 4.2, we use this information to prove Theorem 1.1.1, which describes the low-dimensional representations of G. In Section 4.3, we use D. White's [76] results to describe the Brauer characters of G which lie in unipotent blocks. Finally, in Section 4.4, we give the distribution of ordinary characters of G into non-unipotent blocks and use the results of Section 4.1 and results of various authors to describe the Brauer characters in these blocks.

4.1 On Bonnafé-Rouquier's Morita Equivalence

Recall that in [9], Bonnafé and Rouquier show that when $C_{\underline{G}^*}(t)$ is contained in an F^*-stable Levi subgroup, $\underline{L}^*$, of $\underline{G}^*$, then Deligne-Lusztig induction R_L^G yields a Morita equivalence between $\mathcal{E}_\ell(L,(t))$ and $\mathcal{E}_\ell(G,(t))$, where $L = (\underline{L})^F$ and $(\underline{L}, F)$ is dual to $(\underline{L}^*, F^*)$. Also, recall that when $G = Sp_6(q)$, q even, with $G = \underline{G}^F$ and $(\underline{G}^*, F^*)$ in duality with $(\underline{G}, F)$, each semisimple conjugacy class (s) of $G^* = (\underline{G}^*)^{F^*}$ satisfies that $|s|$ is odd. Hence by [20, Lemma 13.14(iii)], the centralizer $C_{\underline{G}^*}(s)$ is connected.

While applying Deligne-Lusztig theory to $Sp_{2n}(q)$ with q even, it will be convenient to view $G = Sp_{2n}(q)$ as $SO_{2n+1}(q) \cong Sp_{2n}(q)$, so that $G^* = Sp_{2n}(q)$.

Lemma 4.1.1. *Let $G^* = Sp_6(q)$, q even, with $G = \underline{G}^F$ and $(\underline{G}^*, F^*)$ in duality with $(\underline{G}, F)$. The nontrivial semisimple conjugacy classes (s) of G^* each satisfy $C_{\underline{G}^*}(s) = \underline{L}^*$ for an F^*-stable Levi subgroup $\underline{L}^*$ of $\underline{G}^*$ with $C_{G^*}(s) = (\underline{L}^*)^{F^*} =: L^*$. In* *particular, Bonnafé-Rouquier's theorem [9] implies that there is a Morita equivalence $\mathcal{E}_\ell(L,(1)) \leftrightarrow \mathcal{E}_\ell(G,(t))$ given by Deligne-Lusztig induction (composed with tensoring by a suitable linear character) when $t \neq 1$ is a semisimple ℓ'-element, where $L = \underline{L}^F$ and $(\underline{L}, F)$ is dual to $(\underline{L}^*, F^*)$.*

Proof. Write $G^* = (\underline{G}^*)^{F^*}$, as above. Direct calculation shows that for each semisimple element $s \neq 1$ of G^*, $C_{\underline{G}^*}(s) \leq C_{\underline{G}^*}(S)$ for some F^*-stable torus S in $\underline{G}^*$ containing s. (Each such s is conjugate in $\underline{G}^*$ to a diagonal matrix $s' = gsg^{-1}$, $g \in \underline{G}^*$, whose centralizer in $\underline{G}^*$ depends only on the number of distinct entries different than 1 and their multiplicities. Hence we may choose S to be $g^{-1}S'g$, where S' is the torus consisting of all diagonal matrices in $\underline{G}^*$ with the same form as s'.) Therefore, $C_{\underline{G}^*}(s) = C_{\underline{G}^*}(S)$, which is an F^*-stable Levi subgroup of $\underline{G}^*$.

Let t be a semisimple ℓ'-element of G^*. Writing $\underline{L}^* = C_{\underline{G}^*}(t)$, we see that $t \in Z(\underline{L}^*)$ and therefore $t \in Z(L^*)$. But then by [20, Proposition 13.30], tensoring with a suitable linear character yields a Morita equivalence of $\mathcal{E}_\ell(L,(t)) \leftrightarrow \mathcal{E}_\ell(L,(1))$. Hence there is a Morita equivalence $\mathcal{E}_\ell(G,(t)) \leftrightarrow \mathcal{E}_\ell(L,(t)) \leftrightarrow \mathcal{E}_\ell(L,(1))$ by this fact and Bonnafé-Rouquier's theorem [9]. $\square$

Proposition 4.1.2. *In the notation of Lemma 4.1.1, let t be a semisimple ℓ'-element of G^*. Let $\theta \in \mathcal{E}_\ell(G,(t))$ be an irreducible Brauer character. Then $\theta(1) = [G^* : C_{G^*}(t)]_{2'}\varphi(1)$ for some $\varphi \in \mathrm{IBr}_\ell(L)$ lying in a unipotent block of L. Moreover, if $\theta(1) = [G^* : C_{G^*}(t)]_{2'}$, then the equivalence given by Lemma 4.1.1 maps θ to the principal Brauer character of $C_{G^*}(t)$ and θ lifts to a complex character.*

Proof. From Lemma 4.1.1, Deligne-Lusztig induction R_L^G provides a Morita equivalence between $\mathcal{E}_\ell(L,(1))$ and $\mathcal{E}_\ell(G,(t))$. Hence R_L^G gives a bijection between ordinary characters in $\mathcal{E}_\ell(L,(1))$ and $\mathcal{E}_\ell(G,(t))$ and also a bijection between ℓ-Brauer characters in these two unions of blocks, which preserve the decomposition matrices for these two unions of blocks.

Let B be a unipotent block in L, and let $\varphi_1, ..., \varphi_m$ be the irreducible Brauer characters in B. Let $\chi_1, ..., \chi_s$ be the irreducible ordinary characters in B. Then we can write $\widehat{\chi}_i = \sum_{j=1}^m d_{ij}\varphi_j$, where (d_{ij}) is the decomposition matrix of the block B. Writing ψ^* for the image of an ordinary or Brauer character, ψ, of L under Deligne-Lusztig induction R_L^G, we therefore also have $\widehat{\chi}_i^* = \sum_{j=1}^m d_{ij}\varphi_j^*$.

Moreover, we may write $\varphi_k = \sum_{i=1}^s a_{ki}\widehat{\chi}_i$ for some integers a_{ki}. We claim that $\varphi_k^* = \sum_{i=1}^s a_{ki}\widehat{\chi}_i^*$ as well. Indeed,

$$\varphi_k = \sum_{i=1}^s a_{ki}\widehat{\chi}_i = \sum_{i=1}^s a_{ki}\left(\sum_{j=1}^m d_{ij}\varphi_j\right) = \sum_{j=1}^m \varphi_j \left(\sum_{i=1}^s a_{ki}d_{ij}\right),$$

so $\sum_{i=1}^s a_{ki}d_{ij} = \delta_{kj}$ is the Kronecker delta by the linear independence of irreducible Brauer characters. Now,

$$\sum_{i=1}^s a_{ki}\widehat{\chi}_i^* = \sum_{i=1}^s a_{ki}\left(\sum_{j=1}^m d_{ij}\varphi_j^*\right) = \sum_{j=1}^m \varphi_j^* \left(\sum_{i=1}^s a_{ki}d_{ij}\right) = \sum_{j=1}^m \varphi_j^*\delta_{kj} = \varphi_k^*,$$

proving the claim.

Note that $\chi_i^*(1) = [G : L]_{2'}\chi_i(1)$ for $1 \leq i \leq s$. Letting $\theta = \varphi_k^*$, we can write $\theta = \sum_{i=1}^s a_{ki}\widehat{\chi}_i^*$, and hence

$$\theta(1) = \sum_{i=1}^s a_{ki}\widehat{\chi}_i^*(1) = [G : L]_{2'}\sum_{i=1}^s a_{ki}\widehat{\chi}_i(1) = [G : L]_{2'}\varphi_k(1) = [G^* : C_{G^*}(t)]_{2'}\varphi_k(1),$$

which completes the proof of the first statement.

For the last statement, we further note that the principal character $1_{C_{G^*}(s)}$ is the only Brauer character of the group $C_{G^*}(s)$ with degree 1 lying in a unipotent block.

□

The following lemma records the semisimple classes of $Sp_6(q)$ whose index of the centralizer have smallest $2'$-part.

Lemma 4.1.3. *Let $q \geq 4$ be even and let $s \in G^* = Sp_6(q)$ be a noncentral semisimple element. Then either $[G^* : C_{G^*}(s)]_{2'} \geq (q-1)^2(q^2+1)(q^4+q^2+1)$, or s is a member of*

Table 4.1: Semisimple Classes of $G^* = Sp_6(q)$ with Small $[G^* : C_{G^*}(s)]_{2'}$

Semisimple Class (s)	$[G^* : C_{G^*}(s)]_{2'}$	$C_{G^*}(s)$
$c_{4,0}$	$\frac{q^6-1}{q+1}$	$Sp_4(q) \times GU_1(q)$
$c_{3,0}$	$\frac{q^6-1}{q-1}$	$Sp_4(q) \times GL_1(q)$
$c_{6,0}$	$(q^2+1)(q-1)^2(q^2+q+1)$	$GU_3(q)$
$c_{5,0}$	$(q^2+1)(q+1)^2(q^2-q+1)$	$GL_3(q)$
$c_{10,0}$	$(q-1)(q^2+1)(q^4+q^2+1)$	$GU_2(q) \times Sp_2(q)$
$c_{8,0}$	$(q+1)(q^2+1)(q^4+q^2+1)$	$GL_2(q) \times Sp_2(q)$

one of the classes in Table 4.1, which follows the notation of [47] and lists the classes in increasing order of $[G^* : C_{G^*}(s)]_{2'}$. The table also lists the isomorphism class of $C_{G^*}(s)$.

Proof. This is evident from inspection of the list of semisimple classes and the sizes of their centralizers in [47, Tabelles 10 and 14]. □

4.2 Low-Dimensional Representations of $Sp_6(q)$

The purpose of this section is to prove Theorem 1.1.1. We recall the statement of the theorem:

Theorem (1.1.1)**.** *Let $G = Sp_6(q)$, with $q \geq 4$ even, and let $\ell \neq 2$ be a prime dividing $|G|$. Suppose $\chi \in \mathrm{IBr}_\ell(G)$. Then:*

A) If χ lies in a unipotent ℓ-block, then either

1. $\chi \in \left\{1_G, \widehat{\alpha}_3, \widehat{\rho}_3^1 - \delta_1, \widehat{\beta}_3 - \delta_2, \widehat{\rho}_3^2 - \delta_3\right\}$, *where*

$$\delta_1 := \begin{cases} 1_G, & \ell | (q^2+q+1), \\ 0, & \textit{otherwise}, \end{cases} \quad \delta_2 := \begin{cases} 1_G, & \ell | (q+1), \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$\delta_3 := \begin{cases} 1_G, & \ell | (q^3+1), \\ 0, & \textit{otherwise}. \end{cases}$$

2. χ is as in the following table:

Condition on ℓ	χ	*Degree* $\chi(1)$
$\ell \mid (q^3-1)$ *or* $3 \neq \ell \mid (q^2-q+1)$	$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\ell \mid (q^2+1)$	$\widehat{\chi}_6 - 1_G$	$q^2(q^4+q^2+1)-1$
$\ell \mid (q+1)$	$\widehat{\chi}_{28}$ $= \widehat{\chi}_6 - \widehat{\chi}_3 - \widehat{\chi}_2 + 1_G$	$(q^2+q+1)(q-1)^2(q^2+1)$

,

3. *χ is as in the following table:*

Condition on ℓ	χ	*Degree* $\chi(1)$
$\ell \mid (q^3-1)$ *or* $3 \neq \ell \mid (q^2-q+1)$	$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\ell \mid (q^2+1)$	$\widehat{\chi}_7 - \widehat{\chi}_4$	$q^3(q^4+q^2+1) - q(q+1)(q^3+1)/2$
$\ell \mid (q+1)$	$\widehat{\chi}_{35} - \widehat{\chi}_5$ $= \widehat{\chi}_7 - \widehat{\chi}_6 + \widehat{\chi}_3 - \widehat{\chi}_1$	$(q-1)(q^2+1)(q^4+q^2+1) - q(q-1)(q^3-1)/2$

,

or

4. *$\chi(1) \geq D$, where D is as in the table:*

Condition on ℓ	D
$\ell \mid (q^3-1)(q^2+1)$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$
$\ell \mid (q+1)$, $(q+1)_\ell \neq 3$	$\frac{1}{2}q(q^3-2)(q^2+1)(q^2-q+1) - \frac{1}{2}q(q-1)(q^3-1)+1$
$\ell \mid (q+1)$, $(q+1)_\ell = 3$	$\frac{1}{2}q(q^3-2)(q^2+1)(q^2-q+1)+1$
$3 \neq \ell \mid (q^2-q+1)$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1) - \frac{1}{2}q(q-1)^2(q^2+q+1) = \frac{1}{2}q(q^3-1)^2(q-1)$

B) If χ does not lie in a unipotent block, then either

1. $\chi \in \{\widehat{\tau}_3^i, \widehat{\zeta}_3^j\}_{1 \leq i \leq ((q-1)_{\ell'}-1)/2, 1 \leq j \leq ((q+1)_{\ell'}-1)/2}$,

2. $\chi(1) = (q^2+1)(q-1)^2(q^2+q+1)$ *or* $(q^2+1)(q+1)^2(q^2-q+1)$ *(here χ is the restriction to ℓ-regular elements of the semisimple character indexed by a semisimple ℓ' - class in the family $c_{6,0}$ or $c_{5,0}$ respectively, in the notation of [47] - see Table 4.1),*

3. $\chi(1) = (q-1)(q^2+1)(q^4+q^2+1)$ *or* $(q+1)(q^2+1)(q^4+q^2+1)$ *(here χ is the restriction to ℓ-regular elements of the semisimple character indexed by a semisimple ℓ' - class in the family $c_{10,0}$ or $c_{8,0}$ respectively, in the notation of [47] - see Table 4.1), or*

4. $\chi(1) \geq q(q^4+q^2+1)(q-1)^3/2$.

We begin by introducing the Weil characters of $Sp_{2n}(q)$.

4.2.1 Weil Characters of $Sp_{2n}(q)$

It is convenient to view $Sp_{2n}(q)$ as a subgroup of both $GL_{2n}(q)$ and $GU_{2n}(q)$. In [27], Guralnick and Tiep describe the linear-Weil characters and unitary-Weil characters, which are irreducible characters of $Sp_{2n}(q)$ for q even and $n \geq 2$ obtained by restriction from $GL_{2n}(q)$ and $GU_{2n}(q)$.

Consider the action of $GL_{2n}(q)$ on the points of its natural module $\mathbb{F}_q^{2n}$. The irreducible constituents of the permutation representation of this action yield the complex irreducible characters known as Weil characters for $GL_{2n}(q)$, denoted τ_n^i for $0 \leq i \leq q-2$. If $n \geq 2$, these restrict irreducibly to $SL_{2n}(q)$, giving the Weil characters of $SL_{2n}(q)$. The τ_n^i for $i \geq 1$ have degree $(q^{2n}-1)/(q-1)$, and τ_n^0 has degree $(q^{2n}-q)/(q-1)$ (see, for example, [27]). More generally, the largest composition factor of the restriction of the corresponding Weil characters to ℓ-regular elements yield the irreducible ℓ-modular Weil characters of $GL_{2n}(q)$.

Now consider the action of $GL_{2n}(q)$ on the set of 1-spaces of its natural module $\mathbb{F}_q^{2n}$, and let P be the stabilizer of a 1-space $\langle v \rangle_{\mathbb{F}_q}$. Then P has a Levi decomposition $P = UL$, where $U = O_p(P)$ and $L \cong GL_{2n-1}(q) \times GL_1(q)$. (Here p is the characteristic of $\mathbb{F}_q$.) Consider a linear character $\alpha \in \mathrm{Irr}(GL_1(q))$. Note that $GL_1(q) \cong C_{q-1}$, and hence there are $q-1$ such linear characters. The character α extends to a linear character of L by taking $1_{GL_{2n-1}(q)} \times \alpha$, and by the identification $L \cong P/U$, we can then inflate this character to a character of P. By an abuse of notation, we will also denote this character of P by α. Inducing to G, we obtain the character $\mathrm{Ind}_P^G(\alpha)$, which has degree $\frac{q^{2n}-1}{q-1}$. If $\alpha \neq 1_P$, then it turns out that this (complex) character is actually irreducible, and the $q-2$ characters obtained in this way are actually the τ_n^i for $1 \leq i \leq q-2$. If, however, $\alpha = 1_P$, then $\mathrm{Ind}_P^G(\alpha) - 1_G$ is irreducible, and this is the degree-$\frac{q^{2n}-q}{q-1}$ character τ_n^0.

If we write τ_n for the permutation representation of $GL_{2n}(q)$ acting on its natural module, then $\tau_n(g) = q^{\dim_{\mathbb{F}_q} \ker(g-1)}$, where the kernel is taken on the natural module.

Table 4.2: Weil Characters of $Sp_{2n}(q)$ [27, Table 1]

Complex Linear Weil Characters	Degree	ℓ-Modular Linear Weil Characters ($\ell \neq 2$)
ρ_n^1	$\frac{(q^n+1)(q^n-q)}{2(q-1)}$	$\widehat{\rho}_n^1 - \begin{cases} 1, & \ell \mid \frac{q^n-1}{q-1}, \\ 0, & \text{otherwise} \end{cases}$
ρ_n^2	$\frac{(q^n-1)(q^n+q)}{2(q-1)}$	$\widehat{\rho}_n^2 - \begin{cases} 1, & \ell \mid (q^n+1), \\ 0, & \text{otherwise} \end{cases}$
τ_n^i, $1 \leq i \leq (q-2)/2$	$\frac{q^{2n}-1}{q-1}$	$\widehat{\tau}_n^i$ $1 \leq i \leq ((q-1)_{\ell'}-1)/2$
Complex Unitary Weil Characters	Degree	ℓ-Modular Unitary Weil Characters ($\ell \neq 2$)
α_n	$\frac{(q^n-1)(q^n-q)}{2(q+1)}$	$\widehat{\alpha}_n$
β_n	$\frac{(q^n+1)(q^n+q)}{2(q+1)}$	$\widehat{\beta}_n - \begin{cases} 1, & \ell \mid (q+1), \\ 0, & \text{otherwise} \end{cases}$
ζ_n^i, $1 \leq i \leq q/2$	$\frac{q^{2n}-1}{q+1}$	$\widehat{\zeta}_n^i$, $1 \leq i \leq ((q+1)_{\ell'}-1)/2$

The Weil characters of $SU_{2n}(q)$ can be obtained in an analogous manner, defining ζ_n to be the character $\zeta_n(g) = (-q)^{\dim_{\mathbb{F}_{q^2}} \ker(g-1)}$, where now the kernel is taken over the natural module $\mathbb{F}_{q^2}^{2n}$ of $SU_{2n}(q)$. This character then decomposes into the sum of characters ζ_n^i for $0 \leq i \leq q$, which we call the Weil characters of $SU_{2n}(q)$.

Guralnick and Tiep [27] show that the restrictions to $Sp_{2n}(q)$ satisfy $\tau_n^i|_{Sp_{2n}(q)} = \tau_n^{q-1-i}|_{Sp_{2n}(q)}$ and $\zeta_n^j|_{Sp_{2n}(q)} = \zeta_n^{q+1-j}|_{Sp_{2n}(q)}$, and these are irreducible for $1 \leq i \leq \frac{q-2}{2}$ and $1 \leq j \leq \frac{q}{2}$. Also, $\tau_n^0|_{Sp_{2n}(q)}$ decomposes into the sum of two irreducible characters ρ_n^1 and ρ_n^2, and similarly $\zeta_n^0|_{Sp_{2n}(q)} = \alpha_n + \beta_n$ for irreducible characters α_n, β_n. Moreover, [27, Theorems 7.5,7.10], yield that the restrictions of these characters to the ℓ-regular elements of $Sp_{2n}(q)$ are either irreducible Brauer characters or the sum of an irreducible Brauer character and the principal character $1_{Sp_{2n}(q)}$. These nontrivial irreducible Brauer characters are called the ℓ-modular linear-Weil characters and ℓ-modular unitary-Weil characters, and are listed in Table 4.2, which is a recreation of [27, Table 1].

The formulae from [27] for calculating the values for the characters τ_n^i and ζ_n^i in

$SL_{2n}(q)$ and $SU_{2n}(q)$, respectively, are

$$\tau_n^i(g) = \frac{1}{q-1}\sum_{j=0}^{q-2} \tilde{\delta}^{ij} q^{\dim_{\mathbb{F}_q} \ker(g-\delta^j)} - 2\delta_{i,0} \tag{4.2.1}$$

and

$$\zeta_n^i(g) = \frac{1}{q+1}\sum_{j=0}^{q} \tilde{\xi}^{ij} (-q)^{\dim_{\mathbb{F}_{q^2}} \ker(g-\xi^j)}. \tag{4.2.2}$$

Here δ and $\tilde{\delta}$ are fixed primitive $(q-1)$th roots of unity in $\mathbb{F}_q$ and $\mathbb{C}$, respectively. Similarly, ξ, $\tilde{\xi}$ are fixed primitive $(q+1)$th roots of unity in $\mathbb{F}_{q^2}$ and $\mathbb{C}$, respectively. The kernels in the formulae are computed on the natural modules $W := (\mathbb{F}_q)^{2n}$ for $SL_{2n}(q)$ or $\tilde{W} := (\mathbb{F}_{q^2})^{2n}$ for $SU_{2n}(q)$.

4.2.2 The Proof of Theorem 1.1.1

We are now ready to prove Theorem 1.1.1. We do this in the form of two separate proofs - one for part (A) and one for part (B).

Proof of Theorem 1.1.1 (A). Suppose that $\chi \in \mathrm{IBr}_\ell(G)$ lies in a unipotent block. The degrees of irreducible Brauer characters lying in unipotent blocks can be extracted from [76], and we have listed these in Section 4.3. Note that the character χ_2 in the notation of [76] is the Weil character ρ_3^2 in the notation of [27]. Similarly, $\chi_3 = \beta_3$, $\chi_4 = \rho_3^1$, and $\chi_5 = \alpha_3$.

We consider the cases ℓ divides $q-1, q+1, q^2-q+1, q^2+q+1$, and q^2+1 separately. Let D_ℓ denote the bound in part A(4) of Theorem 1.1.1 for the prime ℓ.

First, assume that $\ell|(q-1)$ and $\ell \neq 3$. If $\chi(1) \leq D_\ell = \widehat{\chi}_{11}(1)$, then since $q \geq 4$, χ must be $\widehat{\chi}_1 = 1_G, \widehat{\chi}_2, \widehat{\chi}_3, \widehat{\chi}_4, \widehat{\chi}_5, \widehat{\chi}_6$, or $\widehat{\chi}_7$. Hence we are in situation A(1), A(2), or A(3).

Now let $\ell|(q^2+q+1)$. Note that we are including the case $\ell = 3|(q-1)$. In either case, if $\chi(1) \leq D_\ell = \widehat{\chi}_{11}(1)$, then χ is $1_G, \widehat{\chi}_2, \widehat{\chi}_3, \widehat{\chi}_4 - 1_G, \widehat{\chi}_5, \widehat{\chi}_6$, or $\widehat{\chi}_7$, as $q \geq 4$. Again, we therefore have situation A(1), A(2), or A(3).

If $\ell|(q^2+1)$, then again $D_\ell = \widehat{\chi}_{11}(1)$. A character in a unipotent block has degree smaller than this bound if and only if it is $1_G, \widehat{\chi}_2, \widehat{\chi}_3, \widehat{\chi}_4, \widehat{\chi}_5, \widehat{\chi}_6 - 1_G$, or $\widehat{\chi}_7 - \widehat{\chi}_4$, which gives us situation A(1), A(2), or A(3) in this case.

Now let $\ell|(q^2-q+1)$ with $\ell \neq 3$. Then $D_\ell = \widehat{\chi}_{11}(1) - \widehat{\chi}_5(1)$, and $\chi(1) < D_\ell$ if and only if χ is $1_G, \widehat{\chi}_2 - 1_G, \widehat{\chi}_3, \widehat{\chi}_4, \widehat{\chi}_5, \widehat{\chi}_6$ or $\widehat{\chi}_7$, so we have situation A(1), A(2), or A(3) for this choice of ℓ.

Finally, suppose $\ell|(q+1)$. In this case, $D_\ell = \varphi_7(1)$. Note that from [76], the parameter α which occurs in the description for this Brauer character (see Section 4.3) is 1 if $(q+1)_\ell = 3$ and 2 otherwise. Also, note that in this case, D. White [76] has left 3 unknowns in the decomposition matrix for the principal block. Namely, the unknown β_1 is either 0 or 1 and the unknowns β_2, β_3 satisfy

$$1 \leq \beta_2 \leq (q+2)/2, \quad 1 \leq \beta_3 \leq q/2.$$

Now, using these bounds for β_2 and β_3, we may find a lower bound for $\varphi_{10}(1)$ as follows:

$$\begin{aligned}
\varphi_{10}(1) &= \chi_{30}(1) - \beta_3(\chi_{11}(1) - \chi_5(1)) - (\beta_2 - 1)\chi_{23}(1) - \chi_{28}(1) \\
&= \phi_1^2\phi_3(q^3\phi_4 - \beta_3 q^4/2 + \beta_3 q/2 - \phi_4 - (\beta_2 - 1)q\phi_1\phi_6/2) \\
&\geq \phi_1^2\phi_3(q^3\phi_4 - (q/2)q^4/2 + q/2 - \phi_4 - (q/2)q\phi_1\phi_6/2) \\
&= \phi_1^2\phi_3(q^3\phi_4 - q^5/4 + q/2 - \phi_4 - q^2\phi_1\phi_6/4).
\end{aligned}$$

Here ϕ_j represents the jth cyclotomic polynomial. As this bound is larger than D_ℓ for $q \geq 4$, and the other Brauer characters are known, with the possible exception of $\varphi_2 = \widehat{\chi}_2 - \beta_1 \cdot 1_G$, we see that the only irreducible Brauer characters in a unipotent block with degree less than D_ℓ are $1_G, \widehat{\chi}_2 - \beta_1 \cdot 1_G, \widehat{\chi}_3 - 1_G, \widehat{\chi}_4, \widehat{\chi}_5, \widehat{\chi}_6 - \widehat{\chi}_3 - \widehat{\chi}_2 + 1_G = \widehat{\chi}_{28}$, and $\widehat{\chi}_7 - \widehat{\chi}_6 + \widehat{\chi}_3 - 1_G = \widehat{\chi}_{35} - \widehat{\chi}_5$.

Now, recall that when $\ell|(q^3+1)$, [27, Table 1] gives us that $\widehat{\rho}_3^2 - 1_G$ is an irreducible Brauer character. Since $(q+1)|(q^3+1)$ and $\widehat{\rho}_3^2 = \widehat{\chi}_2$, this implies that in fact the unknown β_1 must be 1.

Hence, we see that we are in one of the situations A(1), A(2), or A(3), and the proof is complete for χ in a unipotent block.

$\square$

Proof of Theorem 1.1.1(B). As χ does not lie in a unipotent block, we have $\chi \in \mathcal{E}_\ell(G,(s))$ for some semisimple ℓ'-element $s \neq 1$. Let B denote the bound $q(q^4+q^2+1)(q-1)^3/2$ in part B(4) of Theorem 1.1.1. Since $(q-1)^2(q^2+1)(q^4+q^2+1) > B$ when $q \geq 4$, it follows from Lemma 4.1.3 and Proposition 4.1.2 that either $\chi(1) > B$ or $\chi \in \mathcal{E}_\ell(G,(s))$ where s lies in one of the classes $c_{3,0}, c_{4,0}, c_{5,0}, c_{6,0}, c_{8,0}$, or $c_{10,0}$ of $G^* = Sp_6(q)$. (Note that we are making the identification $G \cong SO_7(q)$ so that $G^* = Sp_6(q)$ here.) From Table 4.1, we see that in each of these cases, $C_{G^*}(s) = L^*$ is a direct product of groups of the form $Sp_2(q), Sp_4(q), GU_i(q)$, or $GL_i(q)$ for $1 \leq i \leq 3$, and hence is self-dual. That is, $L \cong L^*$ in the notation of Lemma 4.1.1. We will make this identification and consider characters of $C_{G^*}(s)$ as characters of L.

If $s \in c_{3,0}$ or $c_{4,0}$, then $C_{G^*}(s) \cong C \times Sp_4(q)$, where C is a cyclic group of order $q-1$ or $q+1$, respectively. In this case, since $\mathfrak{d}_\ell(Sp_4(q)) = (q-1)(q^2-q)/2$ (see [41]), we have $\chi(1) \geq (q^6-1)(q-1)(q^2-q)/(2(q+1)) = B$ by Proposition 4.1.2, unless χ corresponds to $1_{C_{G^*}(s)}$ in $\mathrm{IBr}_\ell(C_{G^*}(s))$. In the latter case, we are in situation B(1), as χ is one of the characters $\widehat{\tau}_3^i$ or $\widehat{\zeta}_3^j$.

For s in one of the families of classes $c_{5,0}$ or $c_{6,0}$, we have $C_{G^*}(s) \cong GL_3(q)$ or $GU_3(q)$, respectively. Now, nonprincipal characters found in a unipotent ℓ-block of $GL_3(q)$ have degree at least q^2+q-1 (see [35]). Moreover, $\mathfrak{d}_\ell(GU_3(q))$ is at least q^2-q (see, for example, [73]). Hence in either case, for $\chi \in \mathcal{E}_\ell(G,(s))$, we know by Proposition 4.1.2 that either $\chi(1) \geq (q^2+1)(q-1)^2(q^2+q+1)(q^2-q) > B$ or χ corresponds to $1_{C_{G^*}(s)}$ in $\mathrm{IBr}_\ell(C_{G^*}(s))$. In the second case, we have situation B(2).

Next, suppose that $\chi \in \mathcal{E}_\ell(G,s)$ with $s \in c_{8,0}$ or $c_{10,0}$. Here we have $C_{G^*}(s) \cong GL_2(q) \times SL_2(q)$ or $GU_2(q) \times SL_2(q)$, respectively. The smallest possible nontrivial character degree in a unipotent block is therefore at least $q-1$. Since $(q-1)[G^* :$

$C_{G^*}(s)]_{2'} > B$ in either case, we know by Proposition 4.1.2 that either $\chi(1) \geq B$ or situation B(3) holds, and the proof is complete.

□

4.3 Unipotent Blocks of $Sp_6(2^a)$

In this section, we use [76] to describe the Brauer characters of $G = Sp_6(q)$, q even, lying in unipotent blocks, in terms of the restrictions of ordinary characters to ℓ-regular elements.

First let $\ell|(q-1)$. In this case, there are two unipotent blocks, the principal block b_0 and a cyclic block b_1 (in the notation of [76]). Using [76], we see that the irreducible Brauer characters of $G = Sp_6(q)$ can be written as in Table 4.3 if $\ell \neq 3$ and Table 4.4 if $\ell = 3$, where $\widehat{\chi}$ is the restriction of the character $\chi \in \mathrm{Irr}(G)$ to the ℓ-regular elements G° of G.

Now let $\ell|(q+1)$. In this case, from [76], there are again two unipotent blocks: the principal block b_0 and a cyclic block b_1. Using the decomposition numbers found in [76], we see that the irreducible Brauer characters are as shown in Table 4.5. We use the notation ϕ_i for the ith cyclotomic polynomial. Also, $\alpha = 2$ if $(q+1)_\ell \neq 3$ and is 1 if $(q+1)_\ell = 3$. In this case, D. White has left three unknowns β_i for $1 \leq i \leq 3$, which satisfy $1 \leq \beta_2 \leq q/2+1$, and $1 \leq \beta_3 \leq q/2$ (see [76]). Moreover, from [76], the unknown β_1 is either 0 or 1. However, as discussed in the proof of Theorem 1.1.1 in Section 4.2 above, the results of [27] yield that in fact $\beta_1 = 1$.

In the case $\ell|(q^2-q+1)$, where $\ell \neq 3$, there is only one unipotent block of nonzero defect, namely the principal block b_0. Inspection of the decomposition matrices in [76] yields the list of irreducible ℓ-Brauer characters of G in this block to be as in Table 4.6.

Now, suppose $\ell|(q^2+q+1)$, where $\ell \neq 3$. Then there is again only one unipotent block of positive defect, namely the principal block b_0, and from inspection of the decomposition matrices found in [76], we find that the irreducible ℓ-Brauer characters of G in this block can be written as in Table 4.7.

Finally, let $\ell|(q^2+1)$. In this case, all blocks are cyclic and there are two unipotent blocks of positive defect: the principal block, b_0, and the block b_1, and inspection of the decomposition matrices found in [76] yields that the irreducible ℓ-Brauer characters of G in the unipotent blocks can be written as in Table 4.8.

Table 4.3: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell|(q-1)$, $\ell \neq 3$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\widehat{\chi}_1$	1
$\widehat{\chi}_2$	$\frac{1}{2}q(q^2+q+1)(q^2+1)$
$\widehat{\chi}_3$	$\frac{1}{2}q(q^2-q+1)(q^2+1)$
$\widehat{\chi}_4$	$\frac{1}{2}q(q^2-q+1)(q+1)^2$
$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\widehat{\chi}_8$	$\frac{1}{2}q^4(q^2+q+1)(q^2+1)$
$\widehat{\chi}_9$	$\frac{1}{2}q^4(q^2-q+1)(q+1)^2$
$\widehat{\chi}_{10}$	$\frac{1}{2}q^4(q^2-q+1)(q^2+1)$
$\widehat{\chi}_{12}$	q^9

(b) Block b_1

$\varphi \in \mathrm{IBr}(G) \cap b_1$	Degree, $\varphi(1)$
$\widehat{\chi}_5$	$\frac{1}{2}q(q-1)^2(q^2+q+1)$
$\widehat{\chi}_{11}$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$

Table 4.4: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell = 3|(q-1)$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\widehat{\chi}_1$	1
$\widehat{\chi}_2$	$\frac{1}{2}q(q^2+q+1)(q^2+1)$
$\widehat{\chi}_3$	$\frac{1}{2}q(q^2-q+1)(q^2+1)$
$\widehat{\chi}_4 - \widehat{\chi}_1$	$\frac{1}{2}q(q^2-q+1)(q+1)^2-1$
$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\widehat{\chi}_8$	$\frac{1}{2}q^4(q^2+q+1)(q^2+1)$
$\widehat{\chi}_9 - \widehat{\chi}_3$	$\frac{1}{2}q^4(q^2-q+1)(q+1)^2 - \frac{1}{2}q(q^2-q+1)(q^2+1)$
$\widehat{\chi}_{10} - \widehat{\chi}_4 + \widehat{\chi}_1$	$\frac{1}{2}q^4(q^2-q+1)(q^2+1) - \frac{1}{2}q(q^2-q+1)(q+1)^2+1$
$\widehat{\chi}_{12} - \widehat{\chi}_9 + \widehat{\chi}_3$	$q^9 - \frac{1}{2}q^4(q^2-q+1)(q+1)^2 + \frac{1}{2}q(q^2-q+1)(q^2+1)$

(b) Block b_1

$\varphi \in \mathrm{IBr}(G) \cap b_1$	Degree, $\varphi(1)$
$\widehat{\chi}_5$	$\frac{1}{2}q(q-1)^2(q^2+q+1)$
$\widehat{\chi}_{11}$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$

Table 4.5: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell|(q+1)$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\varphi_1 = \widehat{\chi}_1$	1
$\varphi_2 = \widehat{\chi}_2 - \beta_1\widehat{\chi}_1$	$\frac{1}{2}q(q^2+q+1)(q^2+1)-\beta_1$
$\varphi_3 = \widehat{\chi}_3 - \widehat{\chi}_1$	$\frac{1}{2}q(q^2-q+1)(q^2+1)-1$
$\varphi_4 = \widehat{\chi}_5$	$\frac{1}{2}q(q^2+q+1)(q-1)^2$
$\varphi_5 = \widehat{\chi}_{28}$ $= \widehat{\chi}_6 - \widehat{\chi}_3 - \widehat{\chi}_2 + \widehat{\chi}_1$	$(q^2+q+1)(q-1)^2(q^2+1)$
$\varphi_6 = \widehat{\chi}_{35} - \widehat{\chi}_5$ $= \widehat{\chi}_7 - \widehat{\chi}_6 + \widehat{\chi}_3 - \widehat{\chi}_1$	$\phi_1\phi_3(\phi_4\phi_6 - \frac{1}{2}q\phi_1)$
$\varphi_7 = \widehat{\chi}_{22} - (\alpha-1)\widehat{\chi}_5 - \widehat{\chi}_3 + \widehat{\chi}_1$ $= \widehat{\chi}_8 - \widehat{\chi}_7 - \alpha\widehat{\chi}_5 - \widehat{\chi}_3 + \widehat{\chi}_1$	$\frac{1}{2}q\phi_1\phi_3\phi_4\phi_6 - \frac{\alpha-1}{2}q\phi_1^2\phi_3 - \frac{1}{2}q\phi_4\phi_6 + 1$
$\varphi_8 = \widehat{\chi}_{23}$ $= \widehat{\chi}_{10} - \widehat{\chi}_7 + \widehat{\chi}_6 - \widehat{\chi}_3$	$\frac{1}{2}q\phi_1^3\phi_3\phi_6$
$\varphi_9 = \widehat{\chi}_{11} - \widehat{\chi}_5$	$\frac{1}{2}q\phi_1^3\phi_3^2$
$\varphi_{10} = \widehat{\chi}_{30} - \beta_3(\widehat{\chi}_{11} - \widehat{\chi}_5) - (\beta_2-1)\widehat{\chi}_{23} - \widehat{\chi}_{28}$	$\phi_1^2\phi_3(q^3\phi_4 - \frac{\beta_3}{2}q^4 + \frac{\beta_3}{2}q - \phi_4 - \frac{\beta_2-1}{2}q\phi_1\phi_6)$

(b) Block b_1

$\varphi \in \mathrm{IBr}(G) \cap b_1$	Degree, $\varphi(1)$
$\widehat{\chi}_4$	$\frac{1}{2}q(q+1)^2(q^2-q+1)$
$\widehat{\chi}_9 - \widehat{\chi}_4$	$\frac{1}{2}q(q+1)^2(q^2-q+1)(q^3-1)$

Table 4.6: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell|(q^2-q+1)$, $\ell \neq 3$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\widehat{\chi}_1$	1
$\widehat{\chi}_2 - \widehat{\chi}_1$	$\frac{1}{2}q(q^2+q+1)(q^2+1)-1$
$\widehat{\chi}_8 - \widehat{\chi}_2 + \widehat{\chi}_1$	$\frac{1}{2}q^4(q^2+q+1)(q^2+1) - \frac{1}{2}q(q^2+q+1)(q^2+1)+1$
$\widehat{\chi}_{12} - \widehat{\chi}_8 + \widehat{\chi}_2 - \widehat{\chi}_1$	$q^9 - \frac{1}{2}q^4(q^2+q+1)(q^2+1) + \frac{1}{2}q(q^2+q+1)(q^2+1) - 1$
$\widehat{\chi}_5$	$\frac{1}{2}q(q-1)^2(q^2+q+1)$
$\widehat{\chi}_{11} - \widehat{\chi}_5$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1) - \frac{1}{2}q(q-1)^2(q^2+q+1)$

(b) Blocks of Defect 0

$\varphi \in \mathrm{IBr}(G)$	Degree, $\varphi(1)$
$\widehat{\chi}_3$	$\frac{1}{2}q(q^2-q+1)(q^2+1)$
$\widehat{\chi}_4$	$\frac{1}{2}q(q^2-q+1)(q+1)^2$
$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\widehat{\chi}_9$	$\frac{1}{2}q^4(q^2-q+1)(q+1)^2$
$\widehat{\chi}_{10}$	$\frac{1}{2}q^4(q^2-q+1)(q^2+1)$

Table 4.7: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell|(q^2+q+1)$, $\ell \neq 3$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\widehat{\chi}_1$	1
$\widehat{\chi}_4 - \widehat{\chi}_1$	$\frac{1}{2}q(q+1)^2(q^2-q+1)-1$
$\widehat{\chi}_{10} - \widehat{\chi}_4 + \widehat{\chi}_1$	$\frac{1}{2}q^4(q^2+1)(q^2-q+1) - \frac{1}{2}q(q+1)^2(q^2-q+1)+1$
$\widehat{\chi}_3$	$\frac{1}{2}q(q^2+1)(q^2-q+1)$
$\widehat{\chi}_9 - \widehat{\chi}_3$	$\frac{1}{2}q^4(q+1)^2(q^2-q+1) - \frac{1}{2}q(q^2+1)(q^2-q+1)$
$\widehat{\chi}_{12} - \widehat{\chi}_9 + \widehat{\chi}_3$	$q^9 - \frac{1}{2}q^4(q+1)^2(q^2-q+1) + \frac{1}{2}q(q^2+1)(q^2-q+1)$

(b) Blocks of Defect 0

$\varphi \in \mathrm{IBr}(G)$	Degree, $\varphi(1)$
$\widehat{\chi}_2$	$\frac{1}{2}q(q^2+q+1)(q^2+1)$
$\widehat{\chi}_5$	$\frac{1}{2}q(q-1)^2(q^2+q+1)$
$\widehat{\chi}_6$	$q^2(q^4+q^2+1)$
$\widehat{\chi}_7$	$q^3(q^4+q^2+1)$
$\widehat{\chi}_8$	$\frac{1}{2}q^4(q^2+q+1)(q^2+1)$
$\widehat{\chi}_{11}$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$

Table 4.8: $\ell-$Brauer Characters in Unipotent Blocks of $G = Sp_6(q)$, $\ell|(q^2+1)$

(a) Principal Block b_0

$\varphi \in \mathrm{IBr}(G) \cap b_0$	Degree, $\varphi(1)$
$\widehat{\chi}_1$	1
$\widehat{\chi}_6 - \widehat{\chi}_1$	$q^2(q^4+q^2+1)-1$
$\widehat{\chi}_9 - \widehat{\chi}_6 + \widehat{\chi}_1$	$\frac{1}{2}q^4(q+1)^2(q^2-q+1) - q^2(q^4+q^2+1)+1$
$\widehat{\chi}_{11}$	$\frac{1}{2}q^4(q-1)^2(q^2+q+1)$

(b) Block b_1

$\varphi \in \mathrm{IBr}(G) \cap b_1$	Degree, $\varphi(1)$
$\widehat{\chi}_4$	$\frac{1}{2}q(q+1)^2(q^2-q+1)$
$\widehat{\chi}_7 - \widehat{\chi}_4$	$q^3(q^4+q^2+1) - \frac{1}{2}q(q+1)^2(q^2-q+1)$
$\widehat{\chi}_{12} - \widehat{\chi}_7 + \widehat{\chi}_4$	$q^9 - q^3(q^4+q^2+1) + \frac{1}{2}q(q+1)^2(q^2-q+1)$
$\widehat{\chi}_5$	$\frac{1}{2}q(q-1)^2(q^2+q+1)$

(c) Blocks of Defect 0

$\varphi \in \mathrm{IBr}(G)$	Degree, $\varphi(1)$
$\widehat{\chi}_2$	$\frac{1}{2}q(q^2+q+1)(q^2+1)$
$\widehat{\chi}_3$	$\frac{1}{2}q(q^2+1)(q^2-q+1)$
$\widehat{\chi}_8$	$\frac{1}{2}q^4(q^2+q+1)(q^2+1)$
$\widehat{\chi}_{10}$	$\frac{1}{2}q^4(q^2+1)(q^2-q+1)$

4.4 Non-Unipotent Blocks of $Sp_6(2^a)$

4.4.1 Non-Unipotent Block Distributions for $\mathrm{Irr}(Sp_6(2^a))$

We list here the block distribution, in cross-characteristic, for irreducible characters of positive defect lying in non-unipotent blocks of $G = Sp_6(q)$ with q even. Any characters not listed either have defect zero or lie in a unipotent block, whose distribution can be found in [76]. In this section, the notation for characters of G is taken from CHEVIE [26]. The indexing sets are as follows:

For $\epsilon \in \{\pm 1\}$, let $I^0_{q-\epsilon}$ be the set $\{i \in \mathbb{Z} : 1 \leq i \leq q - \epsilon - 1\}$, and let $I_{q-\epsilon}$ be a set of class representatives on $I^0_{q-\epsilon}$ under the equivalence relation $i \sim j \iff i \equiv \pm j \mod (q - \epsilon)$. Let $I^0_{q^2+1} := \{i \in \mathbb{Z} : 1 \leq i \leq q^2\}$ and $I^0_{q^2-1} := \{i \in \mathbb{Z} : 1 \leq i \leq q^2 - 1, (q-1) \not| i, (q+1) \not| i\}$, and let $I_{q^2-\epsilon}$ be a set of representatives for the equivalence relation on $I^0_{q^2-\epsilon}$ given by $i \sim j \iff i \equiv \pm j$ or $\pm qj \mod (q^2 - \epsilon)$. Similarly, let $I^0_{q^3-\epsilon} := \{i \in \mathbb{Z} : 1 \leq i \leq q^3 - \epsilon; (q^2 + \epsilon q + 1) \not| i\}$ and $I_{q^3-\epsilon}$ a set of representatives for the equivalence relation on $I^0_{q^3-\epsilon}$ given by $i \sim j \iff i \equiv \pm j, \pm qj$, or $\pm q^2 j \mod (q^3 - \epsilon)$. Given one of these indexing sets, I_*, we write I^k_* for the elements $(i_1, ..., i_k)$ of $I_* \times I_* ... \times I_*$ (k copies) with none of $i_1, i_2, ..., i_k$ the same and I^{k*}_* for the set of equivalence classes of I^k_* under $(i_1, ..., i_k) \sim (\rho(i_1), ..., \rho(i_k))$ for all $\rho \in S_k$.

We will denote by $B_i(J)$ the ℓ-blocks in $\mathcal{E}_\ell(G,(s))$ of positive defect, where s is conjugate in G^* to the semisimple element $g_i(J)$ in the notation of [47]. (Here J denotes the proper indices. For example, for the family g_6, $J = (i)$ for $i \in I_{q-1}$, and for the family g_{32}, $J = (i,j,k)$ where $(i,j,k) \in I^{3*}_{q+1}$.) In most cases, $C_{G^*}(s)$ has only one unipotent block, and therefore $\mathcal{E}_\ell(G,(s))$ is a single block. However, when multiple such blocks exist, which occurs for $i = 6,7,8,9$ when $\ell | (q^2 - 1)$, we will denote by $B_i(J)^{(0)}$ the the block corresponding in the Bonnafé-Rouquier correspondence (see Lemma 4.1.1) to the principal block of $C_{G^*}(s)$ and by $B_i(J)^{(1)}$ the block corresponding to the unique other block of positive defect. (Indeed, in such cases there are only two blocks of positive defect in $\mathcal{E}_\ell(G,(g_i(J)))$.)

The block distributions listed in this section follow from the theory of central characters (the central characters of G can be obtained from CHEVIE [26]) together with the definition of $\mathcal{E}_\ell(G,(g_i(J)))$ and Broué-Michel's result [12] that this is a union of ℓ-blocks.

4.4.2 $\ell|(q^2+1)$

Let $k \in I_{q-1}, t \in I_{q+1}, s \in I_{q^2+1}$ with $(q^2+1)_\ell|s$ and write $m := (q^2+1)_{\ell'}$.

$$B_6(k) = \{\chi_{13}(k), \chi_{14}(k), \chi_{17}(k), \chi_{18}(k), \chi_{62}(k,r) \colon m|r\}$$

$$B_7(t) = \{\chi_{19}(t), \chi_{20}(t), \chi_{23}(t), \chi_{24}(t), \chi_{65}(r,t) \colon m|r\}$$

$$B_{24}(s)^{(0)} = \{\chi_{55}(r) \colon r \equiv \pm s \text{ or } \pm qs \mod m\}$$

$$B_{24}(s)^{(1)} = \{\chi_{56}(r) \colon r \equiv \pm s \text{ or } \pm qs \mod m\}$$

$$B_{30}(k,s) = \{\chi_{62}(k,r) \colon r \equiv \pm s \text{ or } \pm qs \mod m\}$$

$$B_{33}(s,t) = \{\chi_{65}(r,t) \colon r \equiv \pm s \text{ or } \pm qs \mod m\}$$

4.4.3 $3 \neq \ell|(q^2+q+1)$

In the following, let $k \in I_{q-1}$ and $v \in I_{q^3-1}$ with $(q^3-1)_\ell|v$ and write $n := (q^2+q+1)_{\ell'}$.

$$B_8(k) = \{\chi_{25}(k), \chi_{26}(k), \chi_{27}(k), \chi_{63}(r) \colon r \equiv \pm k(q^2+q+1) \mod (q-1)n\}$$

$$B_{31}(v) = \{\chi_{63}(r)\colon r \equiv \pm v, \pm qv, \text{ or } \pm q^2 v \mod (q-1)n\}$$

4.4.4 $3 \neq \ell | (q^2 - q + 1)$

In the following, let $t \in I_{q+1}$ and $w \in I_{q^3+1}$ with $(q^3+1)_\ell | w$ and write $n := (q^2-q+1)_{\ell'}$.

$$B_9(t) = \{\chi_{28}(t), \chi_{29}(t), \chi_{30}(t), \chi_{66}(r)\colon r \equiv \pm t(q^2 - q + 1) \mod (q+1)n\}$$

$$B_{34}(w) = \{\chi_{66}(r)\colon r \equiv \pm w, \pm qw, \text{ or } \pm q^2 w \mod (q+1)n\}$$

4.4.5 $\ell | (q - 1)$

In the following, let $k_1, k_2, k_3 \in I_{q-1}$ with $\ell^d | k_i$ and none of k_1, k_2, k_3 the same. Let $t_1, t_2, t_3 \in I_{q+1}$ with none of t_1, t_2, t_3 the same, $u \in I_{q^2+1}$, and $s \in I_{q^2-1}$ with $\ell^d | s$, where $\ell^d := (q-1)_\ell$. Let $v \in I_{q^3-1}$ and $w \in I_{q^3+1}$ with $(q^3-1)_\ell | v$. Moreover, let $m := (q-1)_{\ell'}$. When $\ell = 3$, let $n := (q^2 + q + 1)_{3'}$.

$$\begin{aligned} B_6(k_1)^{(0)} = \{&\chi_{13}(r), \chi_{14}(r), \chi_{15}(r), \chi_{16}(r), \chi_{18}(r), \chi_{39}(j,r), \chi_{40}(j,r), \\ &\chi_{41}(j,r), \chi_{42}(j,r), \chi_{57}(r,j,i)\colon r \equiv \pm k_1 \mod m, m|j, m|i\} \end{aligned}$$

$$B_6(k_1)^{(1)} = \{\chi_{17}(r)\colon r \equiv \pm k_1 \mod m\}$$

$$\begin{aligned} B_7(t_1) = \{&\chi_{19}(t_1), \chi_{20}(t_1), \chi_{21}(t_1), \chi_{22}(t_1), \chi_{24}(t_1), \chi_{43}(r,t_1), \chi_{44}(r,t_1), \\ &\chi_{47}(r,t_1), \chi_{48}(r,t_1), \chi_{58}(r,j,t_1)\colon m|r, m|j\} \end{aligned}$$

(Note: $\mathcal{E}_\ell(G, g_7(t_1))$ also contains the defect-zero block $\{\chi_{23}(t_1)\}$.)

$$B_8(k_1) = \begin{cases} \{\chi_{25}(r_1), \chi_{26}(r_1), \chi_{27}(r_1), \chi_{39}(r_1, r_2), \chi_{40}(r_1, r_2), & \text{if } \ell \neq 3 \\ \quad \chi_{57}(r_1, r_2, r_3) \colon r_1, r_2, r_3 \equiv \pm k_1 \mod m\} & \\ & \\ \{\chi_{25}(r_1), \chi_{26}(r_1), \chi_{27}(r_1), \chi_{39}(r_1, r_2), \chi_{40}(r_1, r_2), & \\ \quad \chi_{57}(r_1, r_2, r_3), \chi_{63}(r_4) \colon r_1, r_2, r_3 \equiv \pm k_1 \mod m, & \text{if } \ell = 3 \\ \qquad r_4 \equiv \pm k_1(q^2 + q + 1) \mod mn\} & \end{cases}$$

$$B_9(t_1) = \{\chi_{28}(t_1), \chi_{30}(t_1), \chi_{61}(r, t_1) \colon r \equiv \pm(q-1)t_1 \mod m(q+1)\}$$

(Note: $\mathcal{E}_\ell(G, g_9(t_1))$ also contains the defect-zero block $\{\chi_{29}(t_1)\}$.)

$$B_{11}(k_1) = \{\chi_{31}(r_1), \chi_{32}(r_1), \chi_{33}(r_1), \chi_{34}(r_1), \chi_{41}(r_1, r_2), \chi_{42}(r_1, r_2), \chi_{39}(r_1, j),$$
$$\chi_{40}(r_1, j), \chi_{57}(r_1, r_2, j) \colon r_1 \equiv \pm k_1 \mod m, r_2 \equiv \pm k_1 \mod m, m|j\}$$

$$B_{13}(t_1) = \{\chi_{35}(t_1), \chi_{36}(t_1), \chi_{37}(t_1), \chi_{38}(t_1), \chi_{49}(t_1, r), \chi_{50}(t_1, r), \chi_{45}(j), \chi_{46}(j),$$
$$\chi_{59}(j, r) \colon j \equiv \pm(q-1)t_1 \mod m(q+1), m|r\}$$

$$B_{16}(k_1, k_2) = \{\chi_{39}(r_1, r_2), \chi_{40}(r_1, r_2), \chi_{57}(r_1, j, r_2) \colon r_i \equiv \pm k_i \mod m,$$
$$j \equiv \pm k_1 \mod m\}$$

$$B_{17}(k_1, k_2) = \{\chi_{41}(r_1, r_2), \chi_{42}(r_1, r_2), \chi_{57}(r_1, r_2, j) \colon r_i \equiv \pm k_i \mod m, m|j\}$$

$$B_{18}(k_1, t_1) = \{\chi_{43}(r, t_1), \chi_{44}(r, t_1), \chi_{58}(r, j, t_1) \colon r \equiv \pm k_1 \mod m,$$
$$j \equiv \pm k_1 \mod m\}$$

$$B_{19}(s) = \{\chi_{45}(r), \chi_{46}(r), \chi_{59}(r,j) \colon r \equiv \pm s \text{ or } \pm qs \mod m(q+1), m|j\}$$

$$B_{20}(k_1,t_1) = \{\chi_{47}(r,t_1), \chi_{48}(r,t_1), \chi_{58}(r,j,t_1) \colon r \equiv \pm k_1 \mod m, m|j\}$$

$$\begin{aligned} B_{21}(t_1,k_1) = \{\chi_{49}(t_1,r), \chi_{50}(t_1,r), \chi_{59}(j,r) \colon j \equiv \pm(q-1)t_1 \mod m(q+1), \\ r \equiv \pm k_1 \mod m\} \end{aligned}$$

$$B_{22}(t_1,t_2) = \{\chi_{51}(t_1,t_2), \chi_{52}(t_1,t_2), \chi_{61}(r,t_2) \colon r \equiv \pm(q-1)t_1 \mod m(q+1)\}$$

$$B_{23}(t_1,t_2) = \{\chi_{53}(t_1,t_2), \chi_{54}(t_1,t_2), \chi_{60}(r,t_1,t_2) \colon m|r\}$$

$$B_{24}(u) = \{\chi_{55}(u), \chi_{56}(u), \chi_{62}(r,u) \colon r \equiv 0 \mod m\}$$

$$B_{25}(k_1,k_2,k_3) = \{\chi_{57}(r_1,r_2,r_3) \colon r_i \equiv \pm k_i \mod m\}$$

$$B_{26}(k_1,k_2,t_1) = \{\chi_{58}(r_1,r_2,t_1) \colon r_i \equiv \pm k_i \mod m\}$$

$$B_{27}(s,k_1) = \{\chi_{59}(r,j) \colon r \equiv \pm s \text{ or } \pm qs \mod m(q+1), j \equiv \pm k_1 \mod m\}$$

$$B_{28}(k_1,t_1,t_2) = \{\chi_{60}(r,t_1,t_2) \colon r \equiv \pm k_1 \mod m\}$$

$$B_{29}(s,t_1) = \{\chi_{61}(r,t_1)\colon r \equiv \pm s \text{ or } \pm qs \mod m(q+1)\}$$

$$B_{30}(k_1,u) = \{\chi_{62}(r,u)\colon r \equiv \pm k_1 \mod m\}$$

$$B_{31}(v) = \begin{cases} \{\chi_{63}(r)\colon r \equiv \pm v, \pm qv, \text{ or } \pm q^2 v \mod m(q^2+q+1)\} & \text{if } \ell \neq 3 \\ \{\chi_{63}(r)\colon r \equiv \pm v, \pm qv, \text{ or } \pm q^2 v \mod mn\} & \text{if } \ell = 3 \end{cases}$$

$$B_{32}(t_1,t_2,t_3) = \{\chi_{64}(t_1,t_2,t_3)\} \quad \text{(defect zero)}$$

$$B_{33}(u,t_1) = \{\chi_{65}(u,t_1)\} \quad \text{(defect zero)}$$

$$B_{34}(w) = \{\chi_{66}(w)\} \quad \text{(defect zero)}$$

4.4.6 $\ell|(q+1)$

In the following, let $k_1, k_2, k_3 \in I_{q-1}$ with none of k_1, k_2, k_3 the same. Let $t_1, t_2, t_3 \in I_{q+1}$ with $\ell^d|t_i$ and none of t_1, t_2, t_3 the same, $u \in I_{q^2+1}$, and $s \in I_{q^2-1}$ with $\ell^d|s$, where $\ell^d := (q+1)_\ell$. Let $v \in I_{q^3-1}$ and $w \in I_{q^3+1}$ with $(q^3+1)_\ell|w$. Let $m := (q+1)_{\ell'}$. When $\ell = 3$, write $n := (q^2-q+1)_{3'}$.

$$B_6(k_1) = \{\chi_{13}(k_1), \chi_{15}(k_1), \chi_{16}(k_1), \chi_{17}(k_1), \chi_{18}(k_1), \chi_{47}(k_1,r), \chi_{48}(k_1,r), \\ \chi_{49}(r,k_1), \chi_{50}(r,k_1), \chi_{60}(k_1,r,j)\colon m|r, m|j\}$$

(Note: $\mathcal{E}_\ell(G, g_6(k_1))$ also contains the defect-zero block $\{\chi_{14}(k_1)\}$.)

$$B_7(t_1)^{(0)} = \{\chi_{19}(r), \chi_{21}(r), \chi_{22}(r), \chi_{23}(r), \chi_{24}(r), \chi_{51}(j,r), \chi_{52}(j,r), \\ \chi_{53}(j,r), \chi_{54}(j,r), \chi_{64}(r,j,i)\colon r \equiv \pm t_1 \mod m, m|j, m|i\}$$

$$B_7(t_1)^{(1)} = \{\chi_{20}(r) \colon r \equiv \pm t_1 \mod m\}$$

$$B_8(k_1) = \{\chi_{25}(k_1), \chi_{27}(k_1), \chi_{59}(r, k_1) \colon r \equiv \pm(q+1)k_1 \mod m(q-1)\}$$

(Note: $\mathcal{E}_\ell(G, g_8(k_1))$ also contains the defect-zero block $\{\chi_{26}(k_1)\}$.)

$$B_9(t_1) = \begin{cases} \{\chi_{28}(r_1), \chi_{29}(r_1), \chi_{30}(r_1), \chi_{51}(r_1, r_2), \chi_{52}(r_1, r_2), \\ \chi_{64}(r_1, r_2, r_3) \colon r_1, r_2, r_3 \equiv \pm t_1 \mod m\} & \text{if } \ell \neq 3, \\ \\ \{\chi_{28}(r_1), \chi_{29}(r_1), \chi_{30}(r_1), \chi_{51}(r_1, r_2), \chi_{52}(r_1, r_2), \\ \chi_{64}(r_1, r_2, r_3), \chi_{66}(r_4) \colon r_1, r_2, r_3 \equiv \pm t_1 \mod m, & \text{if } \ell = 3, \\ \qquad r_4 \equiv \pm t_1(q^2 - q + 1) \mod mn\} \end{cases}$$

$$\begin{aligned} B_{11}(k_1) = \{&\chi_{31}(k_1), \chi_{32}(k_1), \chi_{33}(k_1), \chi_{34}(k_1), \chi_{43}(k_1, r), \chi_{44}(k_1, r), \\ &\chi_{45}(j), \chi_{46}(j), \chi_{61}(j, r) \colon j \equiv \pm(q+1)k_1 \mod m(q-1), m|r\} \end{aligned}$$

$$\begin{aligned} B_{13}(t_1) = \{&\chi_{35}(r_1), \chi_{36}(r_1), \chi_{37}(r_1), \chi_{38}(r_1), \chi_{53}(r_1, r_2), \chi_{54}(r_1, r_2), \chi_{51}(r_1, j), \\ &\chi_{52}(r_1, j), \chi_{64}(r_1, r_2, j) \colon r_1 \equiv \pm t_1 \mod m, r_2 \equiv \pm t_1 \mod m, m|j\} \end{aligned}$$

$$B_{16}(k_1, k_2) = \{\chi_{39}(k_1, k_2), \chi_{40}(k_1, k_2), \chi_{59}(r, k_2) \colon r \equiv \pm(q+1)k_1 \mod m(q-1)\}$$

$$B_{17}(k_1, k_2) = \{\chi_{41}(k_1, k_2), \chi_{42}(k_1, k_2), \chi_{58}(k_1, k_2, r) \colon m|r\}$$

$$\begin{aligned} B_{18}(k_1, t_1) = \{&\chi_{43}(k_1, r), \chi_{44}(k_1, r), \chi_{61}(j, r) \colon j \equiv \pm(q+1)k_1 \mod m(q-1), \\ &r \equiv \pm t_1 \mod m\} \end{aligned}$$

$$B_{19}(s) = \{\chi_{45}(r), \chi_{46}(r), \chi_{61}(r,j) \colon r \equiv \pm s \text{ or } \pm qs \mod m(q-1), m|j\}$$

$$B_{20}(k_1,t_1) = \{\chi_{47}(k_1,r), \chi_{48}(k_1,r), \chi_{60}(k_1,r,j) \colon r \equiv \pm t_1 \mod m, m|j\}$$

$$\begin{aligned} B_{21}(t_1,k_1) = \{\chi_{49}(r,k_1), \chi_{50}(r,k_1), \chi_{60}(k_1,r,j) \colon r \equiv \pm t_1 \mod m, \\ j \equiv \pm t_1 \mod m\} \end{aligned}$$

$$\begin{aligned} B_{22}(t_1,t_2) = \{\chi_{51}(r_1,r_2), \chi_{52}(r_1,r_2), \chi_{64}(r_1,j,r_2) \colon r_i \equiv \pm t_i \mod m, \\ j \equiv \pm t_1 \mod m\} \end{aligned}$$

$$B_{23}(t_1,t_2) = \{\chi_{53}(r_1,r_2), \chi_{54}(r_1,r_2), \chi_{64}(r_1,r_2,j) \colon r_i \equiv \pm t_i \mod m, m|j\}$$

$$B_{24}(u) = \{\chi_{55}(u), \chi_{56}(u), \chi_{65}(u,r) \colon r \equiv 0 \mod m\}$$

$$B_{25}(k_1,k_2,k_3) = \{\chi_{57}(k_1,k_2,k_3)\} \quad \text{(defect zero)}$$

$$B_{26}(k_1,k_2,t_1) = \{\chi_{58}(k_1,k_2,r) \colon r \equiv \pm t_1 \mod m\}$$

$$B_{27}(s,k_1) = \{\chi_{59}(r,k_1) \colon r \equiv \pm s \text{ or } \pm qs \mod m(q-1)\}$$

$$B_{28}(k_1,t_1,t_2) = \{\chi_{60}(k_1,r_1,r_2) \colon r_i \equiv \pm t_i \mod m\}$$

$$B_{29}(s,t_1)=\{\chi_{61}(r,j)\colon r\equiv \pm s \text{ or } \pm qs \mod m(q-1), j\equiv \pm t_1 \mod m\}$$

$$B_{30}(k_1,u)=\{\chi_{62}(k_1,u)\} \quad \text{(defect zero)}$$

$$B_{31}(v)=\{\chi_{63}(v)\} \quad \text{(defect zero)}$$

$$B_{32}(t_1,t_2,t_3)=\{\chi_{64}(r_1,r_2,r_3)\colon r_i\equiv \pm t_i \mod m\}$$

$$B_{33}(u,t_1)=\{\chi_{65}(u,r)\colon r\equiv \pm t_1 \mod m\}$$

$$B_{34}(w)=\begin{cases} \{\chi_{66}(r)\colon r\equiv \pm w, \pm qw, \text{ or } \pm q^2 w \mod m(q^2-q+1)\} & \text{if } \ell\neq 3,\\ \{\chi_{66}(r)\colon r\equiv \pm w, \pm qw, \text{ or } \pm q^2 w \mod mn\} & \text{if } \ell = 3\end{cases}$$

4.4.7 Non-Unipotent Brauer Characters for $Sp_6(2^a)$

Tables 4.9 and 4.10 give the irreducible Brauer characters of $G = Sp_6(q)$, q even, listed by the families $\mathcal{E}_\ell(G,(t))$ for ℓ'-semisimple elements $t\in G^*$. The indexing sets for $t = g_k$ are as given in Section 4.4 for B_k. Characters listed in the same set for the same choice of t make up the Brauer characters of a single block. Notation for the characters of G is taken from CHEVIE [26], and the notation for the class representatives $t\in G^*$ is from [47]. As usual, $\widehat{\chi}$ denotes the restriction of $\chi\in \mathrm{Irr}(G)$ to ℓ-regular elements G° of G.

The results in the tables follow from Lemma 4.1.1, Theorem 2.4.2, and the decomposition numbers for the unipotent blocks for the low-rank groups. The decomposition matrices for the unipotent blocks of $SL_2(q)$ (and therefore $Sp_2(q)=SL_2(q), GL_2(q) = C_{q-1}\times SL_2(q)$, and $GU_2(q)=C_{q+1}\times SL_2(q)$) and $GL_3(q)$ can be obtained from [35], and those for $Sp_4(q)$ are found in [75]. Note that the number α found in the description of the Brauer characters of $\mathcal{E}_\ell(G,(t))$ for t in the family g_6 or g_7 when $\ell|(q+1)$

is as in [75], and by [60], we have $\alpha = 1$ when $(q+1)_\ell = 3$ and $\alpha = 2$ otherwise. The decomposition matrices for the unipotent blocks of $GU_3(q)$ were found in [25], up to an unknown in the case $\ell|(q+1)$, which is found in [61].

Table 4.9: $\ell-$Brauer Characters in Non-Unipotent Blocks of $G = Sp_6(2^a)$, $\ell \neq 2$

t $C_{G^*}(t)$	Condition on ℓ	$\mathrm{IBr}_\ell(G) \cap \mathcal{E}_\ell(G,(t))$
$g_6(k_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{13}(k_1), \widehat{\chi}_{14}(k_1), \widehat{\chi}_{15}(k_1), \widehat{\chi}_{16}(k_1), \widehat{\chi}_{18}(k_1)\}$, $\{\widehat{\chi}_{17}(k_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{13}(k_1), \widehat{\chi}_{15}(k_1) - \widehat{\chi}_{13}(k_1), \widehat{\chi}_{16}(k_1) - \widehat{\chi}_{13}(k_1), \widehat{\chi}_{17}(k_1),$ $\widehat{\chi}_{18}(k_1) - \alpha\widehat{\chi}_{17}(k_1) - \widehat{\chi}_{16}(k_1) - \widehat{\chi}_{15}(k_1) + \widehat{\chi}_{13}(k_1)\}$, $\{\widehat{\chi}_{14}(k_1)\}$
$C_{q-1} \times Sp_4(q)$	$\ell \mid (q^2+1)$	$\{\widehat{\chi}_{13}(k_1), \widehat{\chi}_{14}(k_1) - \widehat{\chi}_{13}(k_1), \widehat{\chi}_{18}(k_1) - \widehat{\chi}_{14}(k_1) + \widehat{\chi}_{13}(k_1), \widehat{\chi}_{17}(k_1)\}$, $\{\widehat{\chi}_{15}(k_1)\}, \{\widehat{\chi}_{16}(k_1)\}$
	$\ell \nmid (q^4-1)$	$\{\widehat{\chi}_{13}(k_1)\}, \{\widehat{\chi}_{14}(k_1)\}, \{\widehat{\chi}_{15}(k_1)\}, \{\widehat{\chi}_{16}(k_1)\}, \{\widehat{\chi}_{17}(k_1)\}, \{\widehat{\chi}_{18}(k_1)\}$
$g_7(t_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{19}(t_1), \widehat{\chi}_{20}(t_1), \widehat{\chi}_{21}(t_1), \widehat{\chi}_{22}(t_1), \widehat{\chi}_{24}(t_1)\}$, $\{\widehat{\chi}_{23}(t_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{19}(t_1), \widehat{\chi}_{21}(t_1) - \widehat{\chi}_{19}(t_1), \widehat{\chi}_{22}(t_1) - \widehat{\chi}_{19}(t_1), \widehat{\chi}_{23}(t_1),$ $\widehat{\chi}_{24}(t_1) - \alpha\widehat{\chi}_{23}(t_1) - \widehat{\chi}_{22}(t_1) - \widehat{\chi}_{21}(t_1) + \widehat{\chi}_{19}(t_1)\}$, $\{\widehat{\chi}_{20}(t_1)\}$
$C_{q+1} \times Sp_4(q)$	$\ell \mid (q^2+1)$	$\{\widehat{\chi}_{19}(t_1), \widehat{\chi}_{20}(t_1) - \widehat{\chi}_{19}(t_1), \widehat{\chi}_{24}(t_1) - \widehat{\chi}_{20}(t_1) + \widehat{\chi}_{19}(t_1), \widehat{\chi}_{23}(t_1)\}$, $\{\widehat{\chi}_{21}(t_1)\}, \{\widehat{\chi}_{22}(t_1)\}$
	$\ell \nmid (q^4-1)$	$\{\widehat{\chi}_{19}(t_1)\}, \{\widehat{\chi}_{20}(t_1)\}, \{\widehat{\chi}_{21}(t_1)\}, \{\widehat{\chi}_{22}(t_1)\}, \{\widehat{\chi}_{23}(t_1)\}, \{\widehat{\chi}_{24}(t_1)\}$
$g_8(k_1)$	$3 \neq \ell \mid (q-1)$	$\{\widehat{\chi}_{25}(k_1), \widehat{\chi}_{26}(k_1), \widehat{\chi}_{27}(k_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{25}(k_1), \widehat{\chi}_{27}(k_1) - \widehat{\chi}_{25}(k_1)\}$, $\{\widehat{\chi}_{26}(k_1)\}$
$GL_3(q)$	$\ell \mid (q^2+q+1)$	$\{\widehat{\chi}_{25}(k_1), \widehat{\chi}_{26}(k_1) - \widehat{\chi}_{25}(k_1), \widehat{\chi}_{27}(k_1) - \widehat{\chi}_{26}(k_1) + \widehat{\chi}_{25}(k_1)\}$
	$\ell \nmid (q^3-1)(q+1)$	$\{\widehat{\chi}_{25}(k_1)\}, \{\widehat{\chi}_{26}(k_1)\}, \{\widehat{\chi}_{27}(k_1)\}$
$g_9(t_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{28}(t_1), \widehat{\chi}_{30}(t_1)\}, \{\widehat{\chi}_{29}(t_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{28}(t_1), \widehat{\chi}_{29}(t_1), \widehat{\chi}_{30}(t_1) - 2\widehat{\chi}_{29}(t_1) - \widehat{\chi}_{28}(t_1)\}$
$GU_3(q)$	$3 \neq \ell \mid (q^2-q+1)$	$\{\widehat{\chi}_{28}(t_1), \widehat{\chi}_{29}(t_1), \widehat{\chi}_{30}(t_1) - \widehat{\chi}_{28}(t_1)\}$
	$\ell \nmid (q^3+1)(q-1)$	$\{\widehat{\chi}_{28}(t_1)\}, \{\widehat{\chi}_{29}(t_1)\}, \{\widehat{\chi}_{30}(t_1)\}$
$g_{11}(k_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{31}(k_1), \widehat{\chi}_{32}(k_1), \widehat{\chi}_{33}(k_1), \widehat{\chi}_{34}(k_1)\}$
$GL_2(q) \times Sp_2(q)$	$\ell \mid (q+1)$	$\{\widehat{\chi}_{31}(k_1), \widehat{\chi}_{32}(k_1) - \widehat{\chi}_{31}(k_1), \widehat{\chi}_{33}(k_1) - \widehat{\chi}_{31}(k_1),$ $\widehat{\chi}_{34}(k_1) - \widehat{\chi}_{33}(k_1) - \widehat{\chi}_{32}(k_1) + \widehat{\chi}_{31}(k_1)\}$
	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{31}(k_1)\}, \{\widehat{\chi}_{32}(k_1)\}, \{\widehat{\chi}_{33}(k_1)\}, \{\widehat{\chi}_{34}(k_1)\}$
$g_{13}(t_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{35}(t_1), \widehat{\chi}_{36}(t_1), \widehat{\chi}_{37}(t_1), \widehat{\chi}_{38}(t_1)\}$
$GU_2(q) \times Sp_2(q)$	$\ell \mid (q+1)$	$\{\widehat{\chi}_{35}(t_1), \widehat{\chi}_{36}(t_1) - \widehat{\chi}_{35}(t_1), \widehat{\chi}_{37}(t_1) - \widehat{\chi}_{35}(t_1),$ $\widehat{\chi}_{38}(t_1) - \widehat{\chi}_{37}(t_1) - \widehat{\chi}_{36}(t_1) + \widehat{\chi}_{35}(t_1)\}$
	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{35}(t_1)\}, \{\widehat{\chi}_{36}(t_1)\}, \{\widehat{\chi}_{37}(t_1)\}, \{\widehat{\chi}_{38}(t_1)\}$
$g_{16}(k_1,k_2)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{39}(k_1,k_2), \widehat{\chi}_{40}(k_1,k_2)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{39}(k_1,k_2), \widehat{\chi}_{40}(k_1,k_2) - \widehat{\chi}_{39}(k_1,k_2)\}$
$C_{q-1} \times GL_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{39}(k_1,k_2)\}, \{\widehat{\chi}_{40}(k_1,k_2)\}$
$g_{17}(k_1,k_2)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{41}(k_1,k_2), \widehat{\chi}_{42}(k_1,k_2)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{41}(k_1,k_2), \widehat{\chi}_{42}(k_1,k_2) - \widehat{\chi}_{41}(k_1,k_2)\}$
$(C_{q-1})^2 \times Sp_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{41}(k_1,k_2)\}, \{\widehat{\chi}_{42}(k_1,k_2)\}$
$g_{18}(k_1,t_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{43}(k_1,t_1), \widehat{\chi}_{44}(k_1,t_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{43}(k_1,t_1), \widehat{\chi}_{44}(k_1,t_1) - \widehat{\chi}_{43}(k_1,t_1)\}$
$C_{q+1} \times GL_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{43}(k_1,t_1)\}, \{\widehat{\chi}_{44}(k_1,t_1)\}$
$g_{19}(s)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{45}(s), \widehat{\chi}_{46}(s)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{45}(s), \widehat{\chi}_{46}(s) - \widehat{\chi}_{45}(s)\}$
$C_{q^2-1} \times Sp_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{45}(s)\}, \{\widehat{\chi}_{46}(s)\}$
$g_{20}(k_1,t_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{47}(k_1,t_1), \widehat{\chi}_{48}(k_1,t_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{47}(k_1,t_1), \widehat{\chi}_{48}(k_1,t_1) - \widehat{\chi}_{47}(k_1,t_1)\}$
$C_{q-1} \times C_{q+1} \times Sp_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{47}(k_1,t_1)\}, \{\widehat{\chi}_{48}(k_1,t_1)\}$
$g_{21}(t_1,k_1)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{49}(t_1,k_1), \widehat{\chi}_{50}(t_1,k_1)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{49}(t_1,k_1), \widehat{\chi}_{50}(t_1,k_1) - \widehat{\chi}_{49}(t_1,k_1)\}$
$C_{q-1} \times GU_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{49}(t_1,k_1)\}, \{\widehat{\chi}_{50}(t_1,k_1)\}$
$g_{22}(t_1,t_2)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{51}(t_1,t_2), \widehat{\chi}_{52}(t_1,t_2)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{51}(t_1,t_2), \widehat{\chi}_{52}(t_1,t_2) - \widehat{\chi}_{51}(t_1,t_2)\}$
$C_{q+1} \times GU_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{51}(t_1,t_2)\}, \{\widehat{\chi}_{52}(t_1,t_2)\}$
$g_{23}(t_1,t_2)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{53}(t_1,t_2), \widehat{\chi}_{54}(t_1,t_2)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{53}(t_1,t_2), \widehat{\chi}_{54}(t_1,t_2) - \widehat{\chi}_{53}(t_1,t_2)\}$
$(C_{q+1})^2 \times Sp_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{53}(t_1,t_2)\}, \{\widehat{\chi}_{54}(t_1,t_2)\}$
$g_{24}(u)$	$\ell \mid (q-1)$	$\{\widehat{\chi}_{55}(u), \widehat{\chi}_{56}(u)\}$
	$\ell \mid (q+1)$	$\{\widehat{\chi}_{55}(u), \widehat{\chi}_{56}(u) - \widehat{\chi}_{55}(u)\}$
$C_{q^2+1} \times Sp_2(q)$	$\ell \nmid (q^2-1)$	$\{\widehat{\chi}_{55}(u)\}, \{\widehat{\chi}_{56}(u)\}$

Table 4.10: $\ell-$Brauer Characters in Non-Unipotent Blocks of $G = Sp_6(2^a)$, $\ell \neq 2$, Continued

t $C_{G^*}(t)$	Condition on ℓ	$\mathrm{IBr}_\ell(G) \cap \mathcal{E}_\ell(G,(t))$
$g_{25}(k_1,k_2,k_3)$ $(C_{q-1})^3$	all $\ell \neq 2$	$\{\widehat{\chi}_{57}(k_1,k_2,k_3)\}$
$g_{26}(k_1,k_2,t_1)$ $(C_{q-1})^2 \times C_{q+1}$	all $\ell \neq 2$	$\{\widehat{\chi}_{58}(k_1,k_2,t_1)\}$
$g_{27}(s,k_1)$ $C_{q-1} \times C_{q^2-1}$	all $\ell \neq 2$	$\{\widehat{\chi}_{59}(s,k_1)\}$
$g_{28}(k_1,t_1,t_2)$ $C_{q-1} \times (C_{q+1})^2$	all $\ell \neq 2$	$\{\widehat{\chi}_{60}(k_1,t_1,t_2)\}$
$g_{29}(s,t_1)$ $C_{q+1} \times C_{q^2-1}$	all $\ell \neq 2$	$\{\widehat{\chi}_{61}(s,t_1)\}$
$g_{30}(k_1,u)$ $C_{q-1} \times C_{q^2+1}$	all $\ell \neq 2$	$\{\widehat{\chi}_{62}(k_1,u)\}$
$g_{31}(v)$ C_{q^3-1}	all $\ell \neq 2$	$\{\widehat{\chi}_{63}(v)\}$
$g_{32}(t_1,t_2,t_3)$ $(C_{q+1})^3$	all $\ell \neq 2$	$\{\widehat{\chi}_{64}(t_1,t_2,t_3)\}$
$g_{33}(u,t_1)$ $C_{q+1} \times C_{q^2+1}$	all $\ell \neq 2$	$\{\widehat{\chi}_{65}(u,t_1)\}$
$g_{34}(w)$ C_{q^3+1}	all $\ell \neq 2$	$\{\widehat{\chi}_{66}(w)\}$

Chapter 5

Cross-Characteristic Representations of $Sp_6(2^a)$ and Their Restrictions to Proper Subgroups

Recall Section 1.1, where we provide an overview and motivation of the Aschbacher-Scott program and introduce the main theorems of this chapter.

We begin in Section 5.1 by making some preliminary observations, listing some useful facts, and reviewing some of our notation. In the remaining sections, we prove Theorems 1.1.2 and 1.1.3, first making a basic reduction to rule out a few subgroups, then treating each remaining maximal subgroup H separately to find all irreducible $G-$modules V which restrict irreducibly to H. Finally, in Section 5.6 we treat the case $q = 2$ and prove Theorems 1.1.4 and 1.1.5.

5.1 Some Preliminary Observations

We adapt the notation of [37] for the finite groups of Lie type. In particular, $L_n(q)$ and $U_n(q)$ will denote the groups $PSL_n(q)$ and $PSU_n(q)$, respectively. $O^+_{2n}(q)$ and $O^-_{2n}(q)$ will denote the general orthogonal groups corresponding to quadratic forms of Witt defect 0 and 1, respectively. Moreover, if X acts on a group Y, we denote by $Y : X$ or $Y \rtimes X$ the semidirect product of Y with X. More generally, we may write $Y.X$ if Y is a (not necessarily complemented) normal subgroup with quotient X. If r is a positive integer, we will sometimes write $Y : r$ (or $Y.r$) if $X = C_r$ is the cyclic group of order r, and an elementary abelian group of order r will be denoted by $[r]$.

Given a finite group X, recall that we denote by $\mathfrak{d}_\ell(X)$ the smallest degree larger than one of absolutely irreducible representations of X in characteristic ℓ. Similarly, $\mathfrak{m}_\ell(X)$ denotes the largest such degree. When $\ell = 0$, we write $\mathfrak{m}_0(X) =: \mathfrak{m}(X)$. Given χ a complex character of X, we denote by $\widehat{\chi}$ the restriction of χ to ℓ-regular elements of X, and we will say a Brauer character φ *lifts* if $\varphi = \widehat{\chi}$ for some complex character χ. Throughout the chapter, ℓ will usually denote the characteristic of the representation.

As usual, $\mathrm{Irr}(X)$ will denote the set of irreducible ordinary characters of X and $\mathrm{IBr}_\ell(X)$ will denote the set of irreducible ℓ-Brauer characters of X. Given a subgroup Y and a character $\lambda \in \mathrm{IBr}_\ell(Y)$, we will use $\mathrm{IBr}_\ell(X|\lambda)$ to denote the set of irreducible Brauer characters of X which contain λ as a constituent when restricted to Y. The restriction of the character φ to Y will be written φ_Y or $\varphi|_Y$, and the induction of λ to X will be written λ^X or sometimes $\mathrm{Ind}_Y^X(\lambda)$ for more clarity. We will use the notation $\ker\varphi$ to denote the kernel of the representation affording $\varphi \in \mathrm{IBr}_\ell(X)$.

We begin by making a few general observations, which we will sometimes use without reference:

Lemma 5.1.1. *Let G be a finite group, $H < G$ a proper subgroup, $\mathbb{F}$ an algebraically closed field of characteristic $\ell \geq 0$, and V an irreducible $\mathbb{F}G$-module with dimension greater than 1. Further, suppose that the restriction $V|_H$ is irreducible. Then*

$$\sqrt{|H/Z(H)|} \geq \mathfrak{m}(H) \geq \mathfrak{m}_\ell(H) \geq \dim(V) \geq \mathfrak{d}_\ell(G).$$

Lemma 5.1.2. *Let $\chi \in \mathrm{Irr}(G)$ such that $\widehat{\chi}|_H \in \mathrm{IBr}_\ell(H)$. Then $\chi|_H \in \mathrm{Irr}(H)$.*

Proof. We may write $\chi|_H = \sum_i a_i\varphi_i$ for $\varphi_i \in \mathrm{Irr}(H)$ and non-negative integers a_i. Then $\mathrm{IBr}_\ell(H) \ni \widehat{\chi}_H = \sum_i a_i\widehat{\varphi}_i$, and by the linear independence of irreducible ℓ-Brauer characters and the irreducibility of $\widehat{\chi}_H$, we see that there is exactly one index i for which a_i is nonzero, and it must be that $\chi|_H = \varphi_i$.

□

Lemma 5.1.3. *Let G be a finite group, $H \leq G$ a subgroup, and ℓ a prime. Let $\widehat{H}$ denote the set of irreducible complex characters of degree 1 of H. If $\chi \in \mathrm{Irr}(G)$ such that $\chi|_H - \lambda \notin \mathrm{Irr}(H)$ for any $\lambda \in \widehat{H} \cup \{0\}$, then $\widehat{\chi}|_H - \mu \notin \mathrm{IBr}_\ell(H)$ for any $\mu \in \mathrm{IBr}_\ell(H)$ of degree 1.*

Proof. Write $\chi|_H = \sum_{i=1}^s \theta_i$ where $\theta_i \in \mathrm{Irr}(H)$ are not necessarily distinct. Each $\widehat{\theta_i}$ is a non-negative integer linear combination of irreducible Brauer characters of H. By way of contradiction, suppose $\widehat{\chi}|_H - \mu \in \mathrm{IBr}_\ell(H)$ for some $\mu \in \mathrm{IBr}_\ell(H)$ with $\mu(1) = 1$. Then $\sum_{i=1}^s \widehat{\theta_i} = \widehat{\chi}|_H = \varphi + \mu$ for some $\varphi \in \mathrm{IBr}_\ell(H)$. By the linear independence of irreducible Brauer characters, we conclude that $s \leq 2$ and, allowing for reordering of the θ_i's, either $\widehat{\theta_1} = \varphi$ and $\widehat{\theta_2} = \mu$ or $\widehat{\theta_1} = \varphi + \mu$. In the first case, $\chi|_H = \theta_1 + \theta_2$ with $\theta_2 \in \widehat{H}$, and in the latter case, $\chi|_H = \theta_1 \in \mathrm{Irr}(H)$, yielding a contradiction in either situation. $\square$

Lemmas 5.1.2 and 5.1.3 suggest that in some situations, we will be able to reduce to the case of ordinary representations.

5.1.1 Other Notes on $Sp_6(q)$, q even

We note that $|Sp_6(q)| = q^9(q^2-1)(q^4-1)(q^6-1)$, so if ℓ is a prime dividing $|Sp_6(q)|$ and $\ell \neq 3$, then ℓ must divide exactly one of $q-1$, $q+1$, q^2+1, q^2+q+1, or q^2-q+1. If $\ell = 3$, then it divides $q-1$ if and only if it divides q^2+q+1, and it divides $q+1$ if and only if it divides q^2-q+1. In what follows, it will often be convenient to distinguish between these cases.

D. White [76] has calculated the decomposition numbers for the unipotent blocks of $G = Sp_6(q)$, q even, up to a few unknowns in the case $\ell|(q+1)$. Recall that in Section 4.3 of Chapter 4, we have summarized these results by giving the description (in terms of the restrictions of ordinary characters to ℓ-regular elements) of the ℓ-Brauer characters for G that lie in unipotent blocks, along with their degrees.

5.2 A Basic Reduction

The goal of this section is to eliminate many possibilities for subgroups H yielding triples as in Problem 1. We do this in the form of two theorems, treating $Sp_6(q)$ and $Sp_4(q)$ separately.

Theorem 5.2.1 (Reduction Theorem for $Sp_6(q)$)**.** *Let (G, H, V) be a triple as in Problem 1, with $\ell \neq 2$, $G = Sp_6(q)$, $q \geq 4$ even, and $H < G$ a maximal subgroup. Then H is G-conjugate to either $G_2(q), O_6^{\pm}(q)$, or a maximal parabolic subgroup of G.*

Proof. First note that from [41], $\mathfrak{d}_\ell(G) = (q^3-1)(q^3-q)/(2(q+1))$. Second, by [10] and [37], the maximal subgroups of G are isomorphic to one of the following:

1. $SL_2(q^3).3$
2. $Sp_2(q) \wr S_3$
3. $Sp_4(q) \times Sp_2(q)$
4. $Sp_6(q_0)$, where $q = q_0^m$, some $m > 1$
5. $G_2(q)$
6. $O_6^{\pm}(q)$
7. a maximal parabolic subgroup of G.

If H is as in (1), then by Clifford theory, $\mathfrak{m}(H) \leq 3(q^3+1) < \mathfrak{d}_\ell(G)$, since $\mathfrak{m}(SL_2(q^3)) = q^3+1$. Hence by Lemma 5.1.1, H is not of this form.

If H is as in (2), then $(Sp_2(q))^3 \lhd H$ of index 6, so by Clifford theory, $\mathfrak{m}(H) \leq 6(q+1)^3$, which is smaller than $\mathfrak{d}_\ell(G)$ unless $q = 4$. When $q = 4$, we have $6(q+1)^3 < q^2(q^3-1)$, so we can restrict our attention to the Weil characters, by Theorem 1.1.1. Now since the ℓ-modular Weil characters are of the form $\widehat{\chi}$ or $\widehat{\chi} - 1_G$ for some complex Weil character $\chi \in \mathrm{Irr}(G)$ (see Table 4.2), it suffices by Lemma 5.1.3 to note that neither $\chi(1)$ nor $\chi(1)-1$ divides $|H|$ for any complex Weil character χ, so these degrees cannot appear as ordinary character degrees for H. Hence again, H cannot be of this form.

If H is as in (3), then $\mathfrak{m}(H) \leq (q^2+1)(q+1)^3$, since by [49], $\mathfrak{m}(Sp_2(q)) \leq q+1$ and $\mathfrak{m}(Sp_4(q)) \leq (q+1)^2(q^2+1)$. Hence $\mathfrak{m}(H) \leq D$, where D is the bound in part (B) of Theorem 1.1.1, so by Theorem 1.1.1, χ must either lift to an ordinary character or belong to a unipotent block of G.

Moreover, part (A) of Theorem 1.1.1 yields that the only irreducible Brauer characters in a unipotent block that do not lift and have degree at most $\mathfrak{m}(H)$ are $\widehat{\rho_3^2}-1, \widehat{\beta_3}-1$ in the case $\ell|(q+1)$, $\widehat{\rho_3^2}-1$ in the case $\ell|(q^2-q+1)$, or $\widehat{\rho_3^1}-1$ in the case $\ell|(q^2+q+1)$. From [49], we see that none of the degrees corresponding to these characters occur in $\mathrm{Irr}(H)=\mathrm{Irr}(Sp_4(q))\otimes\mathrm{Irr}(Sp_2(q))$, and moreover none of the degrees of characters in $\mathrm{Irr}(G)$ can occur in $\mathrm{Irr}(H)$. Thus by Lemma 5.1.3, there are no possible such modules V for this choice of H.

Finally, suppose H is as in (4). Then

$$\mathfrak{m}(H)=\begin{cases}(q_0^2+1)(q_0^4+q_0^2+1)(q_0+1)^3 & \text{if } q_0>4\\ q_0^2(q_0+1)(q_0^2+1)(q_0^4+q_0^2+1) & \text{if } q_0\le 4\end{cases}$$

by [49], and $\mathfrak{d}_\ell(G)=\frac{(q_0^{3m}-1)(q_0^{3m}-q_0^m)}{2(q_0^m+1)}$. Thus

$$\mathfrak{d}_\ell(G)\ge\frac{(q_0^6-1)(q_0^6-q_0^2)}{2(q_0^2+1)}=\frac{1}{2}q_0^2(q_0^4+q_0^2+1)(q_0^2-1)^2>\mathfrak{m}(H)$$

as long as $q_0\ge 4$, and we have only to consider the case $H=Sp_6(2)$.

Here as long as $q\ge 8$, we also have $\mathfrak{d}_\ell(G)>\mathfrak{m}(H)$, so we are reduced to the case $H=Sp_6(2), G=Sp_6(4)$. Then $\mathfrak{m}(H)=512$ and $\mathfrak{d}_\ell(G)=378$. Moreover, from Theorem 1.1.1, the only irreducible ℓ-Brauer characters of G which have degree less than or equal to $\mathfrak{m}(H)$ are Weil characters, which are all of the form $\widehat{\chi}$ or $\widehat{\chi}-1$ for $\chi\in\mathrm{Irr}(G)$. Now, from GAP's character table library (see [24], [11]), it is clear that the only ℓ-Brauer character of G whose degree also occurs as a degree of H is $\widehat{\alpha_3}$, which has degree 378. However, there is an involutory class of H on which the character of this degree takes the value -30, but there is no such involutory class in G for $\widehat{\alpha_3}$. Thus $\widehat{\alpha_3}$ does not restrict irreducibly to H, and there are no possible triples (G,H,V) with this choice of G and H, by Lemma 5.1.3.

Therefore, we are left only with subgroups H as in (5)-(7), as claimed.

$\square$

Theorem 5.2.2 (Reduction Theorem for $Sp_4(q)$)**.** *Let (G, H, V) be a triple as in Problem 1, with $\ell \neq 2$, $G = Sp_4(q)$, $q \geq 4$ even, and $H < G$ a maximal subgroup. Then H is a maximal parabolic subgroup of G.*

Proof. Let V afford the character $\chi \in \mathrm{IBr}_\ell(G)$. From [41],

$$\mathfrak{d}_\ell(G) = \frac{(q^2-1)(q^2-q)}{2(q+1)} = \frac{1}{2}q(q-1)^2,$$

and by [23] and [10], the maximal subgroups of G are

1. a maximal parabolic subgroup of G (geometrically, the stabilizer of a point or a line)
2. $Sp_2(q) \wr S_2$ (geometrically, the stabilizer of a pair of polar hyperbolic lines)
3. $O_4^\epsilon(q)$, $\epsilon = +$ or $-$
4. $Sp_2(q^2) : 2$
5. $[q^4] : C_{q-1}^2$
6. $Sp_4(q_0)$, where $q = q_0^m$, some $m > 1$
7. $C_{q-1}^2 : D_8$
8. $C_{q+1}^2 : D_8$
9. $C_{q^2+1} : 4$
10. $Sz(q)$ (when $q = 2^m$ with $m \geq 3$ odd)

If H is as in (2), H has an index-2 subgroup K isomorphic to $Sp_2(q) \times Sp_2(q)$, so by Clifford theory, an irreducible character of H must restrict to either an irreducible character of K or the sum of two irreducible characters of K of the same degree. In particular, this must be true of $\chi|_H$, as we are assuming χ is irreducible on H.

Suppose first that $\chi|_K \in \mathrm{IBr}_\ell(K)$. Now by [49], $\mathfrak{m}(K) = (q+1)^2 < \mathfrak{d}_\ell(G)$ unless $q = 4$, in which case $\widehat{\alpha}_2$ is the only character of G with degree sufficiently small. Since $Sp_2(4)$ has ordinary character degrees $1, 3, 4$, and 5, we see that $\alpha_2(1) = 18$ is not an ordinary character degree of K, and therefore by Lemma 5.1.2, $\widehat{\alpha}_2|_K$ is not irreducible. Thus we must have that χ_K is the sum of two irreducible characters of the same degree, and hence $\chi(1)$ is even. Moreover, $\chi(1) \leq 2(q+1)^2$, which is again smaller than $\mathfrak{d}_\ell(G)$ unless $q = 4$. When $q = 4$, using the GAP Character Table Library [11], we see that this leaves the ℓ-modular Weil characters $\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\rho}_2^1$, and $\widehat{\rho}_2^2$, which have degree $18, 34, 34$, and 50, respectively, as the only possibilities for χ. (The characters $\widehat{\rho_2^1} - 1, \widehat{\rho_2^2} - 1$, or $\widehat{\beta_2} - 1$ have odd degree, a contradiction.) Now, using GAP, we see that K has no irreducible character of degree 17 and exactly one irreducible character of degree 25. Inspecting the character values, we see that on classes consisting of elements of order 3, ρ_2^2 does not take twice the value of this degree-25 character of K. Thus β_2, ρ_2^1, and ρ_2^2 do not restrict irreducibly to H. Also, using GAP, we can construct the character table of H to see that there is a unique character of degree 18, but that this character takes the value 0 on one of the classes containing order-4 elements, and α_2 does not. Hence this character is not the restriction of α_2, and by Lemma 5.1.2 H cannot be as in (2).

If H is as in (3), then

$$H = O_4^\epsilon(q) \cong \begin{cases} SL_2(q^2).2 & \text{if } \epsilon = - \\ (SL_2(q) \times SL_2(q)).2 & \text{if } \epsilon = + \end{cases}.$$

Thus $\mathfrak{m}(H) \leq 2(q^2+1)$ or $2(q+1)^2$, which are smaller than $\mathfrak{d}_\ell(G)$ for $q \geq 8$. Now, when $q = 4$, the only members of $\mathrm{IBr}_\ell(G)$ with sufficiently small degree are the ℓ-modular Weil characters corresponding to $\alpha_2, \beta_2, \rho_2^1$, and ρ_2^2, and hence either lift to an ordinary character or are of the form $\widehat{\chi} - 1_G$ for an ordinary character χ of G. Direct calculation using GAP and the GAP Character Table Library ([24], [11]) shows that no ordinary character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi|_H \in \mathrm{Irr}(H)$ or $\chi|_H - 1 \in \mathrm{Irr}(H)$ when $H \cong SL_2(16).2$. Thus by Lemma 5.1.3, H cannot be $O_4^-(4)$.

If $H = O_4^+(4) \cong (SL_2(4) \times SL_2(4)).2$, then as discussed above, χ cannot restrict irreducibly to $K = SL_2(4) \times SL_2(4) \cong Sp_2(4) \times Sp_2(4)$. Repeating the argument from case (2) above, we see that the only possibility for χ is the Weil character $\widehat{\alpha}_2$. Now, computation in GAP shows that α_2 restricts to $SL_2(4) \times SL_2(4)$ as the sum of two irreducible characters of degree 9. However, it is clear by inspecting the character values on the third class of involutions in the character table for $O_4^+(4)$ stored in GAP [24] that the unique character of degree 18 cannot extend to the unique character, α_2, of degree 18 in G. Hence, by Lemma 5.1.2, H cannot be as in (3).

If H is as in (4), then the bounds for $\mathfrak{m}(H)$ are the same as $O_4^-(q)$, and when $q = 4$, the two groups are the same. Thus the same proof as in case (3) when $\epsilon = -$ shows that H cannot be as in (4) either.

If H is as in (5), then it is solvable and by the Fong-Swan theorem, every ℓ-Brauer character lifts to an ordinary character. Hence by Lemma 5.1.2, it suffices to consider the problem when $\chi \in \mathrm{Irr}(G)$ is an ordinary character. H has a normal subgroup of the form $[q^4] : C_{q-1}$ with quotient group C_{q-1}, so by Clifford theory any irreducible character of H has degree $t \cdot \theta(1)$, where t divides $q-1$ and $\theta \in \mathrm{Irr}([q^4] : C_{q-1})$. Moreover, since $[q^4]$ is a normal abelian subgroup of $[q^4] : C_{q-1}$, Ito's theorem (see [33, Theorem (6.15)]) implies that $\theta(1)$ divides $q-1$. It follows that any character of H must have degree dividing $(q-1)^2$, which is smaller than $\mathfrak{d}_\ell(G)$, so H cannot be as in (5).

If H is as in (6), then

$$\mathfrak{m}(H) = \begin{cases} (q_0^2+1)(q_0+1)^2 & \text{if } q_0 > 4 \\ q_0(q_0+1)(q_0^2+1) & \text{if } q_0 = 4 \\ q_0^4 & \text{if } q_0 = 2 \end{cases}$$

by [49], and $\mathfrak{d}_\ell(G) = \frac{(q_0^{2m}-1)(q_0^{2m}-q_0^m)}{2(q_0^m+1)}$. Thus

$$\mathfrak{d}_\ell(G) \geq \frac{(q_0^4-1)(q_0^4-q_0^2)}{2(q_0^2+1)} = \frac{1}{2}q_0^2(q_0^2-1)^2 > \mathfrak{m}(H),$$

and H cannot be as in (6).

If H is as in (7) or (8), then $|H| = 8(q \pm 1)^2$, and therefore by Lemma 5.1.1, $\mathfrak{m}_\ell(H) \leq 2\sqrt{2}(q \pm 1)$, which is smaller than $\mathfrak{d}_\ell(G)$ for $q \geq 4$. If H is as in (9), then $|H| = 4(q^2 + 1)$, so $\mathfrak{m}_\ell(H) \leq 2\sqrt{q^2 + 1}$, which is also smaller than $\mathfrak{d}_\ell(G)$ for $q \geq 4$. Hence, H cannot be as in (7)-(9).

Finally, if H is as in (10), then from [49], $\mathfrak{m}_\ell(H) = (q-1)(q + \sqrt{2q} + 1)$, which is smaller than $\mathfrak{d}_\ell(G)$ as long as q is at least 8. Since this subgroup only exists for $q = 2^m$, $m \geq 3$ odd, this shows that H cannot be as in (10) either, which leaves (1) as the only possibility for H, as stated.

□

5.3 Restrictions of Irreducible Characters of $Sp_6(q)$ to $G_2(q)$

Let q be a power of 2. The purpose of this section is to prove part (2) of Theorem 1.1.2. Viewing $H = G_2(q)$ as a subgroup of $Sp_6(q)$, we solve Problem 1 for the case $G = Sp_6(q)$, $H = G_2(q)$, and V is a cross-characteristic G-module. That is, we completely classify all irreducible ℓ-Brauer characters of $Sp_6(q)$, which restrict irreducibly to $G_2(q)$ when $\ell \neq 2$. As remarked earlier, this provides the "converse" of G. Seitz' theorem [64] for case (iv) when $p = 2$.

For the classes and complex characters of $Sp_6(q)$, we use as reference Frank Lübeck's thesis (see [47]), in which he finds the conjugacy classes and irreducible complex characters of $Sp_6(q)$. For $G_2(q)$, we refer to [22], in which Enomoto and Yamada find the conjugacy classes and irreducible complex characters of $G_2(q)$. For the remainder of this section, we adapt the notation of [22] that $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \mod 3$.

For the ℓ-Brauer characters lying in unipotent blocks of $Sp_6(q)$, we refer to work done by D. White in [76], and for the Brauer characters of $G_2(q)$ we refer to work by G. Hiss and J. Shamash in [29], [32], [66], [67], and [68]. Since many of these references utilize different notations for the same characters, we include a conversion

Table 5.1: Notation of Characters of $Sp_6(q)$

Degree	Guralnick-Tiep [27]	Lübeck [47]	D. White [76]
$\frac{(q^3+1)(q^3-q)}{2(q-1)}$	ρ_3^1	$\chi_{1,4}$	χ_4
$\frac{(q^3-1)(q^3+q)}{2(q-1)}$	ρ_3^2	$\chi_{1,2}$	χ_2
$\frac{q^6-1}{q-1}$	τ_3^i		Type χ_{13}
$\frac{(q^3-1)(q^3-q)}{2(q+1)}$	α_3	$\chi_{1,5}$	χ_5
$\frac{(q^3+1)(q^3+q)}{2(q+1)}$	β_3	$\chi_{1,3}$	χ_3
$\frac{q^6-1}{q+1}$	ζ_3^i		Type χ_{19}

Table 5.2: Notation of Characters of $G_2(q)$

Degree	Guralnick-Tiep [27]	Enomoto-Yamada [22]	Hiss-Shamash [29],[32],[66],[67],[68]
$\frac{(q^3+1)(q^3-q)}{2(q-1)}$	$(\rho_3^1)\|_{G_2(q)}$	θ_2	X_{15}
$\frac{(q^3-1)(q^3-q)}{2(q+1)}$	$(\alpha_3)\|_{G_2(q)}$	θ_2'	X_{17}
$\frac{q^6-1}{q-1}$	$(\tau_3^i)\|_{G_2(q)}$	$\chi_3(i)$	X_{1b}'
$\frac{q^6-1}{q+1}$	$(\zeta_3^i)\|_{G_2(q)}$	$\chi_3'(i)$	X_{2a}'
$\frac{q(q^2+q+1)(q+1)^2}{6}$		θ_1	X_{16}
$\frac{q(q^2-q+1)(q-1)^2}{6}$		θ_1'	X_{18}
$\frac{q(q^4+q^2+1)}{3}$		θ_4	X_{14}

between notations in Tables 5.1 and 5.2.

Our first step is to find the fusion of conjugacy classes from $G_2(q)$ into $Sp_6(q)$.

5.3.1 Fusion of Conjugacy Classes in $G_2(q)$ into $Sp_6(q)$

In this section, we compute the fusion of conjugacy classes from $H = G_2(q)$ into $G = Sp_6(q)$. Table 5.3 summarizes the results.

We begin with the unipotent classes. In the notation of [22] and [47], the unipotent classes of H and G, respectively, are:

Class in $G_2(q)$	A_0	A_1	A_2	A_{31}	A_{32}	A_4	A_{51}	A_{52}
Order	1	2	2	4	4	4	8	8

Class in $Sp_6(q)$	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$	$c_{1,7}$	$c_{1,8}$	$c_{1,9}$	$c_{1,10}$	$c_{1,11}$
Order	1	2	2	2	2	4	4	4	4	4	8	8

Armed with the calculation of commutators of unipotent elements of H and G given in [22] and [47], respectively, explicit calculations shows that for any element $u \in H$ of order 8, u^4 lies in the class A_1. Similarly, any $u \in G$ of order 8 satisfies that u^4 lies in the class $c_{1,2}$. Thus the class A_1 of H must lie in the class $c_{1,2}$ of G.

Now, [40, Proposition 7.6] implies that the characters τ_3^i for $1 \leq i \leq (q-2)/2$ restrict irreducibly from $GL_6(q)$ to the character $\chi_3(i)$ in $G_2(q)$ (in the notation of [22]). Since the only eigenvalues of unipotent elements are 1, we can use the values given in [22] of $\chi_3(i)$ on the different classes and Equation (4.2.1) to find the dimensions of the eigenspaces for each unipotent class in $G_2(q)$. Then using the explicit descriptions of the unipotent class representatives in [47, Sections 1 and 4], we find the dimensions of the eigenspaces for each unipotent class in $Sp_6(q)$ so as to find the values of τ_3^i on the classes. With this information, we see that $c_{1,4}$ is the only conjugacy class of G of involutions on which τ_3^i has the same value, q^2+q+1, as on the class A_2 in H. This tells us that the class A_2 of H must lie in the class $c_{1,4}$ of G.

Moreover, $\chi_3(i) = \tau_3^i|_H$ has the value $q+1$ on all classes of order-4 elements in H. Among the classes of order-4 elements of G, τ_3^i only has this value on the classes $c_{1,5}$ and $c_{1,6}$. Hence A_{31}, A_{32}, and A_4 must sit inside $(c_{1,5} \cup c_{1,6})$. However, the order of the centralizer in $H = G_2(q)$ of an element of A_{31} is $6q^4$ and of A_4 is $3q^4$, so if $\epsilon = 1$ (that is, $q \equiv 1 \mod 3$), then these do not divide $2q^6(q+1)$, which is the order of the centralizer in $G = Sp_6(q)$ of an element in the class $c_{1,6}$. On the other hand, if $\epsilon = -1$, then they do not divide $2q^6(q-1)$, which is the order of the centralizer in $Sp_6(q)$ of an element in the class $c_{1,5}$. Noting that $|C_H(x)|$ must divide $|C_G(x)|$ for

$x \in H$, we deduce that

$$A_{31}, A_4 \subset H \cap \begin{cases} c_{1,5} & \text{if } \epsilon = 1, \\ c_{1,6} & \text{if } \epsilon = -1 \end{cases}.$$

We claim that the class A_{32} does not fuse with the classes A_{31} and A_4 in $Sp_6(q)$. Indeed, suppose otherwise, so that A_{31}, A_{32}, A_4 are all in $\begin{cases} c_{1,5} & \text{if } \epsilon = 1, \\ c_{1,6} & \text{if } \epsilon = -1 \end{cases}$. Consider the character $\chi = \chi_{1,2} \in \mathrm{Irr}(G)$ in the notation of [47]. Note that this character has the same absolute value on all elements of order 8, namely $\frac{q}{2}$. From [22, Tables I-1, II-1], we know the fusion of the Borel subgroup $B = UT$ into the parabolic subgroup P of H and the fusion of P into H. For the convenience of the reader, we have included these fusions and the class sizes for the unipotent classes of B in Table 5.4, following the notation of [22]. Using this information and the fusion of the elements of order 2 and 4 from H into G which we know (or are assuming), together with the fact that $U \lhd B$ is the union of the unipotent conjugacy classes of B, we calculate that $[\chi_U, \chi_U]$ is not an integer, a contradiction. Therefore, A_{32} must not fuse with A_{31} and A_4, so

$$A_{32} \subset H \cap \begin{cases} c_{1,6} & \text{if } \epsilon = 1, \\ c_{1,5} & \text{if } \epsilon = -1 \end{cases}.$$

We return to the remaining unipotent classes (namely, those consisting of elements of order 8) after calculating the fusion of the non-unipotent classes.

Let W and $\tilde{W}$ denote the natural modules for $SL_6(q)$ and $SU_6(q)$, respectively. The eigenvalues of the semisimple elements acting on W or $\tilde{W}$ is clear from the notation for the element in [47] and [22]. Namely, the element $h(z_1, z_2, z_3)$ has eigenvalues $z_1, z_2, z_3, z_3^{-1}, z_2^{-1}, z_1^{-1}$.

For example, the class representative in [22] for the class $D_{21}(i)$ of $G_2(q)$ is $h(\eta^i, \eta^{-i}, 1)$, and the eigenvalues (acting on either W or $\tilde{W}$) of this element are $\eta^i, \eta^i, \eta^{-i}, \eta^{-i}, 1, 1$. The class representative in [47] for the class $c_{10,0}(i_1)$ is $h(\tilde{\xi}_1^{i_1}, \tilde{\xi}_1^{i_1}, 1)$, which has eigenvalues $\tilde{\xi}_1^{i_1}, \tilde{\xi}_1^{i_1}, \tilde{\xi}_1^{-i_1}, \tilde{\xi}_1^{-i_1}, 1, 1$. Now, we see that both η and $\tilde{\xi}$ represent primitive $(q+1)$st roots of unity in $\mathbb{C}$ in the respective papers, and a comparison of notations tells us that $D_{21}(i)$ must sit inside $c_{10,0}(i)$.

A similar analysis of notations yields the results for the other semisimple classes, which can be found in Table 5.3. For the convenience of the reader, we include below the list of semisimple class representatives in each notation, together with the fusions for the semisimple classes:

N	Nth root of unity in [22]	Nth root of unity in [47]
$q-1$	γ	$\widetilde{\zeta}_1$
$q+1$	η	$\widetilde{\xi}_1$
q^2-1	θ	$\widetilde{\zeta}_2$
$(q^2-1)/3$	ω	
q^2+q+1	τ	$\widetilde{\zeta}_3^{(q-1)}$
q^2-q+1	σ	$\widetilde{\xi}_3^{(q+1)}$

Semisimple Class in $G_2(q)$	Representative (from [22])	Class in $Sp_6(q)$	Representative (from [47])
B_0	$h(\omega,\omega,\omega)$	$\begin{cases} c_{5,0} & \epsilon=1, \\ c_{6,0} & \epsilon=-1 \end{cases}$	$\begin{cases} h(\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_1}) & \epsilon=1, \\ h(\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_1}) & \epsilon=-1 \end{cases}$ taking $i_1=(q+\epsilon)/3$
$C_{11}(i)$	$h(\gamma^i,\gamma^{-2i},\gamma^i)$	$c_{14,0}$	$h(\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_2})$ taking $i_2=2i_1$
$C_{21}(i)$	$h(\gamma^i,\gamma^{-i},1)$	$c_{8,0}$	$h(\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_1},1)$
$C(i,j)$	$h(\gamma^i,\gamma^j,\gamma^{-i-j})$	$c_{22,0}$	$h(\widetilde{\zeta}_1^{i_1},\widetilde{\zeta}_1^{i_2},\widetilde{\zeta}_1^{i_3})$ taking $i_3=i_1+i_2$
$D_{11}(i)$	$h(\eta^i,\eta^{-2i},\eta^i)$	$c_{21,0}$	$h(\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_2})$ taking $i_2=2i_1$
$D_{21}(i)$	$h(\eta^i,\eta^{-i},1)$	$c_{10,0}$	$h(\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_1},1)$
$D(i,j)$	$h(\eta^i,\eta^j,\eta^{-i-j})$	$c_{29,0}$	$h(\widetilde{\xi}_1^{i_1},\widetilde{\xi}_1^{i_2},\widetilde{\xi}_1^{i_3})$ taking $i_3=i_1+i_2$
$E_1(i)$	$h(\theta^i,\theta^{(q-1)i},\theta^{-qi})$	$c_{26,0}$	$h(\widetilde{\zeta_2}^{i_1},\widetilde{\zeta}_2^{qi_1},\widetilde{\xi}_1^{i_2})$ taking $i_2=i_1$
$E_2(i)$	$h(\theta^i,\theta^{qi},\theta^{-(q+1)i})$	$c_{24,0}$	$h(\widetilde{\zeta_2}^{i_1},\widetilde{\zeta}_2^{qi_1},\widetilde{\zeta}_1^{i_2})$ taking $i_2=i_1$
$E_3(i)$	$h(\tau^i,\tau^{qi},\tau^{q^2i})$	$c_{28,0}$	$h(\widetilde{\zeta_3}^{i_1},\widetilde{\zeta}_3^{qi_1},\widetilde{\zeta}_3^{q^2i_1})$ taking $i_1=(q-1)i$
$E_4(i)$	$h(\sigma^i,\sigma^{-qi},\sigma^{q^2i})$	$c_{31,0}$	$h(\widetilde{\xi_3}^{i_1},\widetilde{\xi}_3^{qi_1},\widetilde{\xi}_3^{q^2i_1})$ taking $i_1=(q+1)i$

We note that the result for B_0 depends on ϵ since the element ω of $\mathbb{F}_{q^2}^{\times}$ in the notation of [22] is $\eta^{(q+1)/3}$ if $\epsilon = -1$ and $\gamma^{(q-1)/3}$ if $\epsilon = 1$, where η is a $(q+1)$th root of unity and γ is a $(q-1)$th root of unity.

Now, for arbitrary elements, we use the fact that conjugate elements must have conjugate semisimple and unipotent parts. In the cases of the classes $c_{14,1}(i)$, $c_{21,1}(i)$ in $Sp_6(q)$, these are the only non-semisimple classes with semisimple part in the appropriate class, from which we deduce

$$C_{12}(i) \subset c_{14,1}(i) \cap H \quad \text{and} \quad D_{12}(i) \subset c_{21,1}(i) \cap H.$$

For $\mathcal{C} = C_{22}(i), D_{22}(i), B_1$ in $G_2(q)$, comparing the dimensions of the eigenspaces of the unipotent parts of the classes in $Sp_6(q)$ that have semisimple part in the same class as that of the representative for $\mathcal{C}$, we obtain only one possibility in each case, yielding

$$C_{22}(i) \subset c_{8,3}(i), \quad D_{22}(i) \subset c_{10,3}(i), \quad B_1 \subset \begin{cases} c_{5,1} & \text{if } \epsilon = 1 \\ c_{6,1} & \text{if } \epsilon = -1 \end{cases}$$

This leaves only the classes $B_2(0), B_2(1), B_2(2)$, and the classes of elements of order 8 in $G_2(q)$. For these classes, we again utilize the fact that the scalar product of characters must be integral. Note that the character ρ_3^1 is the character $\chi_{1,4}$ in the notation of [47] and the character α_3 is the character $\chi_{1,5}$ in the notation of [47], and that for the classes whose fusions have been calculated so far, these characters agree with the characters θ_2 and θ_2' of $G_2(q)$, respectively, in the notation of [22]. Also note that to compute $\left[\rho_3^1|_{G_2(q)}, \rho_3^1|_{G_2(q)}\right]$ or $\left[\alpha_3|_{G_2(q)}, \alpha_3|_{G_2(q)}\right]$, the fusion of the order-8 classes is not needed, since the absolute value of each of these characters is the same on all such elements of $Sp_6(q)$.

Suppose that any of $B_2(0), B_2(1)$, or $B_2(2)$ fuses with B_1 in $Sp_6(q)$. Then for $\epsilon = 1$, $\left[\rho_3^1|_{G_2(q)}, \rho_3^1|_{G_2(q)}\right]$ is not an integer since $[\theta_2, \theta_2]$ is an integer. If $\epsilon = -1$, then $\left[\alpha_3|_{G_2(q)}, \alpha_3|_{G_2(q)}\right]$ is not an integer, using the fact that $[\theta_2', \theta_2']$ is an integer. Since there is only one other non-semisimple conjugacy class in $Sp_6(q)$ with the same semisimple part, this contradiction yields that $B_2(0), B_2(1)$, and $B_2(2)$ must fuse in

$Sp_6(q)$, and

$$B_2(0) \cup B_2(1) \cup B_2(2) \subset \begin{cases} c_{5,2} & \text{if } \epsilon = 1, \\ c_{6,2} & \text{if } \epsilon = -1 \end{cases} \cap G_2(q)$$

Finally, we may return to the order-8 unipotent classes. If the two classes A_{51}, A_{52} fused in $Sp_6(q)$, then we would have that ρ_3^1 agrees with the character θ_2 on all conjugacy classes of $G_2(q)$ except either A_{51} or A_{52}. Using this fact, we can calculate $\left[\rho_3^1|_{G_2(q)}, \theta_2\right]$ to see that it is not an integer, so these two classes cannot fuse. If A_{51} was contained in $c_{1,11}$ and A_{52} was in $c_{1,10}$, we would again see that $\left[\rho_3^1|_{G_2(q)}, \theta_2\right]$ is not an integer, so we must have

$$A_{51} \subset c_{1,10} \cap G_2(q) \quad \text{and} \quad A_{52} \subset c_{1,11} \cap G_2(q),$$

which completes the calculation of the fusions of classes of $G_2(q)$ into $Sp_6(q)$.

Table 5.3: The Fusion of Classes from $G_2(q)$ into $Sp_6(q)$

(a)

Class in $G_2(q)$	Class in $Sp_6(q)$
A_0	$c_{1,0}$
A_1	$c_{1,2}$
A_2	$c_{1,4}$
A_{31}	$\begin{cases} c_{1,5} & \text{if } \epsilon = 1, \\ c_{1,6} & \text{if } \epsilon = -1 \end{cases}$
A_{32}	$\begin{cases} c_{1,6} & \text{if } \epsilon = 1, \\ c_{1,5} & \text{if } \epsilon = -1 \end{cases}$
A_4	$\begin{cases} c_{1,5} & \text{if } \epsilon = 1, \\ c_{1,6} & \text{if } \epsilon = -1 \end{cases}$
A_{51}	$c_{1,10}$
A_{52}	$c_{1,11}$

(b)

Class in $G_2(q)$	Class in $Sp_6(q)$
B_0	$\begin{cases} c_{5,0} & \text{if } \epsilon = 1, \\ c_{6,0} & \text{if } \epsilon = -1 \end{cases}$
B_1	$\begin{cases} c_{5,1} & \text{if } \epsilon = 1, \\ c_{6,1} & \text{if } \epsilon = -1 \end{cases}$
$B_2(0)$	$\begin{cases} c_{5,2} & \text{if } \epsilon = 1, \\ c_{6,2} & \text{if } \epsilon = -1 \end{cases}$
$B_2(1)$	$\begin{cases} c_{5,2} & \text{if } \epsilon = 1, \\ c_{6,2} & \text{if } \epsilon = -1 \end{cases}$
$B_2(2)$	$\begin{cases} c_{5,2} & \text{if } \epsilon = 1, \\ c_{6,2} & \text{if } \epsilon = -1 \end{cases}$

(c)

Class in $G_2(q)$	Class in $Sp_6(q)$
$C_{11}(i)$	$c_{14,0}$
$C_{12}(i)$	$c_{14,1}$
$C_{21}(i)$	$c_{8,0}$
$C_{22}(i)$	$c_{8,3}$
$C(i,j)$	$c_{22,0}$
$D_{11}(i)$	$c_{21,0}$
$D_{12}(i)$	$c_{21,1}$
$D_{21}(i)$	$c_{10,0}$
$D_{22}(i)$	$c_{10,3}$
$D(i,j)$	$c_{29,0}$
$E_1(i)$	$c_{26,0}$
$E_2(i)$	$c_{24,0}$
$E_3(i)$	$c_{28,0}$
$E_4(i)$	$c_{31,0}$

Table 5.4: Unipotent Classes of B and Fusion into P and H

Size of Class	Class in B	Class in P	Class in H
1	A_0	A_0	A_0
$q-1$	A_1	A_1	A_1
$q(q-1)$	A_2	A_2	A_1
$q^2(q-1)$	A_3	A_3	A_2
$q^2(q-1)$	A_{41}	A_3	A_2
$\frac{1}{2}q^2(q-1)^2$	A_{42}	A_{41}	A_{31}
$\frac{1}{2}q^2(q-1)^2$	A_{43}	A_{42}	A_{32}
$q^2(q-1)$	A_{51}	A_2	A_1
$\begin{cases} q^2(q-1)^2 & \epsilon=-1 \\ \frac{1}{3}q^2(q-1)^2 & \epsilon=1 \end{cases}$	$A_{52}(i)$, $0 \le i \le 1+\epsilon$	$\begin{cases} A_{42} & & \epsilon=-1 \\ A_{41} & i=0 & \epsilon=1 \\ A_5 & i\neq 0 & \epsilon=1 \end{cases}$	$\begin{cases} A_{32} & & \epsilon=-1 \\ A_{31} & i=0 & \epsilon=1 \\ A_4 & i\neq 0 & \epsilon=1 \end{cases}$
$q^2(q-1)^2$ each	$A_{53}(t)$, $t \in \mathbb{F}_q$	$\begin{cases} A_3 & t=0 \\ A_{42} & t\in\Omega_1 \\ A_{41} & t\in\Omega_2 \\ A_5 & t\in\Omega_3 \end{cases}$	$\begin{cases} A_2 & t=0 \\ A_{32} & t\in\Omega_1 \\ A_{31} & t\in\Omega_2 \\ A_4 & t\in\Omega_3 \end{cases}$
$q^3(q-1)$	A_{61}	A_{61}	A_2
$\frac{1}{2}q^3(q-1)^2$	A_{62}	A_{62}	A_{31}
$\frac{1}{2}q^3(q-1)^2$	A_{63}	A_{63}	A_{32}
$\frac{1}{2}q^4(q-1)^2$	A_{71}	A_{71}	A_{51}
$\frac{1}{2}q^4(q-1)^2$	A_{72}	A_{72}	A_{52}

5.3.2 The Complex Case

In this section, we consider irreducible ordinary characters $\chi \in \mathrm{Irr}(Sp_6(q))$ which restrict irreducibly to $G_2(q)$. We also discuss the decompositions of the linear Weil character, ρ_3^2 and unitary Weil character, β_3, which are reducible over $G_2(q)$.

Theorem 5.3.1. *Let $G = Sp_6(q)$, $H = G_2(q)$ with $q \geq 4$ even. Suppose that V is an absolutely irreducible ordinary G-module. Then V is irreducible over H if and only if V affords one of the Weil characters*

- ρ_3^1, *of degree* $\frac{1}{2}q(q+1)(q^3+1)$,
- τ_3^i, $1 \leq i \leq ((q-1)_{\ell'} - 1)/2$, *of degree* $(q^2+q+1)(q^3+1)$,
- α_3, *of degree* $\frac{1}{2}q(q-1)(q^3-1)$,
- ζ_3^i, $1 \leq i \leq ((q+1)_{\ell'} - 1)/2$, *of degree* $(q^2-q+1)(q^3-1)$.

Proof. Assume $V|_H$ is irreducible. Using [49] to compare character degrees of H and G, we see that the degrees of the characters $\rho_3^1, \tau_3^i, \alpha_3, \zeta_3^i$ are the only irreducible complex character degrees of G which also occur as irreducible character degrees of H. Moreover, we see that these character degrees appear in G with exactly the multiplicity given in the statement of the theorem. Thus it suffices to show that each such character indeed is irreducible when restricted to H.

Note that from [40], the characters τ_3^i for $1 \leq i \leq (q-2)/2$ actually restrict irreducibly from $GL_6(q)$ to $G_2(q)$, and $\tau_3^i|_{G_2(q)} = \chi_3(i)$ in the notation of [22].

We use the fusion of the classes of H into G found in Section 5.3.1 to compute the character values of ζ_3^i on each class. The class representatives for G found in [47] are given in their Jordan-Chevelley decompositions, from which we can find the eigenvalues from the semisimple part (as discussed briefly in Section 5.3.1), and the total number of Jordan blocks (and therefore the dimensions of the eigenspaces over $\mathbb{F}_{q^2}$ for the relevant eigenvalues) from the unipotent part. We then obtain the values of

ζ_3^i by using the formula (4.2.2). We conclude that $\zeta_3^i|_H$ agrees with the character $\chi_3'(i)$ of H in the notation of [22], and therefore is irreducible on H for each $1 \leq i \leq q/2$.

The cases of the characters ρ_3^1 and α_3 are easier, since we see that in the notation of [47], ρ_3^1 is the unipotent character $\chi_{1,4}$ and α_3 is the unipotent character $\chi_{1,5}$. Given the fusion of classes found in Section 5.3.1, we see that $\chi_{1,4}|_H$ agrees with the character θ_2 in [22] and $\chi_{1,5}|_H$ agrees with the character θ_2' in [22], meaning that ρ_3^1 and α_3 are therefore irreducible when restricted to $G_2(q)$.

□

Note that Theorem 5.3.1 tells us that the only characters which restrict irreducibly from $Sp_6(q)$ to $G_2(q)$ are Weil characters. Before moving on to the ℓ-modular case, we briefly discuss the restriction to $G_2(q)$ of the Weil characters missing from Theorem 5.3.1 and show that they restrict as the sum of two irreducible characters.

Theorem 5.3.2. *Let q be a power of 2. Then*

1. *the linear Weil character ρ_3^2 in $\mathrm{Irr}(Sp_6(q))$ decomposes over $G_2(q)$ as*

$$(\rho_3^2)|_{G_2(q)} = \theta_1 + \theta_4,$$

and

2. *the unitary Weil character β_3 in $\mathrm{Irr}(Sp_6(q))$ decomposes over $G_2(q)$ as*

$$(\beta_3)|_{G_2(q)} = \theta_1' + \theta_4,$$

where $\theta_1, \theta_1', \theta_4 \in \mathrm{Irr}(G_2(q))$ are the characters of degrees $\frac{1}{6}q(q+1)^2(q^2+q+1)$, $\frac{1}{6}q(q-1)^2(q^2-q+1)$, and $\frac{1}{3}q(q^4+q^2+1)$, respectively, as in the notation of Enomoto and Yamada, [22].

Proof. This follows from the fusion of conjugacy classes found in Section 5.3.1 and the character tables in [47] and [22], noting that the character ρ_3^2 and β_3 are given by $\chi_{1,2}$ and $\chi_{1,3}$, respectively, in the notation of [47]. (Indeed, by comparing degrees, and using [49] for the multiplicities of degrees in $Sp_6(q)$, we know that these are the correct characters.) $\square$

5.3.3 The Modular Case

In this section, we consider more generally the irreducible Brauer characters $\chi \in \mathrm{IBr}_\ell(Sp_6(q))$ in characteristic $\ell \neq 2$ which restrict irreducibly to $G_2(q)$.

Theorem 5.3.3. *Let $G = Sp_6(q)$, $H = G_2(q)$ with $q \geq 4$ even. Let $\ell \neq 2$ and suppose $\chi \in \mathrm{IBr}_\ell(G)$ is one of the following:*

- $\widehat{\rho}_3^1 - \begin{cases} 1, & \ell | \frac{q^3-1}{q-1}, \\ 0, & otherwise \end{cases},$
- $\widehat{\tau}_3^i$, $1 \leq i \leq ((q-1)_{\ell'} - 1)/2$,
- $\widehat{\alpha}_3$,
- $\widehat{\zeta}_3^i$, $1 \leq i \leq ((q+1)_{\ell'} - 1)/2$.

Then $\chi|_H \in \mathrm{IBr}_\ell(H)$.

Proof. We may assume that $\ell || G|$, since otherwise the result follows from Theorem 5.3.1. We consider the cases ℓ divides $(q-1), (q+1), (q^2-q+1), (q^2+q+1)$, and (q^2+1) separately.

If $\ell|(q-1)$, then $(\rho_3^1)|_H = X_{15}$ in [32],[29]. From [32, Table I], we see that if $\ell = 3$, then indeed $\widehat{X}_{15} - 1_H$ is an irreducible Brauer character of H. From [29], we see that if $\ell \neq 3$, then $\widehat{X}_{15}$ is an irreducible Brauer character. We also see that $(\alpha_3)|_H$ has defect 0, so indeed $(\widehat{\alpha}_3)|_H \in \mathrm{IBr}_\ell(H)$.

By [32] and [29], $(\widehat{\zeta}_3^i)_H = \widehat{X}'_{2a}$ is an irreducible Brauer character, and the $((q-1)_{\ell'} - 1)/2$ characters $(\widehat{\tau}_3^i)|_H = \widehat{X}'_{1b}$ which lie outside the the principal block are also irreducible Brauer characters, completing the proof in the case $\ell|(q-1)$.

Now let $\ell|(q+1)$. In this case, Hiss and Shamash show in [32] and [29] that $(\widehat{\tau}_3^i)|_H = \widehat{X}'_{1b}$ is an irreducible Brauer character and the $\frac{(q+1)_{\ell'}-1}{2}$ characters $(\widehat{\zeta}_3^i)_H = \widehat{X}'_{2a}$ lying outside the principal block are irreducible Brauer characters. Also, from [32, Section 3.3] and [29, Section 2.2], $\widehat{X}_{17} = \widehat{\alpha}_3|_H \in \mathrm{IBr}_\ell(H)$. Finally, note that $(\rho_3^1)|_H$ has defect 0, which completes the proof in the case $\ell|(q+1)$.

Suppose $\ell|(q^2-q+1)$, where $\ell \neq 3$. From [68, Section 2.1], we see that X_{17} lies in the principal block with cyclic defect group and that $\widehat{X}_{17} \in \mathrm{IBr}_\ell(H)$. As this character is the restriction of α_3 to H, we have $(\widehat{\alpha}_3)|_H \in \mathrm{IBr}_\ell(H)$. We see from their degrees that X_{15}, X'_{1b}, and X'_{2a} are all of defect 0, so their restrictions to ℓ-regular elements are irreducible Brauer characters of H. But these are exactly the restrictions to H of the characters ρ_3^1, τ_3^i, and ζ_3^i, respectively, which completes the proof in the case $\ell|(q^2-q+1)$.

Now assume $\ell|(q^2+q+1)$, where $\ell \neq 3$. Then from the Brauer tree for H given in [68, Section 2.1], we see that $\widehat{X}_{15} - 1 \in \mathrm{IBr}_\ell(H)$, and since $(\rho_3^1)|_H = X_{15}$ in Shamash's notation, this shows that $\widehat{\rho_3^1} - 1$ restricts irreducibly to H. Also, X_{17}, X'_{2a}, and X'_{1b} have defect 0, so $\widehat{X}_{17}, \widehat{X}'_{2a}$, and $\widehat{X}'_{1b} \in \mathrm{IBr}(H)$ as well. As $(\alpha_3)|_H = X_{17}, (\zeta_3^k)|_H = X'_{2a}$, and $(\tau_3^k)|_H = X'_{1b}$ in Shamash's notation, it follows that all of the characters claimed indeed restrict irreducibly to H, completing the proof in the case $\ell|(q^2+q+1)$.

Finally, if $\ell|(q^2+1)$, then ℓ does not divide $|H|$, which means that $\mathrm{IBr}_\ell(H) = \mathrm{Irr}(H)$, and the result is clear from Theorem 5.3.1.

$\square$

Theorem 5.3.4. *Let $G = Sp_6(q)$, $H = G_2(q)$ with $q \geq 4$ even. Suppose that V is an absolutely irreducible G-module in characteristic $\ell \neq 2$. Then V is irreducible over H if an only if the ℓ-Brauer character afforded by V is one of the Weil characters*

- $\widehat{\rho}_3^1 - \begin{cases} 1, & \ell|\frac{q^3-1}{q-1}, \\ 0, & \textit{otherwise} \end{cases},$
- $\widehat{\tau}_3^i$*,* $1 \leq i \leq ((q-1)_{\ell'}-1)/2$*,*

- $\widehat{\alpha}_3$,
- $\widehat{\zeta}_3^i$, $1 \leq i \leq ((q+1)_{\ell'} - 1)/2$.

Proof. If V affords one of the characters listed, then V is irreducible on H by Theorem 5.3.3. Conversely, assume that V is irreducible on H and let $\chi \in \mathrm{IBr}_\ell(G)$ denote the ℓ-Brauer character afforded by V. If χ lifts to a complex character, then the result follows from Theorem 5.3.1, so we assume χ does not lift. We may therefore assume that ℓ is an odd prime dividing $|G|$. We note that $\chi(1) \leq \mathfrak{m}(H) \leq (q+1)^2(q^4+q^2+1)$ by [49], and if $q = 4$, then $\mathfrak{m}(H) = q(q+1)(q^4+q^2+1)$.

Since $(q-1)(q^2+1)(q^4+q^2+1) > \mathfrak{m}(H)$ when $q \geq 4$, it follows from part (B) of Theorem 1.1.1 that either χ lifts to an ordinary character or χ lies in a unipotent block of G. In the first situation, Theorem 5.3.1 and Lemma 5.1.2 imply that χ is in fact one of the characters listed in the statement. Therefore, we may assume that χ lies in a unipotent block of G and does not lift to a complex character. Again we treat each case for ℓ separately.

Since $\mathfrak{m}(H)$ is smaller than the degree of each of the characters listed in situation A(3) of Theorem 1.1.1, we see that the only irreducible Brauer characters which do not lift to a complex character and whose degree does not exceed $\mathfrak{m}(H)$ are $\widehat{\rho}_3^2 - 1_G$ and $\widehat{\beta}_3 - 1_G$ when $\ell|(q+1)$, $\widehat{\rho}_3^2 - 1_G$ in the case $3 \neq \ell|(q^2-q+1)$, $\widehat{\rho}_3^1 - 1_G$ in the case $\ell|(q^2+q+1)$, and $\widehat{\chi}_6 - 1_G$ when $\ell|(q^2+1)$.

From Theorem 5.3.2, we know that $(\rho_3^2)|_{G_2(q)} = \theta_1 + \theta_4$ and $(\beta_3)|_{G_2(q)} = \theta_1' + \theta_4$ in the notation of [22]. Also, $\theta_4 = X_{14}, \theta_1 = X_{16}$, and $\theta_1' = X_{18}$ in the notation of Shamash and Hiss.

Suppose $\ell|(q+1)$. From [29, Section 2.2], we know that $\widehat{X}_{14} - 1 \in \mathrm{IBr}_\ell(H)$ when $\ell \neq 3$, and therefore neither $\widehat{\rho_3^2} - 1$ nor $\widehat{\beta}_3 - 1$ can restrict irreducibly to $\mathrm{IBr}_\ell(H)$. If $\ell = 3$, then by [32, Section 3.3], $\widehat{X}_{14} + \widehat{X}_{18} - 1 \notin \mathrm{IBr}_\ell(H)$, since this is $\varphi_{14} + 2\varphi_{18}$ in the notation of [32, Table II]. Similarly, $\widehat{X}_{14} + \widehat{X}_{16} - 1 \notin \mathrm{IBr}_\ell(H)$, so we have shown that if $\ell = 3$, again neither $\widehat{\rho_3^2} - 1$ nor $\widehat{\beta}_3 - 1$ can restrict irreducibly to $\mathrm{IBr}_\ell(H)$.

Suppose $\ell|(q^2-q+1)$, where $\ell \neq 3$. From [68, Section 2.1], the Brauer character $\widehat{X}_{16}-1$ is irreducible, and X_{14} is defect zero, meaning that $\widehat{X}_{14}$ is also irreducible. But this implies that $\widehat{X}_{14}+\widehat{X}_{16}-1$ is not irreducible. Recalling again that $X_{14}=\theta_4$ and $X_{16}=\theta_1$, this shows that $\widehat{\rho}_3^2-1_G$ does not restrict irreducibly to H.

If $\ell|(q^2+q+1)=\frac{q^3-1}{q-1}$, then we know that $\widehat{\rho}_3^1-1_G$ is irreducible, by Theorem 5.3.3, so we are done in this case.

Finally, if $\ell|(q^2+1)$, then ℓ cannot divide $|H|$, which means that $\mathrm{IBr}_\ell(H)=\mathrm{Irr}(H)$, and every irreducible Brauer character of H lifts to $\mathbb{C}$. Since the degree of $\widehat{\chi}_6-\widehat{\chi}_1$ is not the degree of any element of $\mathrm{Irr}(H)$, we know χ cannot be $\widehat{\chi}_6-\widehat{\chi}_1$, and the proof is complete.

□

5.3.4 Descent to Subgroups of $G_2(q)$

We now consider subgroups H of $Sp_6(q)$ such that $H<G_2(q)$. In [55], Nguyen finds all triples as in Problem 1 when $G=G_2(q)$ and H is a maximal subgroup. Noting that none of the representations described in [55] to give triples for $G=G_2(q)$ come from the Weil characters listed in Theorem 5.3.4, it follows that there are no proper subgroups of H of $G_2(q)$ that yield triples as in Problem 1 for $G=Sp_6(q)$.

5.4 Restrictions of Irreducible Characters of $Sp_6(q)$ to the Subgroups $O_6^{\pm}(q)$

In this section, let $q\geq 4$ be a power of 2, $G=Sp_6(q)$, and $H^{\pm}\cong O_6^{\pm}(q)$ as a subgroup of G. Since q is even, we have $H^{\pm}=\Omega_6^{\pm}(q).2\cong L_4^{\pm}(q).2$ (see [37, Chapter 2]). This means that there is an index-2 subgroup of $H^{\pm}$, which we will denote $K^{\pm}$, which satisfies

$$K^{\epsilon}\cong L_4^{\epsilon}(q)=\begin{cases} SL_4(q) & \epsilon=+\\ SU_4(q) & \epsilon=-.\end{cases}$$

We at times may simply refer to H and K rather than $H^{\pm}$ and $K^{\pm}$ if the result is true in either case.

The following lemma describes the order-2 automorphisms of $K^{\pm}$ inside $H^{\pm}$, which will be useful when applying Clifford theory to these groups.

Lemma 5.4.1. *The order-2 automorphism of $K^- = \Omega_6^-(q)$ identified with $L_4^-(q)$ inside $H^- = O_6^-(q) = \Omega_6^-(q).2$ identified with $L_4^-(q).2$ is given by $\tau\colon (a_{ij}) \mapsto (a_{ij}^q)$. The order-2 automorphism of $K^+ = \Omega_6^+(q)$ identified with $L_4^+(q)$ inside $H^+ = O_6^+(q) = \Omega_6^+(q).2$ identified with $L_4^+(q).2$ is given by $\sigma\colon A \mapsto (A^{-1})^T$.*

Proof. The first statement can be seen easily since $\mathrm{Out}(K^-)$ is cyclic (see, for example, [37, Section 2.3]) so K^- has only one order-2 outer automorphism.

From [37, Chapter 2], we see that $\Omega_6^+(q)$ is the index-2 subgroup of $O_6^+(q)$ composed of elements that can be written as a product of an even number of reflections. Hence the order-2 automorphism can be given by conjugation by any element of $O_6^+(q)$ which is a product of an odd number of reflections. In particular, the matrix $J_3 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$ can be written as the product of 3 reflections. Namely, J_3 is the product $r_1r_2r_3$ of the reflections r_i switching the standard basis elements e_i and f_i.

Now, from [37, Chapter 2], the identification of $L_4^+(q)$ with $\Omega_6^+(q)$ is given by the action of $L_4^+(q)$ on the second wedge space $\Lambda^2(W)$ of the natural module $W = \mathbb{F}_q^4$ for $L_4^+(q)$. We claim that the automorphism $\sigma\colon A \mapsto (A^{-1})^T$ of $L_4^+(q)$ corresponds to conjugation by J_3 in $O_6^+(q)$ under this identification. Certainly for $g \in \Omega_6^+(q)$, we have $gJ_3g^T = J_3$, so $J_3gJ_3 = (g^{-1})^T$. Hence it suffices to note that by direct calculation, the transpose of an element in $L_4^+(q)$ acting on its natural module corresponds to the transpose of the corresponding action on the wedge space. $\square$

The purpose of this section is to show that restrictions of nontrivial representations of G to H are reducible. We again begin with the complex case.

Theorem 5.4.2. *Let $G = Sp_6(q)$ and $H = O_6^{\pm}(q)$, with $q \geq 4$ even. If $1_G \neq \chi \in \mathrm{Irr}(G)$, then χ_H is reducible.*

Proof. Assume that $\chi|_H$ is irreducible. For the list of irreducible complex character degrees of $K^{\pm} \cong L_4^{\pm}(q)$ and $G = Sp_6(q)$, we refer to [49].

As $[H : K] = 2$, we see from Clifford theory that χ_H has degree $e \cdot \phi(1)$ where $e \in \{1, 2\}$ and $\phi \in \mathrm{Irr}(K^{\pm})$. Inspecting the list of character degrees for $K^{\pm}$ and for G, we see that for $q > 4$ there is only one character degree of G which matches a character degree or twice a character degree for $K^{\pm}$. It follows that the only option for $\chi(1)$ is $(q^2+1)(q^2-q+1)(q+1)^2$ in case $-$ and $(q^2+1)(q^2+q+1)(q-1)^2$ in case $+$, and that $e = 1$. This means that if $\chi|_H$ is irreducible, then $\chi|_K$ must also be irreducible.

From [47], we see that these characters are $\chi = \chi_{8,1}$ and $\chi_{9,1}$, respectively. In Lübeck's notation [47], the characters can be written

$$\chi_{8,1} = \frac{1}{6}(R_{8,1} + 3R_{8,3} + 2R_{8,7})$$

and

$$\chi_{9,1} = \frac{1}{6}(3R_{9,5} + R_{9,8} + 2R_{9,10}).$$

In particular, on unipotent elements, these characters satisfy

$$\chi_{8,1} = \frac{1}{6}(Q_{1,1} + 3Q_{1,3} + 2Q_{1,7})$$

and

$$\chi_{9,1} = \frac{1}{6}(3Q_{1,5} + Q_{1,8} + 2Q_{1,10}),$$

where Q_{ij} is the Green function from [47, Tabelle 16]. (In Lübeck's notation, i is the index of the semisimple element, and j is the index of the torus.) We can use this to see that on the classes of involutions, the values of $\chi_{8,1}$ and $\chi_{9,1}$ are as shown below:

	$c_{1,0}=\{1\}$	$c_{1,1}$	$c_{1,2}$
$\chi_{8,1}$	$(q^2+1)(q^2-q+1)(q+1)^2$	$(q+1)(q^2+1)$	$(q+1)(q^3+q^2+1)$
$\chi_{9,1}$	$(q^2+1)(q^2+q+1)(q-1)^2$	$-(q-1)(q^2+1)$	$(q-1)(q^3-q^2-1)$

	$c_{1,3}$	$c_{1,4}$
$\chi_{8,1}$	$(q+1)(q^2+1)$	q^2+q+1
$\chi_{9,1}$	$-(q-1)(q^2+1)$	q^2-q+1

However, from [58], the characters $\chi_2(k,\ell)$ of $SU_4(q)$ which have degree $(q^2+1)(q^2-q+1)(q+1)^2$, have the value $2q^2+q+1$ on one of the classes of involutions. (Here we may also use the character table for $GU_4(q)\cong C_{q+1}\times L_4^-(q)$ constructed by F. Lübeck for the CHEVIE system [26].) Since this value does not occur on any of the involution classes of G for $\chi_{8,1}$, we therefore see that $\chi_{8,1}$ does not restrict irreducibly to H^-. Similarly, the characters of $SL_4(q)$ of degree $(q^2+1)(q^2+q+1)(q-1)^2$ have the value $2q^2-q+1$ on one of the involution classes (see, for example, the character table for $GL_4(q)\cong C_{q-1}\times L_4^+(q)$ constructed by F. Lübeck for the CHEVIE system [26]), so $\chi_{9,1}$ also does not restrict irreducibly to H^+.

Thus for $q>4$, there are no irreducible characters of G which restrict irreducibly to H.

In the case $q=4$, there are additional character degrees $\phi(1)$ of K for which $2\phi(1)$ is a character degree for G. For $K^-\cong SU_4(4)$, these degrees are $(q^2+1)(q^2-q+1)=221$ and $(q+1)^2(q^2-q+1)=325$, and for $K^+\cong SL_4(4)$, they are $(q^2+q+1)(q-1)^2=189$ and $(q^2+q+1)(q^2+1)=357$. For each of these degrees, there is exactly one character of $Sp_6(4)$ with twice that degree. Using GAP [24] and the GAP Character Table Library [11], we can find the character tables explicitly for $Sp_6(4), SU_4(4)$, and $SL_4(4)$.

There are exactly two characters of degree 221 in $\mathrm{Irr}(SU_4(4))$, and one of degree 442 in G. Namely, this degree-442 character of G is the Weil character β_3. From direct calculation in GAP (in particular using the "PossibleClassFusions" function), we see that the restriction of β_3 to K^- is indeed the sum of these two characters. However, these characters extend to irreducible characters of H^-, which can be seen as follows.

These characters are $\chi_7, \chi_8 \in \mathrm{Irr}(K^-)$ in the notation of the GAP Character Table Library [11]. However, by Lemma 5.4.1, the order-2 automorphism of K^- inside H^- is given by $\tau\colon (a_{ij}) \mapsto (a_{ij}^q)$. Thus if χ_7 and χ_8 do not extend, then we must have $\chi_7^\tau = \chi_8$, since H^-/K^- is cyclic. Now, when using GAP to construct the conjugacy class representatives of $SU_4(4)$:

```
List(ConjugacyClasses(SpecialUnitaryGroup(4,4)), x->Representative(x));
```

we see that classes 48 and 49 must correspond to the set $\{5e, 5f\}$ in the notation of [11], by inspection of the size of the centralizer. We note that $\chi_7 \neq \chi_8$ on the conjugacy classes 5e, 5f. Now, creating a function for τ in GAP by

```
tau:=function(r)
local z;
  z:=[List(r[1], x->x^4),List(r[2], x->x^4),
      List(r[3], x->x^4),List(r[4], x->x^4)];
return z;
end;
```

and using the"IsConjugate" function, we see that elements A of each of these classes are conjugate in $SU_4(4)$ to $\tau(A)$. But this means that $\chi_7^\tau(A) = \chi_7(A) \neq \chi_8(A)$, meaning that χ_7 and χ_8 must be fixed by τ, and therefore must be extendable to $SU_4(4).2$. Thus the restriction of β_3 is reducible, as it restricts to H^- as the sum of two characters.

Along the same lines, there are two characters of degree 189 in $\mathrm{Irr}(SL_4(4))$, and one of degree 378 in G (namely, α_3), and from direct calculation in GAP as above, we see that the restriction of α_3 to K^+ is again the sum of these two characters. However, we claim that these characters again extend to irreducible characters of H^+. Indeed, the characters of K^+ in question are χ_{52} and χ_{63} in the notation of [11]. By Lemma 5.4.1, the order-2 automorphism of K^+ inside H^+ is given by $\sigma\colon A \mapsto (A^{-1})^T$. Thus if χ_{52} and χ_{63} do not extend, then we must have $\chi_{52}^{\sigma} = \chi_{63}$. Now, when using GAP to construct the conjugacy class representatives of $SL_4(4)$:

```
List(ConjugacyClasses(SpecialLinearGroup(4,4)), x->Representative(x));
```

we see that classes 6 and 8 must correspond to the set $\{5a, 5b\}$ in the notation of [11], by inspection of the size of the centralizer. Now, $\chi_{52} \neq \chi_{63}$ on these classes. But again using the "IsConjugate" function, we see that elements A of each of these classes are conjugate in $SL_4(4)$ to $\sigma(A) = (A^{-1})^T$. But this means that $\chi_{52}^{\sigma}(A) = \chi_{52}(A) \neq \chi_{63}(A)$, so χ_{52} and χ_{63} must be fixed by σ, and therefore extend to $SL_4(4).2$. Thus $\alpha_3|_{H^+}$ is the sum of two characters, so is reducible.

There is exactly one character, ϕ, of degree 325 in $\mathrm{Irr}(SU_4(4))$, which means that if $\chi(1) = 650$, then $\chi|_{K^-} = 2\phi$. Now, as H^-/K^- is cyclic and ϕ is H^--invariant, we see that ϕ must extend to a character of H^-, so $\chi|_{K^-} \neq 2\phi$.

Similarly, there is exactly one character, ϕ, of degree 357 in $\mathrm{Irr}(SL_4(4))$, which means that if $\chi(1) = 714$, then the restriction of χ to K^+ is twice this character. Again, as H^+/K^+ is cyclic and ϕ is H^+-invariant, this is not the case.

□

Lemma 5.4.3. *Let $G = Sp_6(q)$ and $H = O_6^{\pm}(q)$, with $q \geq 4$ even and let $\chi \in \mathrm{Irr}(G)$ be one of the characters $\chi_2, \chi_3, \chi_4, \chi_6$ in the notation of D. White [76] (i.e. $\chi_{12}, \chi_{13}, \chi_{14}, \chi_{16}$ in F. Lübeck's [47] notation). If $\chi|_H - \lambda \in \mathrm{Irr}(H)$ for $\lambda \in \widehat{H}$, then the restriction to $K \cong L_4^{\pm}(q)$ also satisfies $\chi|_K - \lambda|_K \in \mathrm{Irr}(K)$.*

Proof. Write $\theta := \chi|_H - \lambda \in \mathrm{Irr}(H)$. Note that since $q \geq 4$, and $\chi(1)$ is divisible by $\frac{1}{2}q$, we know that $\chi(1)$ is even. In particular, $\theta(1) = \chi(1) - \lambda(1) = \chi(1) - 1$ is odd. Since K has index 2 in H, we know by Clifford theory that

$$\theta_K = \sum_{i=1}^{t} \theta_i$$

where $\theta_i \in \mathrm{Irr}(K)$, each θ_i has the same degree, and $t = [H : \mathrm{stab}_H(\theta_1)]$ divides $[H : K] = 2$. That is, t must be either 1 or 2. This means that if θ_K is reducible, then $\theta(1)$ is even, yielding a contradiction. Hence $\chi|_K - \lambda|_K \in \mathrm{Irr}(K)$. $\square$

Lemma 5.4.4. *Let $q \geq 4$ and χ be one of the characters as in Lemma 5.4.3. Then $\chi|_H - \lambda \notin \mathrm{Irr}(H)$ for any $\lambda \in \widehat{H} \cup \{0\}$.*

Proof. Comparing degrees of characters of G and K (see, for example, [49]), we see that neither $\chi(1)$ nor $\chi(1)/2$ occur as a degree of an irreducible character of K for any of these characters. Then by Clifford theory (see the argument in Lemma 5.4.3), we know that $\chi|_H \notin \mathrm{Irr}(H)$. Moreover, $\chi(1) - 1$ does not occur as an irreducible character degree for K, which means that $\chi|_K - \lambda_K \notin \mathrm{Irr}(K)$ for any $\lambda \in \widehat{H}$. Thus by Lemma 5.4.3, $\chi|_H - \lambda \notin \mathrm{Irr}(H)$ for any $\lambda \in \widehat{H}$. $\square$

The above lemma yields the following:

Corollary 5.4.5. *Let $q \geq 4$ and let ℓ be a prime. If $\chi \in \mathrm{Irr}(G)$ is one of the characters $\chi_2, \chi_3, \chi_4, \chi_6$ in D. White's notation, then $\widehat{\chi}_H - 1_H \notin \mathrm{IBr}_\ell(H)$.*

Proof. This follows immediately from Lemma 5.1.3 and Lemma 5.4.4. $\square$

We are now ready to prove the main theorem of this section, which generalizes Theorem 5.4.2 to the modular case:

Theorem 5.4.6. *Let $H \cong O_6^{\pm}(q)$ be a maximal subgroup of $G = Sp_6(q)$, with $q \geq 4$ even, and let $\ell \neq 2$ be a prime. If $\chi \in \mathrm{IBr}_\ell(G)$ with $\chi(1) > 1$, then the restriction $\chi|_H$ is reducible.*

Proof. Suppose that $\chi|_H$ is irreducible. We first note that from Clifford theory,

$$\mathfrak{m}_\ell(H^{\pm}) = \mathfrak{m}_\ell(K^{\pm}.2) \leq 2\mathfrak{m}_\ell(K^{\pm}).$$

Now

$$\mathfrak{m}_\ell(K^{+}) \leq (q+1)^2(q^2+1)(q^2+q+1)$$

and

$$\mathfrak{m}_\ell(K^-) \leq (q+1)^2(q^2+1)(q^2-q+1)$$

(see, for example, [49]).

Note that $q(q^4+q^2+1)(q-1)^3/2 > \mathfrak{m}_\ell(H^-)$ for $q \geq 4$. Moreover, $q(q^4+q^2+1)(q-1)^3/2 > \mathfrak{m}_\ell(H^+)$, except possibly when $q=4$. However, from [49], we can see that if $q=4$, then in fact $\mathfrak{m}_\ell(K^+) \leq 7140$, so $q(q^4+q^2+1)(q-1)^3/2 > \mathfrak{m}_\ell(H^+)$ in this case as well. Thus we know from Theorem 1.1.1 that either χ lifts to a complex character, or χ lies in a unipotent block.

Suppose that χ lies in a unipotent block of G. Then the character degrees listed in situation A(3) of Theorem 1.1.1 are larger than our bound for $\mathfrak{m}_\ell(H^-)$ for $q \geq 4$ and are larger than $\mathfrak{m}_\ell(H^+)$ unless $q=4$ and $\ell|(q+1)$. (Here we have again used the fact that $\mathfrak{m}_\ell(K^+) \leq 7140$.) Hence, by Theorem 1.1.1, χ either lifts to an ordinary character or is of the form $\widehat{\chi}-1_G$ where χ is one of the characters discussed in Lemma 5.4.4 (and therefore do not remain irreducible over H), except possibly in the case $H = O_6^+(4)$ and $\ell = 5$.

If $q=4$ and $\ell=5$, the bound D in part (A) of Theorem 1.1.1 is larger than 14280, so $\widehat{\chi}_{35}-\widehat{\chi}_5$ is the only additional character we must consider. However, the degree of $\widehat{\chi}_{35}-\widehat{\chi}_5$ is $(q^3-1)(q^4-q^3+3q^2/2-q/2+1) = 13545$, which is odd, so by Clifford theory, if it restricts irreducibly to H^+, then it also restricts irreducibly to the index-2 subgroup K^+. But $7140 < 13545$, a contradiction. Hence $\widehat{\chi}_{35}-\widehat{\chi}_5$ is reducible when restricted to H^+.

We have therefore reduced to the case of complex characters, by Lemma 5.1.2, which by Theorem 5.4.2 are all reducible on H.

$\square$

5.5 Restrictions of Irreducible Characters to Maximal Parabolic Subgroups

The purpose of this section is to prove part (1) of Theorem 1.1.2. We momentarily relax the assumption that $G = Sp_6(q)$, and instead consider the more general case $G = Sp_{2n}(q)$ for $n \geq 2$. Let $\{e_1, ..., e_n, f_1, ..., f_n\}$ denote a symplectic basis for the natural module $\mathbb{F}_q^{2n}$. That is, $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$ is the Kronecker delta for $1 \leq i, j \leq n$, so that the Gram matrix of the symplectic form with isometry group G is

$$J_n := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

as defined in Chapter 2. We will use many results from [27] and will keep the notation used there. In particular, $P_j = \mathrm{stab}_G(\langle e_1, ..., e_j \rangle_{\mathbb{F}_q})$ will denote the jth maximal parabolic subgroup, L_j its Levi subgroup, Q_j its unipotent radical, and $Z_j = Z(Q_j)$.

If we reorder the basis as $\{e_1, ..., e_n, f_{j+1}, ..., f_n, f_1, ..., f_j\}$, then the subgroup Q_j can be written as

$$Q_j = \left\{ \begin{pmatrix} I_j & (A^T)J_{n-j} & C \\ 0 & I_{2n-2j} & A \\ 0 & 0 & I_j \end{pmatrix} : \begin{array}{c} A \in M_{2n-2j,j}(\mathbb{F}_q), C \in M_j(q), \\ C + C^T + (A^T)J_{n-j}A = 0 \end{array} \right\}$$

and

$$Z_j = \left\{ \begin{pmatrix} I_j & 0 & C \\ 0 & I_{2n-2j} & 0 \\ 0 & 0 & I_j \end{pmatrix} : C \in M_j(q), C + C^T = 0 \right\}.$$

In particular, note that in the case $j = n$, Q_n is abelian and $Z_n = Q_n$. Also, $L_j \cong Sp_{2n-2j}(q) \times GL_j(q)$ is the subgroup

$$L_j = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & (A^T)^{-1} \end{pmatrix} : A \in GL_j(q), B \in Sp_{2n-2j}(q) \right\}.$$

Linear characters $\lambda \in \mathrm{Irr}(Z_j)$ are in the form

$$\lambda_Y : \begin{pmatrix} I_j & 0 & C \\ 0 & I_{2n-2j} & 0 \\ 0 & 0 & I_j \end{pmatrix} \mapsto (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(YC))}$$

for some $Y \in M_j(q)$. These characters correspond to quadratic forms q_Y on $\mathbb{F}_q^j = \langle f_1, ..., f_j\rangle_{\mathbb{F}_q}$ defined by $q_Y(f_i) = Y_{ii}$ with associated bilinear form having Gram matrix $Y + Y^T$. The P_j-orbit of the linear characters λ_Y of Z_j is given by the rank r and type $\pm$ of q_Y, denoted by $\mathcal{O}_r^\pm$ for $0 \le r \le j$. We will sometimes denote the corresponding orbit sums by $\omega_r^\pm$. From [27], we can see that for $\lambda \in \mathcal{O}_r^\pm$, the stabilizer in L_j is

$$\mathrm{stab}_{L_j}(\lambda) \cong Sp_{2n-2j}(q) \times \left([q^{r(j-r)}]\colon (GL_{j-r}(q) \times O_r^\pm(q))\right),$$

where $[N]$ denotes the elementary abelian group of order N.

We begin with a theorem proved in [72].

Theorem 5.5.1. *Let $G = Sp_{2n}(q)$. Let Z be a long-root subgroup and assume V is a non-trivial irreducible representation of G. Then Z must have non-zero fixed points on V.*

Proof. This is [72, Theorem 1.6] in the case that G is type C_n. $\square$

Theorem 5.5.1 shows that there are no examples of irreducible representations of G which are irreducible when restricted to P_1.

Corollary 5.5.2. *Let V be an irreducible representation of $G = Sp_{2n}(q)$, q even, which is irreducible on $H = P_1 = \mathrm{stab}_G(\langle e_1\rangle_{\mathbb{F}_q})$. Then V is the trivial representation.*

Proof. Suppose that V is non-trivial and let $\chi \in \mathrm{IBr}_\ell(G)$ denote the Brauer character afforded by V. By Clifford theory, $\chi|_{Z_1} = e\sum_{\lambda\in\mathcal{O}}\lambda$ for some P_1-orbit $\mathcal{O}$ on $\mathrm{Irr}(Z_1)$ and positive integer e. But in this case, Z_1 is a long-root subgroup, so Z_1 has non-zero fixed points on V by Theorem 5.5.1. This means that $\mathcal{O} = \{1_{Z_1}\}$, so $Z_1 \le \ker\chi$, a contradiction since G is simple.

$\square$

We can view $Sp_4(q)$ as a subgroup of G under the identification

$$Sp_4(q) \simeq \mathrm{stab}_G(e_3, ..., e_n, f_3, ..., f_n).$$

With respect to the usual ordering of the symplectic basis above, the embedding is given by

$$Sp_4(q) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{n-2} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{n-2} \end{pmatrix} \in Sp_{2n}(q)$$

where A, B, C, D are each 2×2 matrices. To distinguish between subgroups of $Sp_4(q)$ and $Sp_{2n}(q)$, we will write $P_j^{(n)} := \mathrm{stab}_{Sp_{2n}(q)}(\langle e_1, ..., e_j\rangle)$ for the jth maximal parabolic subgroup of $Sp_{2n}(q)$, $P_j^{(2)}$ for the jth maximal parabolic subgroup of $Sp_4(q)$, and similarly for the subgroups Z_j, Q_j, and L_j. Note that $P_2^{(2)} \leq P_n^{(n)}$ since $Sp_4(q)$ fixes $e_3, ..., e_n$. Moreover, $Z_2^{(2)} \leq Z_n^{(n)}$ since

$$Z_2^{(2)} \ni \begin{pmatrix} I_2 & C \\ 0 & I_2 \end{pmatrix} \mapsto \begin{pmatrix} I_2 & 0 & C & 0 \\ 0 & I_{n-2} & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_{n-2} \end{pmatrix} \in Z_n^{(n)}$$

and $C \in M_2(q)$ satisfies $C + C^T = 0$, so $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}^T = 0$ also.

The following theorem will often be useful when viewing $Sp_4(q)$ as a subgroup of G in this manner.

Theorem 5.5.3. *Let q be even and let V be an absolutely irreducible $Sp_4(q)$-module of dimension larger than 1 in characteristic $\ell \neq 2$. Then V is irreducible on $P_2 = \mathrm{stab}_G(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ if and only if V affords the ℓ-Brauer character $\widehat{\alpha}_2$.*

Proof. Let $Z := Z_2^{(2)}$ be the unipotent radical of P_2. First we claim that $\widehat{\alpha}_2$ is indeed irreducible on P_2. Note that $\widehat{\alpha}_2|_Z = \alpha_2|_Z$ since $Z \setminus \{1\}$ consists of 2-elements. Now, $\alpha_2(1) = |\mathcal{O}_2^-|$, and by Clifford theory it suffices to show that $\alpha_2|_Z = \sum_{\lambda \in \mathcal{O}_2^-} \lambda = \omega_2^-$. From the proof of [27, Proposition 4.1], it follows that nontrivial elements of Z belong to the classes A_{31}, A_2, A_{32} of $Sp_4(q)$. The values of ω_2^- are computed in the proof of [27, Proposition 4.1], and the values of α_2 can be found in [21]. (Note that α_2 is the character θ_5 in the notation of [21].) These character values are as follows:

	1	A_{31}	A_2	A_{32}
ω_2^-	$q(q-1)^2/2$	$-q(q-1)/2$	$-q(q-1)/2$	$q/2$
α_2	$q(q-1)^2/2$	$-q(q-1)/2$	$-q(q-1)/2$	$q/2$

Thus $\alpha_2|_Z = \omega_2^-$, and $\widehat{\alpha}_2$ must be irreducible when restricted to P_2.

Conversely, suppose that χ is the Brauer character afforded by V, and $\chi|_{P_2} = \varphi \in \mathrm{IBr}_\ell(P_2)$. By Clifford theory, $\varphi|_Z = e\sum_{\lambda\in\mathcal{O}}\lambda$ for some nontrivial P_2-orbit $\mathcal{O}$ of $\mathrm{Irr}(Z)$. It follows that φ satisfies condition $\mathcal{W}_2^{\pm}$ of [27], so χ is a Weil character of $Sp_4(q)$ by [27, Theorem 1.2].

Now, following the notation of the proof of [27, Proposition 4.1], we have

$$\zeta_2|_Z = 1_Z + (q+1)\omega_1 + (2q+2)\omega_2^-.$$

Since Z consists of 2-elements, [27, Lemma 3.8] implies that $\zeta_2^i|_Z = \alpha_2|_Z + \beta_2|_Z - 1_Z$ for each $1 \le i \le q/2$, so by the definition of ζ_2 (see [27, Section 3]),

$$\zeta_2|_Z = (q+1)\alpha_2|_Z + (q+1)\beta_2|_Z - q\cdot 1_Z.$$

Since we have already shown $\widehat{\alpha}_2|_Z = \omega_2^-$, it follows that $\widehat{\beta}_2|_Z = 1_Z + \omega_1 + \omega_2^-$. Recalling that $\widehat{\zeta}_2^i|_Z = \widehat{\alpha}_2|_Z + \widehat{\beta}_2|_Z - 1_Z = 2\omega_2^- + \omega_1$, this shows that if χ is any of the unitary Weil characters aside from $\widehat{\alpha}_2$, then $\chi|_Z$ contains as constituents multiple P_2-orbits of characters of Z, a contradiction.

Now suppose χ is a linear Weil character. The values of ω_1 and ω_2^+ on Z are obtained in [27, Proposition 4.1], and the values of ρ_2^1, and ρ_2^2 are obtained in [21]. These values are as follows:

	1	A_{31}	A_2	A_{32}
ω_1	q^2-1	q^2-1	-1	-1
ω_2^+	$q(q^2-1)/2$	$-q(q+1)/2$	$q(q-1)/2$	$-q/2$
ρ_2^1	$q(q^2+1)/2$	$-q(q-1)/2$	$q(q+1)/2$	$q/2$
ρ_2^2	$q(q+1)^2/2$	$q(q+1)/2$	$-q(q-1)/2$	$q/2$

From this we can see that

$$\rho_2^1|_Z = \omega_2^+ + q\cdot 1_Z$$

and

$$\rho_2^2|_Z = (q+1)\cdot 1_Z + \omega_1 + \omega_2^+.$$

Moreover, [27, Lemma 3.8] implies that on Z, $\tau_2^i|_Z = \rho_2^1|_Z + \rho_2^2|_Z + 1$ for each $1 \leq i \leq (q-2)/2$. Thus any linear Weil character will also contain multiple P_2-orbits of characters when restricted to Z, a contradiction.

This shows that only $\widehat{\alpha}_2$ can restrict irreducibly to P_2, as stated.

□

The following corollary follows directly from the proof of Theorem 5.5.3.

Corollary 5.5.4. *Let Z_2 be the unipotent radical of $P_2 = \mathrm{stab}_{Sp_4(q)}(\langle e_1, e_2\rangle_{\mathbb{F}_q})$. Then*

$$\alpha_2|_{Z_2} = \sum_{\lambda\in\mathcal{O}_2^-}\lambda, \qquad \beta_2|_{Z_2} = \sum_{\lambda\in\mathcal{O}_2^-}\lambda + \sum_{\lambda\in\mathcal{O}_1}\lambda + 1_{Z_2},$$

and

$$\zeta_2^i|_{Z_2} = 2\sum_{\lambda\in\mathcal{O}_2^-}\lambda + \sum_{\lambda\in\mathcal{O}_1}\lambda,$$

for each $1 \leq i \leq q/2$. Moreover,

$$\rho_2^1|_{Z_2} = q\cdot 1_{Z_2} + \sum_{\lambda\in\mathcal{O}_2^+}\lambda, \qquad \rho_2^2|_{Z_2} = (q+1)\cdot 1_{Z_2} + \sum_{\lambda\in\mathcal{O}_1}\lambda + \sum_{\lambda\in\mathcal{O}_2^+}\lambda,$$

and

$$\tau_2^i|_{Z_2} = (2q+2)\cdot 1_{Z_2} + \sum_{\lambda\in\mathcal{O}_1}\lambda + 2\sum_{\lambda\in\mathcal{O}_2^+}\lambda$$

for each $1 \leq i \leq (q-2)/2$.

The following theorem shows that for any $n \geq 2$ and any characteristic $\ell \neq 2$, the group $G = Sp_{2n}(2^a)$ yields a triple (G, V, H) as in Problem 1 with $H = P_n$.

Theorem 5.5.5. *Let $G = Sp_{2n}(q)$ with q even and $n \geq 2$, and let V be an absolutely irreducible G-module in characteristic $\ell \neq 2$ affording the ℓ-Brauer character $\widehat{\alpha}_n$. Then V is irreducible on $P_n = \mathrm{stab}_G(\langle e_1, ..., e_n\rangle_{\mathbb{F}_q})$.*

Proof. Note that $\mathrm{IBr}_\ell(Z_n) = \mathrm{Irr}(Z_n)$ since Z_n is made up entirely of 2 - elements. Let $\lambda_Y \in \mathrm{Irr}(Z_n)$ be labeled by

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \in M_n(q)$$

with $Y_1 \in M_2(q), Y_4 \in M_{n-2}(q)$. Identifying a symmetric matrix $X \in M_2(q)$ with both

$$\begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \in Z_2^{(2)}$$

and

$$\begin{pmatrix} I_n & X_1 \\ 0 & I_n \end{pmatrix} \in Z_n^{(n)},$$

where

$$X_1 := \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_n(q),$$

we see

$$\lambda_Y(X) = (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(X_1 Y))} = (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(XY_1))} = \lambda_{Y_1}(X).$$

Thus $\lambda_Y|_{Z_2^{(2)}} = \lambda_{Y_1}$. Also, it is clear from the definition that $q_Y|_{\langle f_1, f_2\rangle_{\mathbb{F}_q}} = q_{Y_1}$.

From [27, Proposition 7.2], $\widehat{\alpha}_n|_{Sp_{2n-2}(q)}$ contains $\widehat{\alpha}_{n-1}$ as a constituent, and continuing inductively, we see $\widehat{\alpha}_n|_{Sp_4(q)}$ contains $\widehat{\alpha}_2$ as a constituent. Now, by Theorem 5.5.3, $\widehat{\alpha}_2$ is irreducible when restricted to $P_2^{(2)}$, and $\widehat{\alpha_2}|_{Z_2^{(2)}}$ is the sum of the characters in the orbit $\mathcal{O}_2^-$.

Since $\widehat{\alpha}_2|_{Z_2^{(2)}}$ is a constituent of $\widehat{\alpha}_n|_{Z_2^{(2)}}$, it follows that $\widehat{\alpha}_n|_{Z_n^{(n)}}$ must contain some λ_Y such that q_{Y_1} is rank-2. Since $|\mathcal{O}_2^-| = \alpha_n(1)$ and $|\mathcal{O}_r^\pm| > \alpha_n(1)$ for the other orbits with $r \geq 2$, we know $\widehat{\alpha}_n|_{Z_n^{(n)}} = \sum_{\lambda \in \mathcal{O}_2^-} \lambda$. Therefore $\widehat{\alpha}_n|_{P_n^{(n)}}$ must be irreducible.

$\square$

It will now be convenient to reorder the basis of $G = Sp_{2n}(q)$ as

$$\{e_1, e_2, ..., e_n, f_3, f_4, ..., f_n, f_1, f_2\}.$$

Under this basis, the embedding of $Sp_4(q)$ into G is given by

$$Sp_4(q) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B \\ 0 & I_{2n-4} & 0 \\ C & 0 & D \end{pmatrix} \in Sp_{2n}(q)$$

where A, B, C, D are each 2×2 matrices.

Note that $P_2^{(2)} \leq P_2^{(n)}$ and, moreover, $Z_2^{(2)} = Z_2^{(n)}$. We will therefore simply write Z_2 for this group.

Theorem 5.5.6. *Let $G = Sp_{2n}(q)$ with q even and $n \geq 2$, and let V be an absolutely irreducible G-module with dimension larger than 1 in characteristic $\ell \neq 2$. Then V is absolutely irreducible on $P_2^{(n)}$ if and only if $n = 2$ and V is the module affording the ℓ-Brauer character $\widehat{\alpha}_2$.*

Proof. Assume $n > 2$. Let $\chi \in \mathrm{IBr}_\ell(G)$ denote the ℓ-Brauer character afforded by V, and let $\varphi \in \mathrm{IBr}_\ell(H)$ be the ℓ-Brauer character afforded by V on $H := P_2^{(n)}$. Write $Z := Z_2$. The nontrivial orbits of the action of H on $\mathrm{Irr}(Z)$ and those of $P_2^{(2)}$ on $\mathrm{Irr}(Z)$ are the same, with sizes

$$|\mathcal{O}_1| = q^2 - 1, \quad |\mathcal{O}_2^-| = \frac{1}{2}q(q-1)^2, \quad |\mathcal{O}_2^+| = \frac{1}{2}q(q^2-1).$$

By Clifford theory, $\chi|_Z = e\sum_{\lambda \in \mathcal{O}} \lambda$ for one of these orbits $\mathcal{O}$ and some positive integer e. (Note that $\mathcal{O}$ is not the trivial orbit since G is simple, so χ cannot contain Z in its kernel.) It is clear from this that $V|_H$ has the property $\mathcal{W}_2^{\pm}$ in the notation of [27], and therefore by [27, Theorem 1.2], χ is one of the Weil characters from Table 4.2.

If χ is a linear Weil character, then the branching rules found in [27, Propositions 7.7] imply that $\chi|_{Sp_4(q)}$ contains $1_{Sp_4(q)}$ as a constituent, and so $\chi|_Z$ contains 1_Z as a constituent, which is a contradiction.

If χ is a unitary Weil character, then the branching rules found in [27, Proposition 7.2] show that $\chi|_{Sp_4(q)}$ contains $\sum_{k=1}^{q/2} \zeta_2^k - \gamma$, where $\gamma \in \{0, 1\}$ as a constituent. (Note that $\gamma = 1$ in the case $\ell|(q+1)$ and $\chi = \widehat{\beta}_n - 1$.) But [27, Lemma 3.8] shows that $\zeta_n^i = \alpha_n + \beta_n - 1$ on Z, so by Corollary 5.5.4, $\chi|_Z$ contains $(q/2)(\omega_1 + 2\omega_2^-) - \gamma$, a contradiction since $\chi|_Z$ can have as constituents Z-characters from only one H-orbit.

We therefore see that n must be 2, and the result follows from Theorem 5.5.3.

□

We are now prepared to classify all triples (G, V, H) as in Problem 1 when $G = Sp_4(2^a)$ and H is a maximal parabolic subgroup.

Corollary 5.5.7. *Let q be even. A nontrivial absolutely irreducible representation V of $Sp_4(q)$ in characteristic $\ell \neq 2$ is irreducible on a maximal parabolic subgroup if and only if the subgroup is P_2 and V affords the character $\widehat{\alpha}_2$.*

Proof. This is immediate from Theorem 5.5.6 and Corollary 5.5.2.

□

Note that we have now completed the proof of Theorem 1.1.3.

We will now return to the specific group $G = Sp_6(q)$. Let $H = P_3 = \mathrm{stab}_G(\langle e_1, e_2, e_3\rangle_{\mathbb{F}_q})$ be the third maximal parabolic subgroup, and note that here $Z_3 = Q_3$ is elementary abelian of order q^6. We will simply write Z for this group. The sizes of the four nontrivial orbits of $\mathrm{Irr}(Z)$ and the corresponding L_3-stabilizers are

$$|\mathcal{O}_1| = q^3 - 1, \quad |\mathrm{stab}_{L_3}(\lambda)| = q^3(q-1)(q^2-1);$$

$$|\mathcal{O}_2^{\pm}| = \frac{1}{2}q(q \pm 1)(q^3 - 1), \quad |\mathrm{stab}_{L_3}(\lambda)| = 2q^2(q-1)(q \mp 1);$$

and

$$|\mathcal{O}_3| = q^2(q-1)(q^3-1), \quad |\mathrm{stab}_{L_3}(\lambda)| = q(q^2-1).$$

We begin by considering the ordinary case, $\ell = 0$.

Theorem 5.5.8. *Let V be a nontrivial absolutely irreducible ordinary representation of $G = Sp_6(q)$, $q \geq 4$ even. Then V is irreducible on $H = P_3$ if and only if it affords the Weil character α_3.*

Proof. Note that α_3 is irreducible on H by Theorem 5.5.5. Conversely, suppose that $\chi \in \mathrm{Irr}(G)$ is irreducible when restricted to H. Since $Z \lhd H$ is abelian, it follows from Ito's theorem (see [33, Theorem 6.15]) that $\chi(1)$ divides $[H : Z]$, which is

$$[H : Z] = |L_3| = |GL_3(q)| = q^3(q-1)(q^2-1)(q^3-1).$$

Moreover, by Clifford theory, if $\lambda \in \mathrm{Irr}(Z)$ such that $\chi|_H \in \mathrm{Irr}(H|\lambda)$, then $\chi(1)$ is divisible by the size of the orbit $\mathcal{O}$ containing λ. In particular, this means that $q^3 - 1$ must divide $\chi(1)$. (Note that $\lambda \neq 1$, since G is simple and thus Z cannot be contained in the kernel of χ.) However, from inspection of the character degrees given in [49], it is clear that the only irreducible ordinary character of G satisfying these conditions is α_3.

□

Given any $\varphi \in \mathrm{IBr}_\ell(H)$ and a nontrivial irreducible constituent λ of $\varphi|_Z$, we know by Clifford theory that $\varphi = \psi^H$ for some $\psi \in \mathrm{IBr}_\ell(I|\lambda)$, where $I := \mathrm{stab}_H(\lambda)$. Then $\psi|_Z = \psi(1) \cdot \lambda$ and therefore $\ker\lambda \leq \ker\psi$. Note that $|Z/\ker\lambda| = 2$ since Z is elementary abelian and λ is nontrivial. Viewing ψ as a Brauer character of $I/\ker\psi$, we see

$$\psi(1) \leq \sqrt{|I/\ker\psi|} \leq \sqrt{|I/\ker\lambda|} = \left(\frac{|Z| \cdot |\mathrm{stab}_{L_3}(\lambda)|}{|\ker\lambda|}\right)^{1/2} = \sqrt{2|\mathrm{stab}_{L_3}(\lambda)|}$$

Now, $\varphi(1) = \psi(1) \cdot |\mathcal{O}|$ where $\mathcal{O}$ is the H-orbit of $\mathrm{Irr}(Z)$ which contains λ. If $\lambda \in \mathcal{O}_1$, this yields

$$\varphi(1) \leq (q^3-1)\sqrt{2q^3(q-1)(q^2-1)} = (q-1)(q^3-1)\sqrt{2q^3(q+1)},$$

and we will denote this upper bound by B_1.

If $\lambda \in \mathcal{O}_2^{\pm}$, then we see similarly that

$$\varphi(1) \leq \frac{1}{2}q(q\pm 1)(q^3-1)\sqrt{4q^2(q-1)(q\mp 1)}.$$

We will denote this bound by $B_2^{\pm}$, so

$$B_2^- := q^2(q-1)(q^3-1)\sqrt{q^2-1}, \quad \text{and} \quad B_2^+ := q^2(q^2-1)(q^3-1).$$

For $\lambda \in \mathcal{O}_3$, we have $I = Z\colon Sp_2(q)$. If we denote $K := \ker\psi$, then

$$(K \cdot Sp_2(q))/K \leq I/K.$$

But

$$(K \cdot Sp_2(q))/K \cong Sp_2(q)/(K \cap Sp_2(q)),$$

which must be isomorphic to $Sp_2(q)$ or $\{1\}$ since $Sp_2(q)$ is simple for $q \geq 4$. Thus either $I/\ker\psi$ contains a copy of $Sp_2(q)$ as a subgroup of index at most 2 or $\psi(1) = 1$. Moreover, $(ZK)/K \lhd I/K$. But

$$(ZK)/K \cong Z/(Z \cap K) = Z/\ker\lambda \cong \mathbb{Z}/2\mathbb{Z},$$

and thus I/K contains a normal subgroup of size 2. Assuming we are in the case that I/K contains a copy of $Sp_2(q)$, we know this normal subgroup intersects $Sp_2(q)$ trivially, and thus $I/K \cong \mathbb{Z}/2 \times Sp_2(q)$. In either case, $\psi(1) \leq \mathfrak{m}(Sp_2(q)) = q+1$, and therefore

$$\varphi(1) \leq (q+1)q^2(q-1)(q^3-1) = q^2(q^2-1)(q^3-1),$$

which we will denote by B_3. Note that $B_3 = B_2^+ > B_2^- > B_1$ for $q \geq 4$.

Theorem 5.5.9. *Let $G = Sp_6(q)$, $q \geq 4$ even, and let $H = P_3$. Then a nontrivial absolutely irreducible G-module V in characteristic $\ell \neq 2$ is irreducible on H if and only if V affords the ℓ-Brauer character $\widehat{\alpha}_3$.*

Proof. That $\widehat{\alpha}_3$ is irreducible on H follows from Theorem 5.5.5. Conversely, suppose that V affords $\chi \in \mathrm{IBr}_\ell(G)$ and that $\chi|_H = \varphi \in \mathrm{IBr}_\ell(H)$. We claim that χ must lift to an ordinary character, so that the result follows from Theorem 5.5.8. We will keep the notation from the above discussion.

First suppose that χ does not lie in a unipotent block. As the bound $q(q-1)^3(q^4+q^2+1)/2$ in part (B) of Theorem 1.1.1 is larger than B_2^- and is larger than B_3 unless $q=4$, it follows that either χ lifts to an ordinary character or $q=4$ and $\lambda \in \mathcal{O}_3$ or $\mathcal{O}_2^+$.

Now let $q=4$. We identify G with $SO_7(4)$ so that $G^* = Sp_6(4)$. As noted in Proposition 4.1.2, if χ corresponds to $1_{C_{G^*}(s)}$ in $\mathrm{IBr}_\ell(C_{G^*}(s))$, then it lifts by the Morita equivalence guaranteed by Lemma 4.1.1. Let $\mathfrak{u}(C_{G^*}(s))$ denote the smallest degree larger than 1 of an irreducible Brauer character lying in a unipotent block of $C_{G^*}(s)$ for a semisimple element s. Using the same argument as in the proof of part (B) of Theorem 1.1.1, we will show that for a nontrivial semisimple element $s \in G^*$, $\mathfrak{u}(C_{G^*}(s))[G^* : C_{G^*}(s)]_{2'} > B_3$ unless s belongs to a class in the family $c_{3,0}$ or $c_{4,0}$.

Indeed, if s is any semisimple element in a class other than $c_{3,0}, c_{4,0}, c_{5,0}, c_{6,0}, c_{8,0}$, or $c_{10,0}$, then $[G^* : C_{G^*}(s)]_{2'} > B_3$ by Lemma 4.1.3. If s is in $c_{5,0}$, then $C_{G^*}(s) \cong GL_3(q)$ and from [35], $\mathfrak{u}(C_{G^*}(s)) \geq q^2+q-1$, so

$$\mathfrak{u}(C_{G^*}(s))[G^* : C_{G^*}(s)]_{2'} \geq (q^2+q-1)[G^* : C_{G^*}(s)]_{2'}$$

$$= (q^2+q-1)(q+1)^2(q^2+1)(q^2-q+1) > B_3.$$

If s is in $c_{6,0}$, then $C_{G^*}(s) \cong GU_3(q)$, so $\mathfrak{d}_\ell(C_{G^*}(s)) \geq \lfloor \frac{q^3-q}{q+1} \rfloor = q^2-q$ (see, for example, [71]), so

$$\mathfrak{u}(C_{G^*}(s))[G^* : C_{G^*}(s)]_{2'} \geq (q^2-q)[G^* : C_{G^*}(s)]_{2'}$$

$$= (q^2-q)(q^2+1)(q-1)^2(q^2+q+1) > B_3.$$

If s is in $c_{8,0}$ or $c_{10,0}$, then $C_{G^*}(s) \cong GL_2^{\pm}(q) \times Sp_2(q)$ so

$$\mathfrak{u}(C_{G^*}(s))[G^* : C_{G^*}(s)]_{2'} \geq (q-1)[G^* : C_{G^*}(s)]_{2'}$$

$$= (q-1)^2(q^2+1)(q^4+q^2+1) > B_3.$$

Hence we may assume s belongs to a class in the family $c_{3,0}$ or $c_{4,0}$. In this case, $C_{G^*}(s) \cong Sp_4(q) \times C$ for a cyclic group C. Now, the Brauer character tables of $Sp_4(4)$ are available in the GAP Character Table Library, [24],[11]. We can see that the smallest nonprincipal character degree of $Sp_4(4)$ for any $\ell \neq 2$ is 18. This corresponds to $\widehat{\alpha}_2$, which clearly lifts to $\mathbb{C}$, so by the Morita equivalence guaranteed by Lemma 4.1.1, χ also lifts if it corresponds to this character. The next smallest degree is 33 if $\ell = 5$ and 34 if $\ell = 3$ or 17. If $s \in c_{3,0}$, then $[G^* : C_{G^*}(s)]_{2'} = 1365$, and $1365 \cdot 33 = 45045 > 15120 = B_3$. If $s \in c_{4,0}$, then $[G^* : C_{G^*}(s)]_{2'} = 819$, and $819 \cdot 33 = 27027 > 15120 = B_3$. It follows that in this case, χ must again lift to an ordinary character.

Next, assume χ lies in a unipotent block. Note that the bound D in part (A) of Theorem 1.1.1 is larger than B_3 for $q \geq 4$. Hence, χ must be as in situations A(1), A(2), or A(3) of Theorem 1.1.1. Also, note that $\chi(1)$ must be divisible by (q^3-1), as $|\mathcal{O}_1|$, $|\mathcal{O}_2^{\pm}|$, and $|\mathcal{O}_3|$ are all divisible by (q^3-1). Therefore, χ cannot be any of the characters $\widehat{\rho}_3^1 - 1, \widehat{\rho}_3^2 - 1$, $\widehat{\beta}_3 - 1$, $\widehat{\chi}_6 - 1$ or $\widehat{\chi}_7 - \widehat{\chi}_4$. Thus in the case $\ell | (q^3-1)(q^2+1)$ or $3 \neq \ell | (q^2-q+1)$, we know from Theorem 1.1.1 that χ lifts to an ordinary character.

Now assume $\ell | (q+1)$ and that χ does not lift to an ordinary character. Then by the above remarks, χ must be $\widehat{\chi}_{35} - \widehat{\chi}_5$, which has degree larger than B_2^- and is odd. Since $|\mathcal{O}_3|$ and $|\mathcal{O}_2^+|$ are each even, this shows our χ cannot be this character. So, χ must again lift to an ordinary character.

This completes the proof, by Lemma 5.1.2 and Theorem 5.5.8.

□

Corollary 5.5.10. *Let $G = Sp_6(q)$ with $q \geq 4$ even. A nontrivial absolutely irreducible G-module V in characteristic $\ell \neq 2$ is irreducible on a maximal parabolic subgroup P if and only if $P = P_3$ and V affords the ℓ-Brauer character $\widehat{\alpha}_3$.*

Proof. This follows directly from Corollary 5.5.2, Theorem 5.5.6, and Theorem 5.5.9.

□

5.5.1 Descent to Subgroups of P_3

Let $Z = Z_3$ be the unipotent radical of P_3, and let $R \le Z$ be the subgroup $[q^3]$ given by matrices $C \in Z$ with zero diagonal. That is,

$$R = \left\{ \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} : C \in M_3(q), C + C^T = 0,\ C \text{ has diagonal } 0 \right\}.$$

Note that the subgroups $L_3 \cong GL_3(q)$ and $L_3' \cong SL_3(q)$ of P_3 each act transitively on $R \setminus 0$.

Let $\lambda = \lambda_Y$ be an irreducible character of Z corresponding to the matrix $Y \in M_3(q)$, and write $\lambda|_R = \mu = \mu_Y$. If λ' is another such character corresponding to Y' and $\lambda'|_R = \mu'$, then we have $\mu = \mu'$ if and only if $(Y + Y') + (Y + Y')^T = 0$. (Note that unlike characters of Z, we do not require that Y, Y' have the same diagonal.) Hence $\mu_Y = \mu_{X^TYX}$ for $X \in GL_3(q)$ if and only if $X^T(Y + Y^T)X = Y + Y^T$. That is, X is in the isometry group of the form with Gram matrix $Y + Y^T$. As the action of $X \in L_3 \cong GL_3(q)$ on μ_Y is given by $(\mu_Y)^X = \mu_{X^TYX}$, this means that $\mathrm{stab}_{L_3}(\mu)$ is this isometry group..

In particular, if λ is in the P_3-orbit $\mathcal{O}_2^-$ of linear characters of Z, then this means that $\mathrm{stab}_{L_3}(\lambda|_R) = [q^2] : (\mathbb{F}_q^\times \times Sp_2(q)) = [q^2] : GL_2(q)$. Recall that from the proof of Theorem 5.5.5, $\alpha_3|_Z = \omega_2^-$ is the orbit sum corresponding to $\mathcal{O}_2^-$. Hence we have

$$\mathrm{stab}_{L_3}(\mu) = [q^2] : GL_2(q)$$

if μ is a constituent of $\alpha_3|_R$. Taking the elements of this stabilizer with determinant one, we also see

$$\mathrm{stab}_{L_3'}(\mu) = [q^2] : SL_2(q).$$

Lemma 5.5.11. *The Brauer character $\widehat{\alpha}_3$ is irreducible on the subgroup $P_3' = Z : SL_3(q)$ of P_3.*

Proof. Let λ be an irreducible constituent of $\widehat{\alpha}_3|_Z$, so that $\lambda \in \mathcal{O}_2^-$. Recall that the stabilizer in $L_3 \cong GL_3(q)$ is $\mathrm{stab}_{L_3}(\lambda) \cong [q^2] : (\mathbb{F}_q^\times \times O_2^-(q))$. Taking the elements in this group with determinant 1, we see that the stabilizer in $SL_3(q)$ is isomorphic to $[q^2] : (O_2^-(q))$, and hence the P_3'-orbit has length

$$\frac{q^9(q^2-1)(q^3-1)}{2q^8(q+1)} = \frac{1}{2}q(q-1)(q^3-1) = |\mathcal{O}_2^-| = \alpha_3(1).$$

Therefore, $\widehat{\alpha}_3|_{P_3'}$ is irreducible.

□

Lemma 5.5.12. *Let $G = Sp_6(q)$ with $q \geq 4$ even, and let V be an absolutely irreducible G-module V which affords the Brauer character $\widehat{\alpha}_3$. Write $Z = Z_3$ for the unipotent radical of the parabolic subgroup P_3 and $L = L_3$ for the Levi subgroup. If $H < P_3$ with $V|_H$ irreducible, then ZH contains $P_3' = Z : L' = Z : SL_3(q)$.*

Proof. Note that $HZ/Z \cong H/(Z \cap H)$ is a subgroup of $P_3/Z \cong GL_3(q)$. As $\alpha_3(1) = q(q-1)(q^3-1)/2$, we know that $|H|_{2'}$ is divisible by $(q-1)(q^3-1)$. Moreover, HZ/Z must act transitively on the $q^3 - 1$ elements of $R \setminus 0$. Therefore, by [40, Proposition 3.3], there is some power of q, say q^s, such that $M := HZ/Z$ satisfies one of the following:

1. $M \rhd SL_a(q^s)$ with $q^{sa} = q^3$ for some $a \geq 2$
2. $M \rhd Sp_{2a}(q^s)'$ with $q^{2sa} = q^3$ for some $a \geq 2$
3. $M \rhd G_2(q^s)'$ with $q^{6s} = q^3$, or
4. $M \cdot (Z(GL_3(q))) \leq \Gamma L_1(q^3)$.

Now, the conditions that $q^{2as} = q^3$ or $q^{6s} = q^3$ imply that H cannot satisfy (2) or (3). As $(q-1)(q^3-1)$ must divide $|M|$, H also cannot satisfy (4). Hence, H is as in (1). But then the conditions $q^{as} = q^3$ and $q \geq 2$ imply that $a = 3$ and $s = 1$. Therefore, $SL_3(q) \lhd M = HZ/Z$.

□

Lemma 5.5.13. *A nontrivial $SL_3(q)$-invariant proper subgroup of Z must be R.*

Proof. Let $D < Z$ be nontrivial and invariant under the $SL_3(q)$-action, which is given by XCX^T for $C \in Z$ and $X \in SL_3(q)$. Note that here we have made the identifications

$$C \leftrightarrow \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \quad \text{and} \quad X \leftrightarrow \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^T \end{pmatrix}.$$

Now, note that $SL_3(q)$ acts transitively on $R \setminus 0$, so $D \cap R$ must be either R or 0. (Indeed, the action of $SL_3(q)$ on R is the second wedge $\Lambda^2(U) \simeq U^*$ of the action on the natural module U for $SL_3(q)$.) Moreover, $SL_3(q)$ acts transitively on $(Z/R) \setminus 0$, so either $DR/R = Z/R$ or $DR = R$. (Indeed, the action of $SL_3(q)$ on Z/R is the Frobenius twist $U^{(2)}$ of the action of $SL_3(q)$ on the natural module U.) If $R < D$, then $D/R = Z/R$, so $D = Z$, a contradiction. Hence either $D = R$ or $D \cap R = 0$.

If $D \cap R = 0$, then $DR \neq R$, so $DR = Z$ and D is a complement in Z for R. Hence no two elements of D can have the same diagonal. Let

$$g = \begin{pmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

be the element in D with diagonal $(1, 0, 0)$, which must exist since $SL_3(q)$ acts transitively on nonzero elements of $DR/R = Z/R$. If g is diagonal, then any matrix of the form $\mathrm{diag}(a, 0, 0)$, $\mathrm{diag}(0, a, 0)$, or $\mathrm{diag}(0, 0, a)$ for $a \neq 0$ is in the orbit of g. Thus since D is an $SL_3(q)$-invariant subgroup, D contains the group of all diagonal matrices. As D is a complement for R, it follows that in fact D is the group of diagonal matrices, a contradiction since this group is not $SL_3(q)$-invariant. Therefore, g has nonzero nondiagonal entries. We claim that there is some $X \in SL_3(q)$ which stabilizes the coset $g + R$ but does not stabilize g. That is, g and XgX^T have the same diagonal, but are not the same element, yielding a contradiction. Indeed, if at least one of a, b is nonzero, then any $X = \mathrm{diag}(1, s, s^{-1})$ with $s \neq 1$ satisfies the claim. If $a = b = 0$

and $c \neq 0$, we can take X to be

$$X = \begin{pmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $r \neq 0$, proving the claim. We have therefore shown that $D = R$.

$\square$

Theorem 5.5.14. *Let $G = Sp_6(q)$ with $q \geq 4$ even, and let V be an absolutely irreducible G-module V which affords the Brauer character $\widehat{\alpha}_3$. Then $V|_H$ is irreducible for some $H < P_3$ if and only if H contains $P_3' = Z : SL_3(q)$.*

Proof. First, if H contains P_3', then $V|_H$ is irreducible by Lemma 5.5.11. Conversely, suppose that $V|_H$ is irreducible for some $H < P_3$. Assume by way of contradiction that H does not contain P_3'. By Lemma 5.5.12, HZ contains P_3', so $H \cap Z$ is $SL_3(q)$-invariant. Therefore, by Lemma 5.5.13, $H \cap Z$ must be $1, R$, or Z. Since H does not contain P_3', it follows that $H \cap Z = 1$ or R.

Write $H_1 := H \cap P_3'$. Then $H_1 Z = P_3'$. (Indeed, $P_3' \leq ZH$, so any $g \in P_3'$ can be written as $g = zh$ with $z \in Z, h \in H$. Hence $z^{-1}g = h \in H \cap P_3' = H_1$, and $g \in H_1 Z$. On the other hand, $H_1 Z \leq P_3' Z = P_3'$.)

Now, if $H \cap Z = 1$, then $H_1 \cap Z = 1$ and $H_1 \cong P_3'/Z = SL_3(q)$. Since $V|_H$ is irreducible and H/H_1 is cyclic of order $q-1$, we see by Clifford theory that $H_1 \cong SL_3(q)$ has an irreducible character of degree $\alpha_3(1)/d$ for some d dividing $q-1$. Then $SL_3(q)$ has an irreducible character degree divisible by $q(q^3-1)/2$, a contradiction, as $\mathfrak{m}(SL_3(q)) < q(q^3-1)/2$.

Hence we have $H \cap Z = R$. Then $H_1 \cap Z = R$ as well, so $(H_1/R) \cap (Z/R) = 1$, and H_1/R is a complement for $(Z/R) = [q^3]$ in $P_3'/R \cong [q^3] : SL_3(q)$. As the first cohomology group $H^1(SL_3(q), \mathbb{F}_q^3)$ is trivial (see, for example [17, Table 4.5]), any complement for Z/R in P_3'/R is conjugate in P_3'/R to H_1/R. In particular, writing $K_1 := R : L_3' = R : SL_3(q)$, we see that K_1/R is also a complement for Z/R in P_3'/R.

Hence, K_1/R is conjugate to H_1/R in P_3'/R, so K_1 is conjugate to H_1 in P_3', and we may assume for the remainder of the proof that $H_1 = R : L_3' = R : SL_3(q)$.

As H/H_1 is cyclic of order dividing $q-1$, we know by Clifford theory that if $\alpha_3|_H$ is irreducible, then there is some $d|(q-1)$ so that each irreducible constituent of $\alpha_3|_{H_1}$ has degree $\alpha_3(1)/d$. Let $\beta \in \mathrm{Irr}(H_1)$ be one such constituent, and let μ be a constituent of $\beta|_R$. Then since $L_3' = SL_3(q)$ acts transitively on $\mathrm{Irr}(R) \setminus \{1_R\}$, $I_{H_1}(\mu) := \mathrm{stab}_{H_1}(\mu) = R : ([q^2] : SL_2(q))$. By Clifford theory, we can write $\beta|_{H_1} = \psi^{H_1}$ for some $\psi \in \mathrm{Irr}(I_{H_1}(\mu)|\mu)$. Hence $\beta(1) = [H : I_H(\mu)] \cdot \psi(1) = (q^3-1)\psi(1)$.

We can view ψ as a character of $I_H(\mu)/\ker\mu$, as $\psi|_R = e\cdot\mu$ for some integer e. But $I_H(\mu)/\ker\mu \cong C_2 \times ([q^2] : SL_2(q))$, as R is elementary abelian and μ is nontrivial. If ψ is nontrivial on $[q^2]$, then $\psi|_{[q^2]}$ is some integer times an orbit sum for some $SL_2(q)$-orbit of characters of $[q^2]$, again by Clifford theory. However, as $SL_2(q)$ is transitive on $[q^2] \setminus 0$, it follows that $\psi(1)$ is divisible by q^2-1, a contradiction since $\beta(1)$ is not divisible by q^2-1.

Hence ψ is trivial on $[q^2]$, so ψ can be viewed as a character of $C_2 \times SL_2(q)$. As $q \geq 4$, $\psi(1) = q(q-1)/2d$ is even. Now, the only even irreducible character degree of $SL_2(q)$ is q, but $q \neq q(q-1)/2d$, which contradicts the existence of this β. Therefore, $\alpha_3|_H$ cannot be irreducible, so neither is $\widehat{\alpha}_3|_H$.

□

We have now completed the proof of Theorem 1.1.2.

5.6 The case $q = 2$

In this section, we prove Theorems 1.1.4 and 1.1.5. To do this, we use the computer algebra system GAP, [24]. In particular, we utilize the character table library [11], in which the ordinary and Brauer character tables for $Sp_6(2)$ and $Sp_4(2) \cong S_6$, along with all of their maximal subgroups, are stored. The maximal subgroups of $Sp_6(2)$

are as follows:

$U_4(2).2, \quad A_8.2, \quad 2^5 : S_6, \quad U_3(3).2, \quad 2^6 : L_3(2), \quad 2.[2^6] : (S_3{\times}S_3), \quad S_3{\times}S_6, \quad L_2(8).3,$

and the maximal subgroups of $Sp_4(2) \cong S_6$ are

$$A_6, \quad A_5.2 = S_5, \quad O_4^-(2) \cong S_5, \quad S_3 \wr S_2, \quad 2 \times S_4, \quad S_2 \wr S_3$$

The ordinary and Brauer character tables for each of these maximal subgroups are stored as well, with the exception of $2^5 : S_6$ and $2^6 : L_3(2)$, for which we only have the ordinary character tables. In addition, the command `PossibleClassFusions(c1,c2)` gives all possible fusions from the group whose (Brauer) character table is c1 and the group whose (Brauer) character table is c2. Using this command, it is straightforward to find all Brauer characters which restrict irreducibly from c2 to c1. Below is a sample of code utilizing this technique:

```
cth:=CharacterTable("[maxsubgroup]");
ctg:=CharacterTable("S6(2)");
    #cth:=CharacterTable(cth, p);
    #ctg:=CharacterTable(ctg,p); for p-Brauer character tables
irrg:=Irr(ctg);
irrh:=Irr(cth);
fus:=PossibleClassFusions(cth,ctg)[1];

for i in [1..Length(irrg)]
do
     for j in [1..Length(irrh)]
     do
   if irrg[i][1]>1 and irrg[i][1]=irrh[j][1] then
```

```
        for m in [1..Length(fus)]
    do
      if not irrg[i][fus[m]]=irrh[j][m] then
     irredsmatch:=false;
     break;
   else
     irredsmatch:=true;
   fi;
    od;
  fi;
    od;
od;
```

From here, we note that for any given $\ell \neq 2$, $\widehat{\alpha}_3$, $\widehat{\beta}_3 - \{0,1\}$, and $\widehat{\rho}_3^1 - \{0,1\}$ are the only irreducible ℓ-Brauer characters of $Sp_6(2)$ with their respective degrees. Also, from [27, Lemma 3.8], we know $\zeta_3^1(g) = \alpha_3(g) + \beta_3(g) - 1$ on 2-elements, which allows us to distinguish between ζ_3^1 and the other character of degree 21. We also know $\rho_3^2 - 1$ should restrict to an irreducible 3-Brauer character, so from this we can distinguish between ρ_3^2 and the other character of degree 35, which restricts to ℓ-regular elements as an irreducible Brauer character for all ℓ (this is needed, for example, in the case $H = A_8.2$).

Now, in the case $H = P_3 = 2^6 : L_3(2)$ or $2^5 : S_6$ and $G = Sp_6(2)$, we need additional techniques, as the Brauer character tables for these choices of H are not stored in the GAP character table library. However, in the case $H = 2^5 : S_6$, we can use the above technique to see that there are no ordinary irreducible characters of G which restrict irreducibly to H, and moreover, there is no $\chi|_H - \lambda$ for $\chi \in \mathrm{Irr}(G), \lambda \in \widehat{H}$ which is irreducible on H. But we can also see that $\mathfrak{m}(H) = 45$, and any $\varphi \in \mathrm{IBr}_\ell(G)$ with $\varphi(1) \leq 45$ either lifts to a complex character or is $\widehat{\chi} - 1$ for some complex character χ. Thus by Lemma 5.1.3, there are no irreducible Brauer characters of G which restrict to an irreducible Brauer character of H, for any choice of $\ell \neq 2$.

We are therefore left with the case $H = P_3$. In this case, it is clear from GAP that the only ordinary characters which restrict irreducibly to H are α_3 and χ_4, where χ_4 is the unique irreducible character of degree 21 which is not ζ_3^1. Moreover, there is again no $\chi \in \mathrm{Irr}(G)$, $\lambda \in \widehat{H}$ such that $\chi|_H - \lambda \in \mathrm{Irr}(H)$. Referring to the notation of Section 5.5, we have $|\mathcal{O}_1| = 7$, $|\mathcal{O}_2^-| = 7, |\mathcal{O}_2^+| = 14$, and $|\mathcal{O}_3| = 28$. So any irreducible Brauer character of H must have degree divisible by 7. Also, $\mathfrak{m}(H) = 56$. We can see from the Brauer character table of G that if $\chi \in \mathrm{IBr}_\ell(G)$ has $\chi(1) \leq 56$, then either χ or $\chi + 1$ lifts to $\mathbb{C}$. Thus by Lemma 5.1.3, the only possibilities are $\widehat{\alpha}_3$ and $\widehat{\chi_4}$. Now $\widehat{\alpha}_3|_H$ is irreducible since $\alpha_3(1) = |\mathcal{O}_1| = |\mathcal{O}_2^-|$, and these are the smallest orbits of characters in Z_3. Therefore it remains only to show that $\widehat{\chi}_4$ is indeed also irreducible on H.

Since we know that $\chi_4|_H \in \mathrm{Irr}(H)$, we know that $\chi_4|_{Z_3}$ must contain only one H-orbit of Z_3-characters as constituents, which means that $\chi_4|_{Z_3} = 3\omega_1$ or $3\omega_2^-$, continuing with the notation of Section 5.5. Since Z_3 consists of 2-elements, we know $\widehat{\chi_4}|_{Z_3}$ can be written in the same way. Moreover, since $q = 2$, $\mathrm{stab}_{L_3}(\lambda)$ is solvable for $\lambda \neq 1$, so we know that if λ is a constituent of $\chi_4|_{Z_3}$, then any $\psi \in \mathrm{IBr}_\ell(I|\lambda)$ lifts to an ordinary character. Since by Clifford theory, any irreducible constituent of $\widehat{\chi}_4|_H$ can be written ψ^H for such a ψ, it follows that if $\widehat{\chi}_4|_H$ is reducible, then it can be written as the sum of some $\widehat{\varphi}_i$ for $\varphi_i \in \mathrm{Irr}(H|\lambda)$. In particular, each of these φ_i must have degree 7 or 14. By inspection of the columns of the ordinary character table of H corresponding to 3-regular and 7-regular classes, it is clear that $\chi_4|_H$ cannot be written as such a sum on ℓ-regular elements, and therefore $\widehat{\chi}_4|_H$ is irreducible.

This completes the proof of Theorems 1.1.4 and 1.1.5, and therefore the classification of triples as in Problem 1 when $G = Sp_6(q)$ or $Sp_4(q)$ with q even.

Chapter 6
Restrictions of Complex Representations of Finite Unitary Groups of Low Rank to Certain Subgroups

In this short chapter, we begin a discussion of the pair $(PSU_{n-1}(q), PSU_n(q))$ in Seitz' list, as listed in Section 1.1. (We note, however, that this pair was omitted from Seitz' original list.) We consider triples (G, V, H) as in Problem 1 in the case where $G = SU_n(q)$, H is a particular subgroup isomorphic to $GU_{n-1}(q)$, and V is an ordinary representation of G. For the purposes of this chapter, it will be convenient to alter our notation, letting $S := SU_n(q)$ and $G := GU_n(q)$. We will eventually deal with two subgroups H and K of G with $H \cong GU_{n-1}(q) \cong K$. The problem we focus on in this chapter is the following:

Problem 2. Let $S = SU_n(q)$ and $K \leq G$ be the subgroup

$$K := \left\{ \begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix} : X \in GU_{n-1}(q) \right\}.$$

Classify all $\chi \in \mathrm{Irr}(S)$ so that $\chi|_K \in \mathrm{Irr}(K)$.

While we conjecture that no triples as in Problem 1 exist for our choice of G and H, this chapter serves merely as a start to the discussion for Problem 2, and we by no means solve the problem here.

One of the main results of this chapter, which we prove in Section 6.1, shows that no characters can exist as in Problem 2 except possibly in the case that $(q+1)$ divides n. (Note that we have included this condition in our version of Seitz' list in Section 1.1.) In Section 6.2 we show that Problem 2 can be reduced to a question about the irreducible characters of $GU_n(q)$. We study the problem in detail for $n = 5$ in Section 6.3 and show that in this case, no such characters exist. In Section 6.4, we also mention the same result for the remaining $4 \leq n \leq 7$ and begin considering the case $n = 8, 9$.

Throughout this chapter, let $G := GU_n(q)$ and $S := SU_n(q)$. Recall that $S \lhd G$ with $G/S \cong C_{q+1}$, $Z(G) = \{cI_n \colon c \in \mathbb{F}_{q^2}; c^{q+1} = 1\} \cong C_{q+1}$, and $Z(S) = S \cap Z(G) \cong C_{\gcd(n,q+1)}$. We will be interested in certain subgroups of G and S, which we now define. Let K be the subgroup of $SU_n(q)$ given by:

$$K := \left\{ \begin{pmatrix} X & 0 \\ 0 & (\det X)^{-1} \end{pmatrix} : X \in GU_{n-1}(q) \right\}.$$

Define $\widetilde{H}$ to be the subgroup of $GU_n(q)$ given by

$$\widetilde{H} := \left\{ \begin{pmatrix} X & 0 \\ 0 & a \end{pmatrix} : X \in GU_{n-1}(q); a \in \mathbb{F}_{q^2}^{\times} \text{ s.t. } a^{q+1} = 1 \right\}$$

Clearly K is a subgroup of $\widetilde{H}$, and in fact $K = SU_n(q) \cap \widetilde{H}$. So, since $SU_n(q)$ is normal in $GU_n(q)$, we have that $K \lhd \widetilde{H}$. Finally, let H be the subgroup of $\widetilde{H}$ given by

$$H := \left\{ \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} : X \in GU_{n-1}(q) \right\}.$$

H is normal in $\widetilde{H}$, since

$$\begin{pmatrix} X & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} XYX^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in H.$$

So we have the following lattice of subgroups, where a double line denotes a normal subgroup:

$$\begin{tikzcd}
 & G \arrow[dl, dash, double] \arrow[dr, dash] & & \\
S \arrow[dr, dash] & & \widetilde{H} \arrow[dl, dash, double] \arrow[dr, dash, double] & \\
 & K & & H
\end{tikzcd}$$

We will keep the same notation for the subgroups $K, H, \widetilde{H}$, as well as the notation $S = SU_n(q), G = GU_n(q)$ throughout this chapter.

6.1 A Condition On (n, q)

Recall that our main problem in this chapter is to understand the irreducible complex characters of $S = SU_n(q)$ which when restricted to K are still irreducible. In this section, our goal is to show that if $(q+1)$ does not divide n, then no such characters exist. We begin with two lemmas:

Lemma 6.1.1. *Let S be any finite group and let $K \leq S$ be a subgroup. Suppose that $\chi \in \mathrm{Irr}(S)$ is a faithful irreducible character and that $\chi|_K \in \mathrm{Irr}(K)$. Then*

$$C_S(K) = Z(S).$$

Proof. Let V be an irreducible $\mathbb{C}S$-module which affords the character χ. Since $V|_K$ is also an irreducible $\mathbb{C}K$-module, we have

$$\mathrm{End}_S(V) = \mathbb{C} \cdot 1 = \mathrm{End}_K(V)$$

by Schur's Lemma. Now let $x \in C_S(K)$. Then left multiplication by x is a K-endomorphism of V, and therefore is also an S-endomorphism of V. Hence for any $s \in S$ and $v \in V$,

$$sx(v) = x(sv) = xsv = xs(v).$$

If $\mathfrak{X}$ is the representation of S affording V, then we know that $\mathfrak{X}(x)$ is defined by the multiplication of x on V, so that

$$\mathfrak{X}(x)\mathfrak{X}(s) = \mathfrak{X}(s)\mathfrak{X}(x).$$

In other words, $\mathfrak{X}(x)$ commutes with all elements in the image $\mathfrak{X}(S)$. Then we know that $\mathfrak{X}(x) = \lambda I$ for some $\lambda \in \mathbb{C}$, and it follows that $x \in Z(\chi)$ (see, for example, [33, (2.25) and (2.27)]). Since χ is faithful, we know that $Z(\chi) = Z(S)$, so in fact $x \in Z(S)$. This yields the containment $C_S(K) \leq Z(S)$ and therefore,

$$C_S(K) = Z(S)$$

since $Z(S)$ is certainly contained in $C_S(K)$.

□

Lemma 6.1.2. *Let $S = SU_n(q)$ and $K = \left\{ \begin{pmatrix} X & 0 \\ 0 & (\det X)^{-1} \end{pmatrix} : X \in GU_{n-1}(q) \right\}$. If $C_S(K) = Z(S)$, then $(q+1)$ divides n.*

Proof. We prove the contrapositive. Suppose that $(q+1)$ does not divide n. Then there exists $\alpha \in \mathbb{F}_{q^2}$ such that $\alpha^{q+1} = 1$ but $\alpha^n \neq 1$ (in particular, take α of order $q+1$ in $\mathbb{F}_{q^2}^\times$). Then the element

$$A = \begin{pmatrix} \alpha I_{n-1} & 0 \\ 0 & \alpha^{1-n} \end{pmatrix}$$

commutes with the elements of K and is contained in S, as $\det A = \alpha^{n-1}\alpha^{1-n} = 1$. However, since

$$Z(S) = \{\lambda I \colon \lambda \in \mathbb{F}_{q^2}, \lambda^{q+1} = 1 = \lambda^n\},$$

we know that A is not in $Z(S)$, for this would imply that $\alpha = \alpha^{1-n}$, and therefore $\alpha^n = 1$, a contradiction. Then $A \in C_S(K)$ but $A \notin Z(S)$, and therefore $C_S(K) \neq Z(S)$.

□

Proposition 6.1.3. *Let $S = SU_n(q)$ and K as above, with $(n, q^2) \notin \{(2,4), (2,9), (3,4)\}$. If $(q+1)$ does not divide n, then there are no $\chi \in \mathrm{Irr}(S)$ such that $\chi|_K \in \mathrm{Irr}(K)$.*

Proof. For $(n, q^2) \neq (2,4), (2,9), (3,4)$, we have that S is perfect and $S/Z(S) = PSU_n(q)$ is a nonabelian simple group. Now if $\chi \in \mathrm{Irr}(S)$ then we know that we can view χ as a faithful character on $S/\ker\chi$. Moreover, we have that $\ker\chi \leq Z(S) \leq K \leq S$. Hence it suffices to show that $C_{S/\ker\chi}(K/\ker\chi) \neq Z(S/\ker\chi)$, as then Lemma 6.1.1 implies that χ is not irreducible when restricted to $K/\ker\chi$, and therefore that $\chi|_K \notin \mathrm{Irr}(K)$.

Now since $q+1$ does not divide n, we have by Lemma 6.1.2 that $C_S(K) \neq Z(S)$. Also, note that if A is the matrix from the proof of Lemma 6.1.2, then $A \notin Z(S)$

by the proof of Lemma 6.1.2, and therefore $A \cdot Z(S)$ is nontrivial as an element of $S/Z(S)$. However, since this element commutes with all of $K/Z(S)$, we see that $C_{S/Z(S)}(K/Z(S)) \neq Z(S/Z(S)) = \{1\}$.

Now suppose $s \in S$ satisfies that $\overline{s} := s \cdot \ker(\chi) \in S/\ker\chi$ is an element of $Z(S/\ker\chi)$. Then since $\ker\chi \leq Z(S)$, we know that

$$S/Z(S) \cong (S/\ker(\chi))/(Z(S)/\ker\chi)$$

so $S/Z(S)$ is a quotient of $S/\ker(\chi)$. Then $\overline{s} \cdot (Z(S)/\ker\chi)$ is contained in the center of $(S/\ker(\chi))/(Z(S)/\ker\chi)$, which is trivial since this is a simple group. This means that $s \cdot Z(S) \in Z(S/Z(S)) = \{1\}$ and so $s \in Z(S)$.

In particular, since our element A is not in the center of S, we know that $A \cdot \ker\chi$ is not in $Z(S/\ker\chi)$. But $A \cdot \ker\chi \in C_{S/\ker\chi}(K/\ker\chi)$, which shows that

$$C_{S/\ker\chi}(K/\ker\chi) \neq Z(S/\ker\chi),$$

and therefore $\chi|_K \notin \mathrm{Irr}(K)$.

$\square$

6.2 Reducing the Problem

As discussed above, we wish to find all $\chi \in \mathrm{Irr}(SU_n(q))$ such that $\chi|_K \in \mathrm{Irr}(K)$. In general, the character table for $S = SU_n(q)$ is not known. As significantly more is known about the character table for $G = GU_n(q)$, we wish to reduce our problem to one regarding this group instead, which is the goal of this section.

More specifically, in this section we show that if $\chi \in \mathrm{Irr}(SU_n(q))$ such that $\chi|_K$ is an irreducible character of K, then there is a character $\theta \in \mathrm{Irr}(GU_n(q))$ such that $\theta|_H = \varphi_1 + ... + \varphi_m$ where m divides $(n, q+1)$, and each φ_i has the same degree. Recall here that $H \leq GU_n(q)$ is the subgroup

$$H := \left\{ \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} : X \in GU_{n-1}(q) \right\} \cong GU_{n-1}(q)$$

This gives us hope of using this fact to prove our conjecture that no such $\chi \in \mathrm{Irr}(S)$ exists by showing that no $\theta \in \mathrm{Irr}(G)$ exists with this property.

Proposition 6.2.1. *Let S, K, G, and H be as above. If $\chi \in \mathrm{Irr}(S)$ such that $\chi|_K \in \mathrm{Irr}(K)$, then there is some $\theta \in \mathrm{Irr}(G)$ such that $\theta|_H$ is the sum of m irreducible characters of H, each of the same degree, where $m|(n, q+1)$.*

Proof. Let $\chi \in \mathrm{Irr}(S)$ with $\chi|_K = \psi \in \mathrm{Irr}(K)$ and let $\theta \in \mathrm{Irr}(G)$ be an irreducible constituent of χ^G. By Clifford's theorem, we have

$$\theta|_S = e \sum_{\chi^g \in Orb_G(\chi)} \chi^g.$$

Recall that G/S is cyclic, and thus $e = 1$ by [33, Corollary (11.22)]. Then $\theta|_S$ is the sum of all of the elements in the orbit of χ under G. Now, note that we can choose the representatives $\{g \colon \chi^g \in Orb_G(\chi)\}$ from G/S. But we have

$$G/S \cong \{\alpha \in (\mathbb{F}_{q^2})^\times \colon \alpha^{q+1} = 1\}$$

where the isomorphism is given by $g \cdot S \mapsto \det(g)$. That is, there is a complete set of coset representatives for G/S given by a set $\mathcal{T}$ of elements of G which satisfy

$$\{\det g \colon g \in \mathcal{T}\} = \{\alpha \in (\mathbb{F}_{q^2})^\times \colon \alpha^{q+1} = 1\}.$$

In particular, we can choose

$$\mathcal{T} := \left\{ \begin{pmatrix} I_{n-1} & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in (\mathbb{F}_{q^2})^\times ; \alpha^{q+1} = 1 \right\}.$$

Then

$$\theta|_S = \sum_{i=1}^{[G:I_G(\chi)]} \chi^{g_i} \quad \text{and so} \quad \theta|_K = \sum_{i=1}^{[G:I_G(\chi)]} \psi^{g_i},$$

where $g_i \in \mathcal{T}$. But for $x = \begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix} \in K$ and $g = \begin{pmatrix} I & 0 \\ 0 & \alpha \end{pmatrix} \in \mathcal{T}$ we have

$$gxg^{-1} = \begin{pmatrix} I & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} X & 0 \\ 0 & a\det X^{-1}a^{-1} \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix} = x.$$

In particular, this tells us that $\psi^g = \psi$ for all $g \in \mathcal{T}$, so $\theta|_K = [G : I_G(\chi)]\psi$.

Now consider the induced character $\psi^{\widetilde{H}}$, where we recall that $\widetilde{H} \leq G$ is the subgroup

$$\widetilde{H} = \left\{ \begin{pmatrix} X & 0 \\ 0 & a \end{pmatrix} : X \in GU_{n-1}(q); a \in \mathbb{F}_{q^2}^{\times} \text{ s.t. } a^{q+1} = 1 \right\}.$$

If $\varphi \in \mathrm{Irr}(\widetilde{H})$ is an irreducible constituent of $\psi^{\widetilde{H}}$, then since

$$\widetilde{H}/K = \widetilde{H}/(\widetilde{H} \cap S) \cong \widetilde{H}S/S \leq G/S$$

is cyclic, we have by [33, Corollary 11.22] and Clifford's theorem that

$$\varphi|_K = \sum_{i=1}^{[\widetilde{H}:I_{\widetilde{H}}(\psi)]} \psi^{h_i}.$$

But also, $\widetilde{H} = K \times \mathcal{T}$, which means that $\varphi = \beta \otimes \lambda$ where $\beta \in \mathrm{Irr}(K), \lambda \in \mathrm{Irr}(\mathcal{T})$. Then $\varphi|_K = \beta \in \mathrm{Irr}(K)$, and so we have that in fact, $\varphi|_K = \psi$.

Now since $\left(\theta|_{\widetilde{H}}\right)\big|_K = [G : I_G(\chi)]\psi$ and the irreducible constituents of $\psi^{\widetilde{H}}$ are the only irreducible characters of $\widetilde{H}$ which contain ψ as a constituent when restricted to K, we have that

$$\theta|_{\widetilde{H}} = \sum a_i \varphi_i$$

where a_i are some nonnegative integers and φ_i are the irreducible constituents of $\psi^{\widetilde{H}}$. Note that each of these φ_i satisfy $\varphi_i|_K = \psi$ by the above argument. In particular, we have

$$[G : I_G(\chi)]\psi = \theta|_K = \sum a_i \varphi_i|_K = \left(\sum a_i\right)\psi$$

so that $\sum a_i = [G : I_G(\chi)]$. Also, since $\widetilde{H} = H \times \mathcal{T}$, we have that $\varphi_i = \beta_i \otimes \lambda_i$ where $\beta_i \in \mathrm{Irr}(H)$ and $\lambda_i \in \mathrm{Irr}(\mathcal{T})$, so that $\varphi_i|_H = \beta_i$. Then

$$\theta|_H = \left(\theta|_{\widetilde{H}}\right)\big|_H = \sum a_i \varphi_i|_H = \sum a_i \beta_i.$$

Then $\theta|_H$ is the sum of $[G : I_G(\chi)]$ irreducible characters of H which each have the same degree, since $\beta_i(1) = \varphi_i(1) = \psi(1)$ for each i. Hence, it just remains to show that $[G : I_G(\chi)]$ divides $(n, q+1)$.

Let $J := I_G(\chi)$. Now, clearly we have that $Z := Z(G)$ is contained in J, and $S \leq J$, so that $ZS \leq J$. Then $[G : J]$ divides $[G : ZS]$. But

$$[G : ZS] = [G/S : ZS/S] = \frac{q+1}{[ZS : S]}$$

and

$$[ZS : S] = [Z : Z \cap S] = \frac{q+1}{(n, q+1)}.$$

Thus we have that $[G : ZS] = (n, q+1)$, so $[G : J]$ divides $(n, q+1)$, completing the proof. $\square$

6.3 The Case $n = 5$.

Proposition 6.2.1 implies that to show our conjecture that there are no $\chi \in \mathrm{Irr}(S)$ such that $\chi|_K \in \mathrm{Irr}(K)$, it suffices to show there are no irreducible characters χ of $G = GU_n(q)$ such that when restricted to $H \cong GU_{n-1}(q)$, χ can be written

$$\chi|_H = \varphi_1 + ... + \varphi_m$$

where m divides $\gcd(n, q+1)$, and the degrees $\varphi_i(1)$ are the same for each $i = 1, ..., m$. We now consider the case when $n = 5$ and show that in this case, no such χ exist.

Notice that the property $m | \gcd(5, q+1)$ requires that $m = 1, 5$, and m can only be 5 in the case that $q \equiv 4 \mod 5$. If $m = 1$, then this is exactly when $\chi|_H$ is irreducible.

To characterize characters χ of $G = GU_5(q)$ which satisfy $\chi|_H = \varphi_1 + ... + \varphi_m$ with $m|(5, q+1)$ and $\varphi_1(1) = ... = \varphi_m(1)$, we will make use of the character tables for $GU_5(q)$ and $GU_4(q)$ found by Sohei Nozawa in [59] and [58]. The following theorem of Zsigmondy from elementary number theory will also be useful.

Theorem 6.3.1 (Zsigmondy)**.** *Let q, n be integers with $q \geq 2, n \geq 3$. Then provided that as pairs in $\mathbb{Z}^2$, $(q, n) \neq (2, 6)$, there is a prime s such that s divides $q^n - 1$ but s does not divide $q^i - 1$ for any $i < n$.*

Using this theorem, we get the following lemma:

Lemma 6.3.2. *Let $G = GU_5(q)$ and $H \leq G$ the subgroup as before. If $\chi \in \mathrm{Irr}(G)$ and $q^4 - q^3 + q^2 - q + 1$ divides $\chi(1)$, then χ can not satisfy*

$$\chi|_H = \varphi_1 + ... + \varphi_m$$

with $m|(5, q+1)$ and φ_i all the same degree.

Proof. We have that $q^5 + 1 = (q+1)(q^4 - q^3 + q^2 - q + 1)$, and so using Zsigmondy's theorem with q and $n = 10$, we have that there is a prime s dividing $q^{10} - 1$ but not dividing $q^i - 1$ for any $i < 10$. In particular, s does not divide $q^5 - 1$, which means that s must divide $q^5 + 1$. Also, s must not divide $q^3 + 1$ since then it divides $q^6 - 1$ and s can't divide $q + 1$ since then it divides $q^2 - 1$. Thus we have that there exists a prime s dividing $q^5 + 1$ but not dividing $q^4 - 1$, $q^3 + 1$, $q^2 - 1$, or $q + 1$. This prime s must therefore divide $q^4 - q^3 + q^2 - q + 1$ (since it doesn't divide $q + 1$), and therefore $s|\chi(1)$, but cannot divide $|H| = |GU_4(q)| = q^6(q+1)(q^2-1)(q^3+1)(q^4-1)$. This implies that s cannot divide $\varphi(1)$ for any $\varphi \in \mathrm{Irr}(H)$, and hence $\chi|_H$ is not irreducible.

Now, if $q \equiv 4 \mod 5$, then suppose by way of contradiction that $\chi|_H = \varphi_1 + ... + \varphi_5$ with $\varphi_i \in \mathrm{Irr}(H)$ for $1 \leq i \leq 5$, all of the same degree. Then $\chi(1) = 5\varphi(1)$ for $\varphi = \varphi_1$. As s divides $\chi(1)$ but not $\varphi(1)$, we see $s = 5$. But this is a contradiction, since $q \equiv 4 \mod 5$, and therefore $s = 5$ divides $q + 1$, but we have already seen that s does not divide $q + 1$. Therefore, $\chi|_H$ cannot be of this form. $\square$

The next lemma will be useful in bounding the degrees of characters for G and H.

Lemma 6.3.3. *Let $P(x)$ be a monic polynomial of degree r which is a product of monic polynomials whose nonzero coefficients are from $\{\pm 1\}$. Then*

$$(q-1)^r \leq P(q) \leq (q+1)^r$$

for any integer $q \geq 2$.

Proof. Without loss of generality, we may assume that $P(q)$ is a monic polynomial of degree r whose nonzero coefficients are all ± 1, since if each factor of this form satisfies the inequality, then the product will also. The second inequality, $P(q) \leq (q+1)^r$, is obvious, since the binomial expansion tells us that the coefficients of $(q+1)^r$ are all at least 1. For the first inequality, we proceed by induction.

If $r = 2$, then $(q-1)^2 = q^2 - 2q + 1 \leq q^2 - q - 1 \leq P(q)$, and so the inequality is satisfied. Now suppose that for polynomial degrees less than r, the inequality holds. We know that $P(q) \geq q^r - q^{r-1} - q^{r-2} - ... - q - 1$. We claim that $q^r - q^{r-1} - ... - q - 1 \geq (q-1)(q^{r-1} - q^{r-2} - ... - q - 1)$. We have that

$$(q-1)(q^{r-1} - q^{r-2} - ... - q - 1) = q^r - 2q^{r-1} + 1$$

and so $q^r - q^{r-1} - ... - q - 1 \geq (q-1)(q^{r-1} - q^{r-2} - ... - q - 1)$ exactly when $q^{r-1} - q^{r-2} - ... - q - 2 \geq 0$. We know that $2^{r-1} - 2^{r-2} - ... - 2 - 2 = 0$, and by Descartes rule of signs, this is the only positive real root of the polynomial. Then since for $q = 3$, $q^{r-1} - q^{r-2} - ... - q - 2 \geq 0$, we know that this is true whenever $q \geq 2$, proving the claim.

This yields that

$$\begin{aligned} P(q) \geq q^r - q^{r-1} - q^{r-2} - ... - q - 1 &\geq (q-1)(q^{r-1} - q^{r-2} - ... - q - 1) \\ &\geq (q-1)(q-1)^{r-1} = (q-1)^r \end{aligned}$$

by the induction hypothesis, and hence

$$(q-1)^r \leq P(q) \leq (q+1)^r,$$

as stated.

□

Proposition 6.3.4. *Let $G = GU_5(q)$ and $H \cong GU_4(q)$ be the subgroup of G as above. Suppose that $\chi \in \mathrm{Irr}(G)$ satisfies $\chi|_H = \varphi_1 + ... + \varphi_m$ with $m|(5, q+1)$ and φ_i all the same degree. Then $\chi(1) = 1$.*

Proof. Assume that $\chi(1) > 1$. From Lemma 6.3.2, we see that in the notation of Nozawa [59], χ is a member of one of the following families of characters in $\mathrm{Irr}(G)$: $A_{11}(i), A_{12}(i), A_{14}(i), A_{16}(i), F(i)$, as these are the only irreducible characters of G with degree not divisible by $q^4 - q^3 + q^2 - q + 1$.

Moreover, from [59] and [58] we see that every character degree of $GU_5(q)$ or $GU_4(q)$ is the product of monic polynomials in q whose nonzero coefficients are all ± 1. We will apply Lemma 6.3.3 to the situation where $P(q)$ is the degree of $\chi \in \mathrm{Irr}(G)$ or $\varphi \in \mathrm{Irr}(H)$. From [58] we note that the highest degree of such a polynomial $P(q) = \varphi(1)$ where $\varphi \in \mathrm{Irr}(H)$ is 6, and hence $\mathfrak{m}(H) \leq (q+1)^6$.

First, suppose $\chi \in \mathrm{Irr}(G)$ is a member of the family $A_{11}(i)$. Then from [59] we have $\chi(1) = q^{10}$, and thus if $\chi|_H \in \mathrm{Irr}(H)$, then we would have that there is some $\varphi \in \mathrm{Irr}(H)$ with $\chi|_H = \varphi$, so

$$(q+1)^6 \geq \chi(1) = q^{10}.$$

But for any $q \geq 2$, this is impossible, and thus $\chi|_H$ is not irreducible. Hence we must have $\chi(1) = 5\varphi(1)$ for some $\varphi \in \mathrm{Irr}(H)$. Now if $q \equiv 4 \mod 5$, then $\chi(1) = q^{10} \equiv 4^{10} \equiv 1 \mod 5$, and therefore 5 does not divide $\chi(1)$, a contradiction.

Now suppose that $\chi \in \mathrm{Irr}(G)$ is a member of the family $A_{12}(i)$. [59] tells us that $\chi(1) = q^6(q-1)(q^2+1)$, which as a polynomial in q has degree 9. Then by Lemma 6.3.3, if $\chi|_H = \varphi \in \mathrm{Irr}(H)$, we have

$$(q-1)^9 \leq \chi(1) \leq (q+1)^6,$$

so that q must be 2 or 3. If $q = 3$, then we see that $\chi(1) = 14580 > 4096 = 4^6 = (q+1)^6$, violating Lemma 6.3.3. Then it must be that $q = 2$, which we by computation in GAP (see Section 6.3.1) is a contradiction. So $\chi|_H \notin \mathrm{Irr}(H)$, and it must be that $q \equiv 4 \mod 5$. Then $\chi(1) \equiv 1 \mod 5$ and therefore 5 cannot divide $\chi(1)$, a contradiction.

For $\chi \in \mathrm{Irr}(G)$ a member of the family $A_{14}(i)$, we have that $\chi(1) = q^3(q^2 - q + 1)(q^2+1)$. We compare this degree to all 22 possible character degrees in $\mathrm{Irr}(H)$, given in [58]. For each $\varphi \in \mathrm{Irr}(H)$, we use the rational roots test to help find any possible integers $q = p^k$ which can satisfy $\varphi(1) = \chi(1)$. It turns out that for this family, no such q exists for any φ, and therefore $\chi|_H \notin \mathrm{Irr}(H)$. Also, as in the cases above, when $q \equiv 4 \mod 5$ we have $\chi(1)$ is not divisible by 5, again yielding a contradiction.

If $\chi \in \mathrm{Irr}(G)$ is a member of the family $A_{16}(i)$, then as above, when $q \equiv 4 \mod 5$, 5 does not divide $\chi(1)$. Proceeding as in the family $A_{14}(i)$, we see that since $\chi(1) = q(q-1)(q^2+1)$, the only possibility of $\varphi \in \mathrm{Irr}(H)$ such that $\varphi(1) = \chi(1)$ are characters of H of the family $\chi_{17}(i)$, which have degree $q(q-1)^2(q^2+1)$, when $q = 2$. Again, computation in GAP (see Section 6.3.1) shows that these characters are not the restriction of any character of G, and $\chi|_H$ is reducible.

Finally, let χ be a member of the family $F(i)$, we notice that $\chi(1) = (q+1)(q^2 - 1)(q^3+1)(q^4-1)$, which is larger than $(q+1)^6$ for all $q \geq 2$. Hence χ is reducible when restricted to H. Now for each $\varphi \in \mathrm{Irr}(H)$ given by [58], we again use the rational roots test to determine which q allow for $\chi(1) = 5\varphi(1)$. It turns out that none of the irreducible characters of H satisfy this condition for any $q \equiv 4 \mod 5$, and therefore χ cannot be in this family.

Hence, it must be that χ is a linear character of G.

□

We are now ready to solve Problem 2 in the case $n = 5$:

Proposition 6.3.5. *There are no nonlinear irreducible characters χ of $S = SU_5(q)$ such that the restriction $\chi|_K$ to $K \cong GU_4(q)$, is irreducible.*

Proof. This is immediate from Proposition 6.3.4 and Proposition 6.2.1.

□

We note that to prove Proposition 6.3.5, we could have noted that when $q = 2$, the result follows from Proposition 6.1.3, so that the computations in GAP were not necessary. However, we provide the discussion of these computations in the next section for completion.

6.3.1 GAP Computation for $q = 2$

In order for $\chi \in \mathrm{Irr}(G)$ to satisfy $\chi|_H \in \mathrm{Irr}(H)$, we must have that there is $\varphi \in \mathrm{Irr}(H)$ so that $\chi(1) = \varphi(1)$. For $q = 2$, to find all such pairs of G- and H-characters, we use the following code:

```
G:=GeneralUnitaryGroup(5,2);
H:=GeneralUnitaryGroup(4,2);
c:=CharacterTable(G);
d:=CharacterTable(H);
irrG:=Irr(c);
irrH:=Irr(d);

for k in [1..Length(irrG)]
do
if Degree(irrG[k])>1 then
   for i in [1..Length(irrH)]
   do
      if Degree(irrG[k])=Degree(irrH[i]) then
```

```
        Print("X");Print(k);Print(",");
        Print("Y");Print(i); Print(" \n");

      fi;
   od;
fi;
od;
```

This code compares all degrees of irreducible characters of G to all degrees of irreducible characters of H, and prints out all pairs `Xk,Yj` (where `Xk` is the kth character of G and `Yj` is the jth character of H) which have the same degree. The resulting pairs are:

```
X4,Y13   X4,Y14   X4,Y15   X4,Y16   X4,Y17   X4,Y18
X5,Y13   X5,Y14   X5,Y15   X5,Y16   X5,Y17   X5,Y18
X6,Y13   X6,Y14   X6,Y15   X6,Y16   X6,Y17   X6,Y18
```

This means that the only possibilities for $\chi \in \mathrm{Irr}(G)$ to be irreducible when restricted to H are if χ is the 4th, 5th, or 6th irreducible character of G in GAP's library, and the corresponding character of H must be the 13th, 14th, 15th, 16th, 17th, or 18th. Now, in order for one of the pairs to actually satisfy `Xk` $=: \chi|_H = \varphi :=$ `Yj`, we must have that every value in the character table for φ is also found in the character table for χ. As it turns out, none of these pairs satisfy this condition, as we find using the following code:

```
for k in [4..6]
do
    for l in [13..18]
```

```
    do
        Print("The ", l, "th character of H and ",
                      k, "th character of G  \n");
        Print("i'th character value of the H-char,
                j'th character value of the G-char \n");
        for i in [1..Length(irrH[l])]
        do
            for j in [1..Length(irrG[k])]
            do
                if irrG[k][j] = irrH[l][i] then
                    Print(i);Print(",");Print(j);
                    Print(" ");Print("\n"); break;
                fi;

            od;
        od;
    od;
od;
```

This code runs through all pairs of G-characters and H-characters which were found to have the same degree above, and then for each such pair runs through all values found in the character tables. If there is at least one character value of `Xk` which matches the ith character value of `Yl`, then GAP will print "i, j" where j is the first index of a character value of `Xk` which matches the character value of `Yl`. Then any character value of `Yl` which is not printed is not found as a `Xk` character value, showing that the H-character `Yl` is not equal to the G-character `Xk` restricted to H.

Indeed, running the code, we find that for each pair of H- and G- characters with the same degree, there is at least one character value for the H-character which does not match any of the character values for the G-character. This shows that for $q = 2$, no nonlinear irreducible character of G is irreducible when restricted to H. Moreover, since $(5, 3) = 1$, this implies that when $q = 2, n = 5$ no nonlinear irreducible character satisfies the condition in Proposition 6.2.1.

6.4 On the Remaining Cases $4 \leq n < 10$

So far what we have shown is that there are no nontrivial $\chi \in \mathrm{Irr}(SU_n(q))$ such that $\chi|_K \in \mathrm{Irr}(K)$ when $(q+1) \not| n$ or when $n = 5$. Certainly, Proposition 6.1.3 yields that for $n = 7$, there are no nontrivial characters of $SU_7(q)$ which restrict irreducibly to K. The following propositions show that for $n = 4$ or 6, there are also no $\chi \in \mathrm{Irr}(SU_n(q))$ which restrict irreducibly to $SU_{n-1}(q)$.

Proposition 6.4.1. *Let q be a prime power and let $\chi \in \mathrm{Irr}(SU_4(q))$ such that $\chi|_K \in \mathrm{Irr}(K)$, where K is as above. Then $\chi = 1_{SU_4(q)}$.*

Proof. Let $G = SU_4(q)$ and suppose $\chi \in \mathrm{Irr}(G)$ is nontrivial and $\chi|_K \in \mathrm{Irr}(K)$. Then by Proposition 6.1.3, we see that q must be 3. From GAP, we see that the only character degree that $K \cong GU_3(3)$ and $G \cong SU_4(3)$ share is 21, and that the character of this degree in G is integer-valued, yielding only two possibilities for $\chi|_K$. Now, observing the character values on classes consisting of elements of order 12, we see that these integer-valued characters of K of degree 21 cannot be the restriction of a degree-21 character of G. Hence we have that $\chi = 1_G$, as stated.

□

Proposition 6.4.2. *Let q be a prime power and let $\chi \in \mathrm{Irr}(SU_6(q))$ such that $\chi|_L \in \mathrm{Irr}(L)$, where $L \leq K$ is the subgroup isomorphic to $SU_5(q)$. Then $\chi = 1_{SU_6(q)}$.*

Proof. Let $G = SU_6(q)$ and suppose $\chi \in \mathrm{Irr}(G)$ is nontrivial and $\chi|_L \in \mathrm{Irr}(L)$. Then by Proposition 6.1.3, we see that q must be 2 or 5.

First suppose that $G = SU_6(2)$ and $L \cong SU_5(2)$. In this case, the only character degree that these groups share is 440. Now, G has exactly one character of this degree, χ_6 in the notation of GAP, and this character is integer-valued. L has 3 characters of this degree, but only one which is integer-valued. However, this integer-valued character of L with degree 440 takes the value 88 on a class of involutions, while no involutions g of G satisfies $\chi_6(g) = 88$. Hence in this case, we must have $\chi = 1_G$.

Now let $G = SU_6(5)$ and $L \cong SU_5(5)$. Then the only degree in common is 1693250, which has multiplicity 1 in G and 5 in L. Observing the character tables for $GU_6(q)$ and $GU_5(q)$ in CHEVIE [26], we see that these restrict from characters in the family $\chi|_8$ of $GU_6(q)$ and $\chi|_{14}$ of $GU_5(q)$, respectively. Observing the values of these characters on unipotent classes, we see that a character in the family $\chi|_{14}$ of $GU_5(q)$ cannot be the restriction of a character in the family $\chi|_8$ of $GU_6(q)$. Hence, we again see that $\chi = 1_G$.

$\square$

We now mention that if $n = 8$ or 9, then there is exactly one possibility for $\chi \in \mathrm{Irr}(SU_n(q))$ which could restrict to a character in $\mathrm{Irr}(SU_{n-1}(q))$. (Though of course, we conjecture that this character does not restrict irreducibly.)

Indeed, if $n = 8$, then by Proposition 6.1.3, we have that $q = 2$ or 7. From [49], we see that the only irreducible character degree that $SU_8(2)$ and $SU_7(2)$ share is 211904, which has multiplicity 1 for $SU_8(2)$ and multiplicity 4 for $SU_7(2)$. Similarly, the only irreducible character degree in common for $SU_8(7)$ and $SU_7(7)$ is 1450393913575299, with multiplicity 1 in $SU_8(7)$ and 14 in $SU_7(7)$.

If $n = 9$, then Proposition 6.1.3 yields that $q = 2$ or 8. From [49],only irreducible character degree shared by $SU_9(2)$ and $SU_8(2)$ is 29240, which has multiplicity 1 in either group. The only degree shared by $SU_9(8)$ and $SU_8(8)$ is 31771439198720, which also has multiplicity 1 in either group.

Chapter 7

$Sp_6(2^a)$ is "Good" for the McKay, Alperin Weight, and Related Local-Global Conjectures

In this Chapter, we prove Theorem 1.2.1, which shows that $Sp_6(q)$ and $Sp_4(q)$, with q even, satisfy the conditions for the reductions to the McKay, Alperin weight, blockwise Alperin weight, and Alperin-McKay conjectures. Recall the discussion in Section 1.2 in Chapter 1 of these conjectures and the reductions.

7.1 Preliminaries and Notation

Throughout this chapter, ℓ denotes a prime, thought of as the characteristic for a representation. As usual, $\mathrm{Irr}(X)$ will denote the set of irreducible ordinary characters of X and $\mathrm{IBr}_\ell(X)$ will denote the set of irreducible ℓ-Brauer characters of X. Further, $\mathrm{Bl}(X|\chi)$ denotes the block of the group X containing $\chi \in \mathrm{Irr}(X) \cup \mathrm{IBr}_\ell(X)$, $\mathrm{Irr}_0(X|D)$ denotes the set of height-zero characters of X which lie in any block with defect group D, and $\mathrm{dz}(X)$ denotes the set of defect-zero characters of X. Given $\chi \in \mathrm{Irr}(X)$, recall that we denote the central character associated to χ by ω_χ. Further, we will denote by $*$ a fixed isomorphism from the set of ℓ'-roots of unity in $\mathbb{C}$ to $\overline{\mathbb{F}}_\ell^\times$ and set $\lambda_B = \omega_\chi^*$ for $B = \mathrm{Bl}(X|\chi)$, as in [33, Chapter 15]. Given a set $\mathfrak{S}$, write $\mathfrak{S}^+ := \sum_{x\in\mathfrak{S}} x$. If $Y \leq X$ is a subgroup, and $b \in \mathrm{Bl}(Y)$, then the induced block b^X is the unique block B so that $\lambda_b^X(\mathcal{K}^+) = \lambda_B(\mathcal{K}^+)$ for all conjugacy classes $\mathcal{K}$ of X, if such a B exists. (In this situation, recall that b^X is said to be defined.) Recall that $\lambda_b^X(\mathcal{K}^+)$ is given by $\lambda_b\left((\mathcal{K}\cap Y)^+\right)$.

If a group X acts on a set $\mathfrak{S}$ and $\mathfrak{s} \subseteq \mathfrak{S}$, then we denote by $X_\mathfrak{s}$ or $\mathrm{stab}_X(\mathfrak{s})$ the subgroup of X stabilizing $\mathfrak{s}$. If X acts on a group Y, we denote by $Y : X$ or $Y \rtimes X$ the semidirect product of Y with X. We may also say this is the extension of Y by

X. In such situations, if r is a positive integer and p is a prime, we will write $Y : r$ if $X = C_r$ is the cyclic group of order r and $Y : p^r$ if X is elementary abelian of order p^r.

For the remainder of the chapter, ℓ is an odd prime and q is a power of 2. Recall that $|Sp_6(q)| = q^9(q^2-1)(q^4-1)(q^6-1)$, so if ℓ is a prime dividing $|Sp_6(q)|$ and $\ell \neq 3$, then ℓ must divide exactly one of $q-1$, $q+1$, q^2+1, q^2+q+1, or q^2-q+1. If $\ell = 3$, then it divides $q-1$ if and only if it divides q^2+q+1, and it divides $q+1$ if and only if it divides q^2-q+1. When ℓ divides q^6-1, we will write $\epsilon \in \{\pm 1\}$ for the number such that $\ell | (q^3-\epsilon)$. If $\ell \neq 3$, we write d for the integer such that $(q^6-1)_\ell = \ell^d$. (Here r_ℓ is the ℓ-part of the integer r.) If $\ell = 3$, we write d for the integer such that $3^d = (q-\epsilon)_3$, so that $(q^3-\epsilon)_3 = 3^{d+1}$. In any case, we will denote by m the integer $(q-\epsilon)_{\ell'}$.

We will also borrow from CHEVIE [26] the notation for characters of $Sp_6(q)$ and the roots of unity $\zeta_i := \exp\left(\frac{2\pi\sqrt{-1}}{q^i-1}\right)$ and $\xi_i := \exp\left(\frac{2\pi\sqrt{-1}}{q^i+1}\right)$. We will sometimes also use $\widetilde{\zeta_i}$ or $\widetilde{\xi_i}$ to denote a corresponding root of unity in $\overline{\mathbb{F}}_q^\times$.

The following sets for indices will be useful. For $\epsilon \in \{\pm 1\}$, let $I^0_{q-\epsilon}$ be the set $\{i \in \mathbb{Z} : 1 \leq i \leq q-\epsilon-1\}$, and let $I_{q-\epsilon}$ be a set of class representatives on $I^0_{q-\epsilon}$ under the equivalence relation $i \sim j \iff i \equiv \pm j \mod (q-\epsilon)$. Let $I^0_{q^2+1} := \{i \in \mathbb{Z} : 1 \leq i \leq q^2\}$ and $I^0_{q^2-1} := \{i \in \mathbb{Z} : 1 \leq i \leq q^2-1, (q-1) \not| i, (q+1) \not| i\}$, and let $I_{q^2-\epsilon}$ be a set of representatives for the equivalence relation on $I^0_{q^2-\epsilon}$ given by $i \sim j \iff i \equiv \pm j$ or $\pm qj \mod (q^2-\epsilon)$. Similarly, let $I^0_{q^3-\epsilon} := \{i \in \mathbb{Z} : 1 \leq i \leq q^3-\epsilon; (q^2+\epsilon q+1) \not| i\}$ and $I_{q^3-\epsilon}$ a set of representatives for the equivalence relation on $I^0_{q^3-\epsilon}$ given by $i \sim j \iff i \equiv \pm j, \pm qj$, or $\pm q^2 j \mod (q^3-\epsilon)$. Given one of these indexing sets, I_*, we write I^k_* for the elements $(i_1, ..., i_k)$ of $I_* \times I_* ... \times I_*$ (k copies) with none of $i_1, i_2, ..., i_k$ the same and I^{k*}_* for the set of equivalence classes of I^k_* under $(i_1, ..., i_k) \sim (\rho(i_1), ..., \rho(i_k))$ for all $\rho \in S_k$.

Let $G := Sp_6(q)$ and let $\mathcal{E}_1$ denote the set of unipotent characters and $\mathcal{E}_i(J)$ denote the Lusztig series $\mathcal{E}(G, (s))$ for G, where s is conjugate in G^* to the semisimple element $g_i(J)$ in the notation of [47]. Here J denotes the proper indices (for example, for the family g_6, $J=(i)$ for $i \in I_{q-1}$, and for the family g_{32}, $J=(i,j,k)$ where $(i,j,k) \in I_{q+1}^{3*}$).

D. White [76] has calculated the decomposition numbers for the unipotent blocks of G, up to a few unknowns in the case $\ell|(q+1)$, and we have used this information in Section 4.3 to describe the irreducible Brauer characters in these blocks. Moreover, recall that in Section 4.4, we have used the theory of central characters, which are available in the CHEVIE system [26] for G, to determine the block distribution of the remaining complex characters and that we have described the Brauer characters of G in terms of the restrictions of ordinary characters to G°. In particular, recall that the set $\mathcal{E}\left(G,(g_i(J))\right)$ forms a basic set for the blocks of $\mathcal{E}_\ell\left(G,(g_i(J))\right)$ for the semisimple ℓ'-elements $g_i(J)$.

As in Section 4.4, when $\ell|(q^2-1)$, we will denote by $B_i(J)$ the ℓ-blocks (or, in some situations, just the irreducible Brauer characters) in $\mathcal{E}_\ell(G,(s))$ where s is conjugate in G^* to the semisimple element $g_i(J)$ in the notation of [47]. In most cases, $C_{G^*}(s)$ has only one unipotent block, and therefore $\mathcal{E}_\ell(G,(s))$ is a single block. However, when multiple such blocks exist, which occurs for $i=6,7,8,9$ when $\ell|(q^2-1)$, we will denote by $B_i(J)^{(0)}$ the (Brauer characters in) the block corresponding in the Bonnafé-Rouquier correspondence to the principal block of $C_{G^*}(s)$ and by $B_i(J)^{(1)}$ the (characters in the) block corresponding to the unique other block of positive defect. Further, B_0 and B_1 will denote the (Brauer characters in the) principal block and the cyclic unipotent block, respectively, as described in [76].

7.2 Radical Subgroups of $Sp_6(2^a)$ and $Sp_4(2^a)$

In this section we describe the radical subgroups of $Sp_6(q)$ with q even and their normalizers. In [6], J. An describes the radical subgroups for $Sp_{2n}(q)$ with odd q, and his results in the first two sections extend to $Sp_6(q)$ when q is even and $\ell|(q^2-1)$, so we will often refer the reader there. We begin by setting some notation for the subgroups of $Sp_6(q)$ that will be of interest.

Let $G = Sp_{2n}(q)$ with q even and let $\{e_1, e_2, ..., e_n, f_1, f_2, ..., f_n\}$ be the standard symplectic basis for the natural module $\mathbb{F}_q^{2n}$ for G. For $r \leq n$, we can view $Sp_{2r}(q)$ as a subgroup of G by identification with the pointwise stabilizer

$$Sp_{2r}(q) \cong \mathrm{stab}_G(e_{r+1}, ..., e_n, f_{r+1}, .., f_n),$$

and by iterating this, we see that for integers $r_1 \geq r_2 \geq ... \geq r_n \geq 0$ so that $s := \sum_{i=1}^n r_i \leq n$, we may view the direct product $\prod_i Sp_{2r_i}(q)$ as a subgroup of G which stabilizes (point-wise) a $2(n-s)$-dimensional subspace of $\mathbb{F}_q^{2n}$. Moreover, we may further view $GL_r^{\pm}(q)$ as a subgroup of $Sp_{2r}(q)$, so that $\prod_i GL_{r_i}^{\pm}(q) \leq G$ under this embedding. We will also require the embeddings $GL_1^{\pm}(q^3) \leq GL_3^{\pm}(q)$ and $GU_1(q^2) \leq GL_2(q)$. (Here we use the notation $GL_r^{+}(q) := GL_r(q)$ and $GL_r^{-}(q) := GU_r(q)$.)

Now specialize $n = 3$, so $G = Sp_6(q)$, and write $H := Sp_4(q) = \mathrm{stab}_G(e_3, f_3)$. Suppose first that $\ell | (q^2-1)$, and let $\epsilon \in \{\pm 1\}$ be such that $\ell | (q-\epsilon)$, with $(q-\epsilon)_\ell = \ell^d$. (We will also write ϵ for the corresponding sign $\pm$.) Let $r_1 \geq r_2 \geq r_3 \geq 0$ be as in our discussion above, and define $Q_{r_1,r_2,r_3} := \mathbf{O}_\ell\left(Z\left(\prod_{i=1}^3 GL_{r_i}^{\epsilon}(q)\right)\right)$, viewed as an ℓ-subgroup of G under the embedding described above. Then $C_G(Q_{r_1,r_2,r_3}) = Sp_{2(n-s)}(q) \times \prod_{i=1}^3 GL_{r_i}^{\epsilon}(q)$, and if c_i is the number of times r_i appears, then $N_G(Q_{r_1,r_2,r_3}) = Sp_{2(n-s)}(q) \times \prod \left(GL_{r_i}^{\epsilon}(q) : 2\right) \wr S_{c_i}$, where the product is now taken over the i so that each distinct r_i appears only once. (This can be seen from direct calculation, or by arguments similar to those in [6, Sections 1 and 2].) Here we can view $GL_{r_i}^{\epsilon}(q)$ as its image under the map $A \mapsto \mathrm{diag}(A, {}^TA^{-1})$, possibly viewed in the overgroup $Sp_{2r_i}(q^2)$, with the C_2 extension inducing the graph automorphism $\tau\colon A \mapsto {}^TA^{-1}$ on $GL_{r_i}^{\epsilon}(q)$. When $r_i = 0$ for some i, we will suppress the notation, so that we will write, for example, Q_1 rather than $Q_{1,0,0}$, and $Q_{1,1}$ rather than $Q_{1,1,0}$. Hence Q_1, Q_2, and Q_3 are cyclic groups of order ℓ^d, $Q_{1,1}$ and $Q_{2,1}$ are isomorphic to $C_{\ell^d} \times C_{\ell^d}$, and $Q_{1,1,1}$ is isomorphic to $C_{\ell^d} \times C_{\ell^d} \times C_{\ell^d}$. Moreover, notice that $Q_{1,1} \in \mathrm{Syl}_\ell(H)$, and when

$\ell \neq 3$, $Q_{1,1,1} \in \mathrm{Syl}_\ell(G)$.

If $\ell = 3$, let P denote the Sylow subgroup, which is $Q_{1,1,1} \rtimes C_3$, or $C_{\ell^d} \wr C_3$, which we can view inside $Sp_2(q) \wr S_3 \leq G$. Write $Z := Q_3$ and let $R \leq GL_3^\epsilon(q)$ be the embedding of the symplectic-type group which is the central product of Z and an extraspecial group E of order 27 with exponent 3, as in [6, (1A) and (1B)]. That is, $E = \langle x_1, x_2 \rangle$ with $x_i^3 = 1$ for $i = 1, 2$, $[x_1, x_2] = y$, and $\{z \in Z \colon z^3 = 1\} = Z(E) = \langle y \rangle$. Moreover, the group of automorphisms of R which commute with Z is isomorphic to $\mathrm{Inn}(E) \rtimes Sp_2(3)$. (See [6, (1.1)].)

Now suppose that $\ell | (q^4 + q^2 + 1)$, and let ϵ be so that $\ell | (q^2 + \epsilon q + 1)$. Write $Q^{(3)} := \mathbf{O}_\ell(Z(GL_1^\epsilon(q^3)))$. When $\ell \neq 3$, $Q^{(3)}$ is a cyclic Sylow ℓ-subgroup of G of order $(q^2 + \epsilon q + 1)_\ell$. When $\ell = 3$, we have $3 | (q - \epsilon)$ as well, and $(q^2 + \epsilon q + 1)_3 = 3$. In this case, $Q^{(3)}$ is a cyclic group of order 3^{d+1} where $(q - \epsilon)_3 = 3^d$. When $\ell | (q^2 + 1)$, write $Q^{(2)} := \mathbf{O}_\ell(Z(GU_1(q^2)))$ so that $Q^{(2)}$ is a cyclic Sylow ℓ-subgroup of G.

Let $s := s_3, s_2$ be a generator of $Q^{(3)}, Q^{(2)}$, respectively. Write $N := N_G(\langle s \rangle)$ and $C := C_G(\langle s \rangle)$. From the description in [47] of semisimple classes of G, we see that s is conjugate to s^i if and only if $i \in \{\pm q, ..., \pm q^j\}$, where $j = 3, 2$ respectively, so that $N/C = \langle \tau, \beta \rangle$ is generated by $\tau \colon s \mapsto s^{-1}$, $\beta \colon s \mapsto s^q$. Moreover, $C_G(s_3) = C_{q^3 - \epsilon}$ and $C_G(s_2) = C_H(s_2) \times Sp_2(q) = C_{q^2+1} \times Sp_2(q)$, so $N_G(Q^{(3)}) = C_{q^3-\epsilon} : 6$ and $N_G(Q^{(2)}) = N_H(Q^{(2)}) \times Sp_2(q) = C_{q^2+1} : 2^2 \times Sp_2(q)$.

Proposition 7.2.1. *1. Let $G = Sp_6(q)$ with q even and let Q be a nontrivial ℓ-radical subgroup of G for a prime $\ell \neq 2$ dividing $|G|$. Then:*

- *If $3 \neq \ell | (q^2 - 1)$, then Q is G-conjugate to one of $Q_1, Q_2, Q_3, Q_{1,1}, Q_{2,1}$ or $Q_{1,1,1}$.*
- *If $\ell = 3 | (q^2 - 1)$, then Q is G-conjugate to one of $Q_1, Q_2, Q_3, Q_{1,1}, Q_{2,1}$, $Q_{1,1,1}, Q^{(3)}, P$ or R.*
- *If $3 \neq \ell | (q^4 + q^2 + 1)$, then Q is G-conjugate to $Q^{(3)}$.*

- *If $\ell|(q^2+1)$, then Q is G-conjugate to $Q^{(2)}$.*

2. *Let $H = Sp_4(q)$ with q even (viewed as $\mathrm{stab}_G(e_3, f_3)$) and let Q be a nontrivial ℓ-radical subgroup of H for a prime $\ell \neq 2$ dividing $|H|$. Then:*

   - *If $\ell|(q^2-1)$, then Q is H-conjugate to one of Q_1, Q_2, or $Q_{1,1}$.*
   - *If $\ell|(q^2+1)$, then Q is H-conjugate to $Q^{(2)}$.*

   Moreover, no two of the subgroups listed are G-conjugate.

To prove Proposition 7.2.1, we will make use of many arguments in [6], and we present here some of the useful lemmas and arguments found there, specialized to our situation.

As above, when $\epsilon \in \{\pm 1\}$, we abuse notation by using ϵ to denote the appropriate sign $\pm$ as well, and $GL_r^{\pm}(q)$ denotes $GU_r(q)$ in case $\epsilon = -$ and $GL_r(q)$ in case $\epsilon = +$.

Lemma 7.2.2 ((1A) of [6])**.** *Let q be any prime power and let ℓ be a prime with $\ell|(q-\epsilon)$ for $\epsilon \in \{\pm 1\}$. Let E be an extraspecial group of order $\ell^{2\gamma+1}$ and write $G = GL^{\epsilon}_{\ell^{\gamma}}(q)$. Then G contains a unique conjugacy class of subgroups isomorphic to E. Moreover, if $\ell|(q-1)$, then $\mathbb{F}_q$ is a splitting field of E.*

Lemma 7.2.3 ((1B) of [6])**.** *Let q be any prime power and let ℓ be a prime with $\ell|(q-\epsilon)$ for $\epsilon \in \{\pm 1\}$. Let E be an extraspecial group of order $\ell^{2\gamma+1}$ and write $G = GL^{\epsilon}_{\ell^{\gamma}}(q)$. Let $R = ZE$ be an ℓ-subgroup of symplectic type of G, with $Z = \boldsymbol{O}_\ell(Z(G))$. Write $C := C_G(R)$ and $N := N_G(R)$. Then $C = Z(G) = Z(N)$ and if E has exponent ℓ, then $N/RC \cong Sp_{2\gamma}(\ell)$. In addition, if R is radical in G, then E has exponent ℓ. Moreover, each linear character of $Z(N)$ acting trivially on $\boldsymbol{O}_\ell(Z(N))$ has an extension to N which is trivial on R.*

As the reader may have inferred, the above lemmas will be primarily useful in the case $\ell = 3$, with R as described above (so $\gamma = 1$), viewed through the embedding of $GL_3(q)$ in $Sp_6(q)$. However, we will formulate our discussion more generally for

$\ell|(q^2-1)$. Given a semisimple $s \in G = Sp_{2n}(q)$, we will call s *primary*, as in [6], if the characteristic polynomial for s acting on the natural module $\mathbb{F}_q^{2n}$ is of the form $\mathfrak{f}(t)^k \in \mathbb{F}_q[t]$, where $\mathfrak{f}$ is either $(t-1)$, self-check, or the product $g(t)g^{\checkmark}(t)$ of a pair of non-self-check polynomials, as in Section 2.4. (Hence if s is primary, then $C_G(s) \cong Sp_k(q), GU_k(q^{d/2})$, or $GL_k(q^{d/2})$, respectively, where $d = \deg \mathfrak{f}$.) As usual, when $\ell|(q^2-1)$, let $\ell^d = (q^2-1)_\ell$. The next lemma is [6, (1C)], adapted for our purposes.

Lemma 7.2.4. *Let $G = Sp_{2n}(q)$ with $n \leq 3$ and q even, let $\ell|(q-\epsilon)$ with $\epsilon \in \{\pm 1\}$, and let $Z = \langle z \rangle$ be cyclic of order $\ell^{d+\alpha}$ for $\alpha \geq 0$. Let E be extraspecial of order $\ell^{2\gamma+1}$, and let $R = ZE$ a symplectic-type group with $Z(R) = Z$. Suppose that $\widetilde{\cdot}$ and $\overline{\cdot}$ are two embeddings of R into G such that $\widetilde{z}$ and $\overline{z}$ are primary. Then $n = m\ell^{\alpha+\gamma}$ for some $m \geq 1$ and $\widetilde{R}$ and $\overline{R}$ are conjugate in G. (Observing the structure of the Sylow ℓ-subgroups, note that $\alpha = 0$ unless $\ell = 3 = n$, in which case $\alpha = 0$ or 1.)*

Identifying R with $\widetilde{R}$, let $C := C_G(R)$, $N := N_G(R)$, and $N^0 := \{g \in N \colon [g, Z] = 1\}$. Then $C \cong GL_m^\epsilon(q^{\ell^\alpha})$. Further, if R is a radical subgroup of G, then E has exponent ℓ and $N^0 = LC$, where $R \lhd L$, $L \cap C = Z(C) = Z(C_G(z)) = Z(L)$, $L/RZ(L) \cong Sp_{2\gamma}(\ell)$, and $[C, L] = 1$. Moreover, each linear character of $Z(L)$ acting trivially on $\boldsymbol{O}_\ell(Z(L))$ can be extended as a character of L which is trivial on R. Also, $N/N^0 \cong N_G(Z)/C_G(Z)$ is cyclic of order $2\ell^\alpha$.

Proof. We largely follow the proof of [6, (1C)]. Since $\widetilde{z}$ and $\overline{z}$ are both primary elements of G, they must be conjugate. (This can be seen, for example, from the conjugacy class descriptions in [21] and [47].) Hence we may assume $Z(\widetilde{R}) = Z(\overline{R})$, so $\widetilde{E}$ and $\overline{E}$ are subgroups of $H := C_G(\widetilde{z})$. Write $\mathfrak{f}^k$ for the characteristic polynomial of $\widetilde{z}$, in the notation of the above discussion, so that $H \cong GL_k^\epsilon(q^{\ell^\alpha})$. (Indeed, note that if $\ell|(q-1)$, then a root of $\mathfrak{f}$ is an element of $\mathbb{F}_q$ and $\mathfrak{f}$ is a product of non-self-check polynomials, but if $\ell|(q+1)$, then a root of $\mathfrak{f}$ is in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, and $\mathfrak{f}$ is a self-check polynomial.)

Now, $\widetilde{E}$ and $\overline{E}$ can be viewed as embeddings of E into H, with $Z(\widetilde{E})$ and $Z(\overline{E})$ generated by scalar multiples of the identity matrix I_k. Hence $\widetilde{E}$ and $\overline{E}$ are conjugate in H, and $k = m\ell^\gamma$ for some $m \geq 1$, by Lemma 7.2.2 and [6, Remark (1) after (1A)]. Note that in our case $n \leq 3$, we have $H \cong GL_n^\epsilon(q)$, unless $n = 3 = \ell$ and $\alpha = 1$, in which case $H \cong GL_1^\epsilon(q^3)$. (In particular, note that $\gamma = 0$, unless possibly if $n = \ell = 3$ and $\alpha = 0$, in which case $\gamma = 0$ or 1.)

This yields that, $\widetilde{R}$ and $\overline{R}$ are conjugate in G. Now, identify H with $GL^\epsilon_{m\ell^\gamma}(q^{\ell^\alpha})$. Then direct calculation and the discussion so far shows that $C \cong C_H(R) \cong GL_m^\epsilon(q^{\ell^\alpha})$.

Now, certainly if $\gamma = 0$, then $R = Z$, $N^0 = C$ and the remainder of the statement follows trivially, with $L := Z(C)$. Henceforth, we may assume $n = 3 = \ell$ and $\gamma = 1$, so that $\alpha = 0$, $m = 1$ and $C \cong GL_1^\epsilon(q)$.

Now, let $\widehat{E}$ denote the embedding of E in $GL_3^\epsilon(q)$ given by Lemma 7.2.2, and write L for the normalizer of $\widehat{E}$ in $GL_3^\epsilon(q)$. Note that $L = N_H(\widetilde{E})$. Then certainly $R \lhd L \leq N^0 = N_H(R)$, since L normalizes E and centralizes Z. Moreover, $C_H(L) = C_H(E) = C$, and $[C, L] = 1$. (Indeed, from above, $C_G(R) = C_H(R)$, which is $C_H(E)$ since $H = C_G(Z)$. Then $C = C_H(E) \leq C_H(L)$, since $C = Z(H)$ and $C_H(L) \leq C_H(E)$ since $E \leq L$.)

Suppose R is a radical subgroup of G. We claim that R has exponent 3, so that R is in fact the 3-group R as defined for $Sp_6(q)$ at the beginning of this section. By way of contradiction, assume R has exponent 9, so we may assume $R = E$, as otherwise we can replace R by the central product of Z and an extraspecial group of exponent 3. Now, from [6, Proof of (1B)], there is a 3-element $x \in L \setminus \widehat{E}$ which induces an element of $Z(\mathrm{Aut}^\circ(E)/\mathrm{Inn}(E))$, where $\mathrm{Aut}^\circ(E)$ is the subgroup of automorphisms which commute with Z, so we may view x as an element of $L \backslash \widetilde{E}$. Note that $\mathrm{Aut}^\circ(E)/\mathrm{Inn}(E) \cong C_3$, by [6, (1.1)]. Let $Q := \langle x, \widetilde{E} \rangle$. Then $C_H(\widetilde{E}) = C_H(Q) = C$, from above.

Now, since $N^0 \lhd N$, we see that $\widetilde{E} = \mathbf{O}_3(N^0)$, since $[E, Z] = 1$ and $\widetilde{E} = \mathbf{O}_3(N)$ as $\widetilde{E}$ is radical in G. Also, each element of N^0 induces an element of $\mathrm{Aut}^\circ(E)$. We claim that $Q \leq \mathbf{O}_3(N^0)$, which contradicts that R has exponent 9.

Indeed, let $h \in N^0$, and by replacing with an appropriate element $h \cdot e$ for $e \in \widetilde{E}$, we may assume h induces an element of $\mathrm{Aut}^\circ(E)/\mathrm{Inn}(E)$. Hence $[h, x]$ is trivial on E (as x induces an element of $Z(\mathrm{Aut}^\circ(E)/\mathrm{Inn}(E))$), so $c := [h, x]$ is an element of the cyclic group $C = Z(H)$. Moreover, since $C = C_H(Q)$, we know $x^h = cx$ and x commute and are both 3-elements, so c is a 3-element. Hence $c \in \mathbf{O}_3(Z(H)) \leq \widetilde{E}$, so h normalizes Q. Therefore, $Q \leq \mathbf{O}_3(N^0) = \widetilde{E}$, contradicting that $x \notin \widetilde{E}$.

This yields that E has exponent 3. Identifying R with $\widetilde{R}$, we see $LC/C \cong L/(C \cap L) = L/Z(L) \cong \mathrm{Aut}^\circ(E) = \mathrm{Aut}^\circ(R)$, and hence $L/RZ(L) \cong Sp_2(3)$ (see [6, Proof of (1C) and discussion before (1.1)]). Moreover, N^0/C induces a subgroup of $\mathrm{Aut}^\circ(R)$, so we see that $N^0/C \leq LC/C \leq N^0/C$, so $N^0 = LC$. Hence $Z(H) \leq Z(N^0) \leq Z(L) \cdot Z(C) = Z(L) \cdot C$, since C centralizes L. Also, $Z(L) \leq C = Z(H)$, so $Z(L) = Z(H) = Z(C) = C = C \cap L$. (Recall here that C is cyclic.) Now, by Lemma 7.2.3, since $N^0 = N_H(R)$ and $Z(N^0) = Z(L)$ from above, each linear character of $Z(L)/\mathbf{O}_3(Z(L))$ extends to L/R, as $L \leq N^0$.

Finally, $N_G(Z)/C_G(Z)$ is cyclic of order 2. Let $g \in N_G(Z)$ generate $N_G(Z)/C_G(Z)$. (Recall that $C_G(Z) \cong GL_3^\epsilon(q)$, and that g induces the automorphism $\tau \colon A \mapsto {}^T A^{-1}$ on $GL_3^\epsilon(q)$.) Then E and E^g are subgroups of $H = C_G(Z) = GL_3^\epsilon(q)$, each extraspecial of order 27, so must be conjugate in H, by Lemma 7.2.2. Hence $E = E^{gh}$ for some $h \in H$, and $gh \in N = N_G(R)$, since certainly $gh \in N_G(Z)$ and $gh \in N_G(E)$. Hence we see that $N_G(Z)/C_G(Z)$ can be embedded in N/N^0. Moreover, $N \leq N_G(Z)$ and $N^0 = N \cap C_G(Z)$, so we have $N/N^0 = N/(N \cap C_G(Z)) \cong NC_G(Z)/C_G(Z) \leq N_G(Z)/C_G(Z)$, and we have $N/N^0 \cong N_G(Z)/C_G(Z)$, completing the proof.

$\square$

The next lemma is [6, (2A)], again adapted to our situation. As in [6], given $R \leq Sp_{2n}(q)$, by a nondegenerate or isotropic R-module, we mean an R-module which is nondegenerate or isotropic as a subspace of the natural module $V = \mathbb{F}_q^{2n}$

with symplectic form $(\cdot,\cdot)$, for $Sp_{2n}(q)$.

Lemma 7.2.5. *Let $G = Sp_{2n}(q)$ with q even, and let $Q \leq G$ be an ℓ-subgroup for a prime $\ell \neq 2$. Then the natural module $V = \mathbb{F}_q^{2n}$ for G has a Q-module decomposition*

$$V = V_1 \perp V_2 \perp \ldots \perp V_\nu \perp (U_{\nu+1} \oplus U'_{\nu+1}) \perp \ldots \perp (U_\omega \oplus U'_\omega),$$

where V_i is a nondegenerate simple Q-module for $1 \leq i \leq \nu$ and U_i, U'_i are totally isotropic simple Q-modules for $\nu + 1 \leq i \leq \omega$, with $U_i \oplus U'_i$ nondegenerate and containing no proper nondegenerate Q-submodules.

Proof. The proof is exactly as in [5, (1B)], but we present it here for completion.

We induct on $\dim V$. Let W be a simple Q-submodule of V of minimal dimension. Then $\{v \in W : (v, W) = 0\}$ is a submodule of W, so must be either trivial or all of W. That is, either W is nondegenerate or totally isotropic.

If W is nondegenerate, we have $V = W \perp W^\perp$, where $W^\perp := \{v \in V : (v, W) = 0\}$ (see, for example, [37, Lemma 2.1.5(v)]). Then as $W^\perp$ is a nondegenerate Q-submodule, we see by induction that $W^\perp$ has such a decomposition, so V also has such a decomposition and the statement holds.

Now assume W is totally isotropic. Then $W^\perp \supseteq W$ is an Q-submodule of V, and $\dim V = \dim W + \dim W^\perp$ (see, for example, [37, Lemma 2.1.5(ii),(iv)]). Since $(2, \ell) = 1$, Maschke's theorem yields that V is a completely reducible Q-module, so we have $V = W^\perp \oplus W'$, where W' is a Q-submodule of V with $\dim W' = \dim W$. Moreover, $W \oplus W'$ is nondegenerate. (Indeed, if $(w + w', W \oplus W') = 0$ for some $w \in W, w' \in W'$, then in particular, $(w + w', W) = 0$, so $(w', W) = 0$ since $W \subseteq W^\perp$, so $w' \in W^\perp$, a contradiction unless $w' = 0$. Now, as V is nondegenerate, there is $x \in W'$ with $(w, x) \neq 0$, unless $w = 0$ as well.) Hence W' is either a totally isotropic or nondegenerate simple Q-submodule of V. If W' is nondegenerate, we may use the preceding paragraph, with W replaced with W', to see that V has the desired decomposition.

So, we may assume both W and W' are totally isotropic simple Q-modules. Suppose that $Y \subset W \oplus W'$ is a proper nondegenerate Q-submodule. Then Y must be simple, and we may again appeal to the earlier case, with W replaced with Y, to see that V has the desired decomposition. Hence we can assume that $W \oplus W'$ has no proper nondegenerate Q-submodules and is of the desired form $U_i \oplus U_i'$ in the statement. Then since $V = (W \oplus W')\perp(W \oplus W')^{\perp}$, we may apply the induction hypothesis to $(W \oplus W')^{\perp}$ to see that V can be decomposed as claimed.

□

Let $\ell|(q^2-1)$ with ϵ, as usual, such that $\ell|(q-\epsilon)$ and $\ell^d = (q-\epsilon)_\ell$. By a *basic* ℓ-subgroup of $Sp_{2n}(q)$ with $n \leq 3$, we mean a group $R_{m,\alpha,\gamma,c}$ as in [6]. That is, we begin with the embedding $R_{\alpha,\gamma}$ of the symplectic-type group $Z_\alpha E_\gamma$, where Z_α is cyclic of order $\ell^{d+\alpha}$ and E_γ is extraspecial of order $\ell^{2\gamma+1}$, into $Sp_{2\ell^{\alpha+\gamma}}(q)$ via the embedding into $GL^{\epsilon}_{\ell^\gamma}(q^{\ell^\alpha})$ with $Z_\alpha = \mathbf{O}_\ell(Z(GL^{\epsilon}_{\ell^\gamma}(q^{\ell^\alpha})))$. $R_{m,\alpha,\gamma}$ is the m-fold diagonal embedding of $R_{\alpha,\gamma}$ into $Sp_{2m\ell^{\alpha+\gamma}}(q)$, and $R_{m,\alpha,\gamma,c}$ is the wreath product $R_{m,\alpha,\gamma} \wr C_{\ell^c}$ in $Sp_{2m\ell^{\alpha+\gamma+c}}$. (Note that in our situation, the groups $A_{\mathbf{c}}$ in [6] must be trivial, unless $\ell = n = 3$, in which case it can be C_3, so we have simplified the notation here.)

The next lemma is [6, (2D)] in our situation.

Lemma 7.2.6. *Let $G = Sp_{2n}(q)$ with q even and $n \leq 3$, and let $Q \leq G$ be an ℓ-radical subgroup for a prime $\ell|(q^2-1)$. As usual, write $\ell|(q-\epsilon)$. Then the natural module $V = \mathbb{F}_q^{2n}$ for G has a Q-module decomposition*

$$V = V_0 \perp V_1 \perp \ldots \perp V_t$$

and Q can be decomposed into a direct product

$$Q = R_0 \times R_1 \times \cdots \times R_t,$$

where R_0 is the trivial subgroup of $Sp(V_0)$ and R_i for $1 \leq i \leq t$ is a basic subgroup of $Sp(V_i)$. Moreover, the extraspecial components of R_i have exponent ℓ.

Proof. Again, we will follow the proof of [6, (2D)] very closely.

Write $V_0 := C_V(Q)$ and let V_+ denote the set of vectors of V which are moved by Q, so that $V = V_0 \perp V_+$ and $Q = R_0 \times R_+$ where R_0 is the subgroup of elements trivial on V_0 and $R_+ \leq Sp(V_+)$. Then $N_G(Q) = Sp(V_0) \times N_{Sp(V_+)}(R_+)$, and R_+ must be ℓ-radical in $Sp(V_+)$. Hence we may assume inductively that $V = V_+$ and $C_V(R) = 0$.

Now, by Lemma 7.2.5, we can write $V = m_1V_1 \perp m_2V_2 \perp \ldots \perp m_\omega V_\omega$, where each V_i is either a nondegenerate simple $Z(Q)$-submodule or the sum $V_i = U_i \oplus U_i'$, were U_i and U_i' are totally isotropic simple $Z(Q)$-submodules, and m_i is the multiplicity of V_i in V for $1 \leq i \leq \omega$. (Note that $Z(Q) = Q$ except possibly in the case $n = 3 = \ell$. Also, note that as $\dim V_i \geq 2$ for each $1 \leq i \leq \omega$, we must have $\omega \leq n$.) The commuting algebra $D_i := \mathrm{End}_{Z(Q)}(V_i)$ of $Z(Q)$ on V_i is $\mathbb{F}_{q^{2\ell^{\alpha_i}}}$ for some $\alpha_i \geq 0$, in the case V_i is nondegenerate, and $D_i := \mathrm{End}_{Z(Q)}(U_i)$ is $\mathbb{F}_{q^{\ell^{\alpha_i}}}$ for some $\alpha_i \geq 0$ if $V_i = U_i \oplus U_i'$ is the sum of totally isotropic spaces. In either case, we note that $\dim V_i = 2\ell^{\alpha_i}$. Note that we must have $\alpha_i = 0$, except possibly in the case $\ell = 3 = n$, in which case $\alpha_i = 0$ or 1. In the latter situation, we have $V = V_1$.

Write $N^0 := \{g \in N_G(Q) \colon [g, Z(Q)] = 1\}$ and $H := C_G(Z(Q))$, so $N^0 = N_G(Q) \cap H = N_H(Q)$. (Then $N^0 = C_G(Q)$ except in the case $\ell = 3$.) For $h \in H$, $1 \leq i \leq \omega$, we have $h(m_iV_i) = m_iV_i$. (Indeed, for $g \in Z(Q)$, $gh(m_iV_i) = hg(m_iV_i) = h(m_iV_i)$, so $h(m_iV_i)$ is a $Z(Q)$-submodule of V, but similarly, $gh(V_i) = h(V_i)$, so $h(V_i)$ is either trivial or a $Z(Q)$-submodule isomorphic to V_i (by Schur's lemma), and hence m_iV_i is preserved by h.)

Let F_i denote the representation of $Z(Q)$ on V_i. If V_i is nondegenerate, then since $D_i^\times$ is cyclic and $F_i(Z(Q)) \leq D_i^\times$, we see that $F_i(Z(Q))$ is cyclic, generated by some $g_i \in Sp(V_i)$. Similarly, the representation of $Z(Q)$ on U_i' is (up to a field automorphism) the contragredient of the representation of $Z(Q)$ on U_i, so we see that $F_i(Z(Q))$ is again cyclic generated by some $g_i \in Sp(V_i)$ in this case. Then V_i or U_i, in the respective cases, is a simple $\langle g_i \rangle$-module. Hence the action of $Z(Q)$ on m_iV_i is generated by the m_i-fold diagonal action of g_i, which we will denote $\widetilde{g}_i$.

In particular, by the description of conjugacy classes and centralizers of ℓ-elements elements in [47] (see also Section 2.4), we see that H has a decomposition $H_1\times\cdots H_\omega$, where $H_i\cong GL^\epsilon_{m_i}(q^{\ell^{\alpha_i}})\leq Sp(m_iV_i)$ for $1\leq i\leq\omega$.

Now, since Q is ℓ-radical and $N^0\lhd N$, we see that $\mathbf{O}_\ell(N^0)\leq\mathbf{O}_\ell(N_G(Q))=Q$. But also, $Q\lhd N^0$, since $N^0=N_H(Q)$. Therefore $Q\leq\mathbf{O}_\ell(N^0)$, and we see that in fact, $Q=\mathbf{O}_\ell(N^0)=\mathbf{O}_\ell(N_H(Q))$, so Q is ℓ-radical in H.

First, suppose that $n=\ell=3$ and that $\alpha_i=1$, so that $V=V_1$, and $\alpha_1=1$. Then $Z(Q)=\langle g\rangle$ acts cyclicly on V and $H=GL_1(q^3)$. (In fact, we see that g belongs a the class $c_{28,0}$ or $c_{31,0}$ in the notation of [47], in the case $\epsilon=1$ and -1, respectively.) Here $Q=\mathbf{O}_3(H)=\mathbf{O}_3(Z(H))\cong C_{3^{d+1}}$ and $Q=Z(Q)$ is cyclic and is certainly a basic subgroup of $Sp(V)$. (In fact, Q is conjugate to $Q^{(3)}$ in our notation.)

In the other cases, $\alpha_i=0$ for each i, so V is the orthogonal sum of m_iV_i, where $m_i\leq 3$, and V_i are 2-dimensional spaces. For each i, let $R_i:=\langle\widetilde{g_i}\rangle$.

If each $m_i=1$, then as $Q\leq H$ and each H_i is cyclic, we see that Q must be abelian and $Q=\prod R_i$. If $m_1=2$, we have $H_1\cong GL^\epsilon_2(q)$ and (for $n=3$) $H_2\cong GL^\epsilon_1(q)$. Again, we see that Q is abelian, since an ℓ-subgroup of H is abelian, and $Q=R_1\times R_2$ (or just R_1 if $n=2$). In either case, letting $N_i:=N_{H_i}(R_i)$, we see that certainly $R_i\leq\mathbf{O}_\ell(N_i)$ for each i, and $\prod_i\mathbf{O}_\ell(N_i)\leq\mathbf{O}_\ell(N_H(Q))=Q=\prod R_i$, so that $R_i=\mathbf{O}_\ell(N_i)$, and $R_i\cong C_{\ell^d}$ for each i. Certainly in these cases, each R_i is a basic subgroup of $Sp(V_i)$. (Note that the case $m_i=1$ us $Q_{1,1}$ for $Sp_4(q)$ and $Q_{1,1,1}$ for $Sp_6(q)$ and the case $m_1=2$ gives $Q_{2,1}$ for $Sp_6(q)$, or Q_2 for $Sp_4(q)$.)

If $m_1=3$, (i.e. $V=3V_1$ and $n=3$), then $H\cong GL^\epsilon_3(q)$, and since Q is radical in H, we know by [4, (4A)] or [6, (2B)] that Q is a basic subgroup of H with extraspecial part of exponent ℓ. Now, G has a basic subgroup Q' of the same form as Q, with extraspecial part having exponent ℓ. Then $Z(Q)$ and $Z(Q')$ are both generated by primary elements of order ℓ^d in G, and hence are conjugate in G. Then we may suppose that $Q'\leq H$, so Q and Q' are conjugate in H. Then Q is a basic subgroup of G, as desired. (We note that this case yields $Q=Q_3$ when $\ell\neq 3$, and $Q=Q_3,P$, or R if $\ell = 3$.)

$\square$

Proof of Proposition 7.2.1. First, it is clear from the description in [47] of the semisimple classes of G that the listed subgroups each lie in a different conjugacy class of subgroups.

On the other hand, when $\ell | (q^2-1)$, the remainder of the statement can be extracted from Lemma 7.2.6.

When $\ell \nmid (q^2-1)$, a Sylow ℓ-subgroup is cyclic, say generated by the semisimple element s. Then any power s^i of s has the same centralizer, which can be seen from Theorem 2.4.2. Moreover, this centralizer is a cyclic group containing $\langle s \rangle$, unless $\ell | (q^2+1)$ for the group $G = Sp_6(q)$. In the latter case, $C_G(s) \cong C_{q^2+1} \times Sp_2(q)$, which has automorphism group $\mathrm{Aut}(C_{q^2+1}) \times \mathrm{Aut}(Sp_2(q))$ since $(q^2+1, |Sp_2(q)|) = 1$. So, $C_G(s)$ contains an $\mathrm{Aut}(C_G(s))$-invariant cyclic direct factor C containing $\langle s \rangle$. Hence in either case, $\langle s \rangle$ is characteristic in the centralizer of any proper, nontrivial subgroup $\langle s^i \rangle$ of the Sylow subgroup $\langle s \rangle$, so $\langle s^i \rangle$ cannot be ℓ-radical.

$\square$

7.3 Characters of $N_G(Q)$

Let $G = Sp_6(q)$. In this section, we describe the characters of $N_G(Q)$ that will be of interest, and in particular the defect-zero characters of $N_G(Q)/Q$, for radical subgroups Q of G. Recall that for $\ell | (q^2-1)$, we have radical subgroups $Q_1, Q_2, Q_3, Q_{1,1}, Q_{2,1}$, and $Q_{1,1,1}$, with the additional subgroups $Q^{(3)}, P$, and R when $\ell = 3$. So, when referring to $Q_1, Q_2, Q_3, Q_{1,1}, Q_{2,1}$, and $Q_{1,1,1}$ we will assume $\ell | (q^2-1)$, without necessarily assuming that $\ell \neq 3$, unless otherwise stated. When referring to P or R, we assume $\ell = 3$, when referring to $Q^{(3)}$, we assume $\ell | (q^4+q^2+1)$ (with the possibility that $\ell = 3$), and when referring to $Q^{(2)}$, we assume $\ell | (q^2+1)$. Throughout this section, we continue to let $\epsilon \in \{\pm 1\}$ be such that $\ell | (q^3 - \epsilon)$, if such an ϵ exists, and let d and m be as in Section 7.1.

Let Q be an ℓ-radical subgroup, and write $N := N_G(Q)$ and $C := C_G(Q)$. The characters of N that we are interested in are those which are defect-zero characters of N/Q or height-zero characters of N with defect group Q. In either case, these characters will be $\chi \in \mathrm{Irr}(N)$ with $\chi(1)_\ell = |N/Q|_\ell$. So if $Q = Q_1, Q_2$, or Q_3, we have $\chi(1)_\ell = \ell^{2d}$, except in the case Q_3 when $\ell = 3$, in which case $\chi(1)_3 = 3^{2d+1}$. If $Q = Q_{1,1}$ or $Q_{2,1}$, then $\chi(1)_\ell = \ell^d$, and if Q is a Sylow subgroup, $\chi(1)_\ell = 1$. If $Q = Q_{1,1,1}, Q^{(3)}$ or R when $\ell = 3 | (q^2 - 1)$, then $\chi(1)_3 = 3$. In most cases, it will suffice for our purposes to describe the constituents of χ when restricted to C, and to keep in mind the action of N/C on C and its characters.

In many of the groups we are concerned with, we have an extension of a subgroup by C_2. Suppose that $X = Y : 2$, with the order-two automorphism on Y denoted by τ. By Clifford theory, a character $\chi \in \mathrm{Irr}(X)$ satisfies $\chi|_Y = \theta + \theta^\tau$ if an irreducible constituent θ of $\chi|_Y$ is not invariant under the automorphism τ, and in this case, $\chi = \theta^X = (\theta^\tau)^X$. Since X/Y is cyclic, if a constituent θ is invariant under τ, then $\chi|_Y = \theta$. In this case, Gallagher's theorem tells us that there are two such characters χ, namely χ and $\chi\lambda$ where λ is the nonprincipal character of $X/Y \cong C_2$. In particular, $\chi \in \mathrm{Irr}(X)$ has degree $\chi(1) = 2\theta(1)$ or $\theta(1)$ for some $\theta \in \mathrm{Irr}(Y)$. In general, when a character θ of $Y \lhd X$ extends to X, we will sometimes write $\theta^{(\nu)}$ for the character $\theta\nu$ of X with $\nu \in X/Y$ by Gallagher's theorem.

We note that from the discussions below for $N_G(Q)$, it will also be easy to see the characters of interest for $N_H(Q)$ with $H = Sp_4(q)$ by similar arguments.

7.3.1 Characters of Some Relevant Subgroups

From Section 7.2, we see that when $\ell | (q^4 - 1)$, the characters of the groups $GL_r^\epsilon(q) : 2$, for $r = 1, 2, 3$, $Sp_4(q)$, and $SL_2(q) = Sp_2(q)$ will play a large role for many of the radical subgroups, so we discuss the characters of these groups here. Recall that the C_2 extension of $GL_r^\epsilon(q)$ acts on $GL_r^\epsilon(q)$ via $\tau\colon A \mapsto {}^TA^{-1}$.

First let $\ell|(q-\epsilon)$ for $\epsilon \in \{\pm 1\}$. Let $\varphi_i \in \mathrm{Irr}(C_{q-\epsilon}) = \mathrm{Irr}(GL_1^\epsilon(q))$ denote the linear character which maps $\tilde{\zeta} \mapsto \zeta^i$, where $(\zeta, \tilde{\zeta}) = (\zeta_1, \tilde{\zeta}_1)$ or $(\xi_1, \tilde{\xi}_1)$, in the cases $\epsilon = 1$ or -1, respectively. Then $\varphi_i^\tau = \varphi_{-i}$, so φ_i is invariant under τ exactly when $(q-\epsilon)|i$, i.e., when $\varphi_i = 1$. Hence an irreducible character of $GL_1^\epsilon(q) : 2$ which is nontrivial on $GL_1^\epsilon(q)$ can be identified by a constituent of its restriction to $GL_1^\epsilon(q)$, and therefore can be labeled by φ_i for $i \in I_{q-\epsilon}$. Moreover, there are two characters of $GL_1^\epsilon(q) : 2$ which are trivial on $GL_1^\epsilon(q)$, corresponding to the two characters $\{\pm 1\}$ of C_2, by Gallagher's theorem, which we will sometimes denote by $1^{(1)}$ and $1^{(-1)}$.

As $2 \not| q-\epsilon$, we may write $GL_2^\epsilon(q) \cong C_{q-\epsilon} \times SL_2(q)$ and note that τ induces an inner automorphism of $SL_2(q)$, so fixes all characters of $SL_2(q)$, and the action of τ on $C_{q-\epsilon}$ is the same as above. So, we will write $\varphi = (\varphi_i, \psi)$ for the character of $GL_2^\epsilon(q) \cong C_{q-\epsilon} \times SL_2(q)$, with φ_i as above, and $\psi \in \mathrm{Irr}(SL_2(q))$. Now, the only series of characters of $SL_2(q) = Sp_2(q)$ with degree divisible by ℓ is $\chi_4(j)$ when $\epsilon = 1$ and $\chi_3(j)$ when $\epsilon = -1$, with degrees $q - \epsilon$ and indexing $j \in I_{q+\epsilon}$ (see, for example, the character table information in CHEVIE [26]). When the context is clear, we will write $\chi_*(j)$ for the proper character $\chi_4(j)$ or $\chi_3(j)$ of $SL_2(q)$. (Also, when $\ell|(q^2+1)$, note that no character of $SL_2(q)$ has degree divisible by ℓ.)

Now consider $GL_3^\epsilon(q) : 2$. The characters $\chi_8(i)$ of $GL_3^\epsilon(q)$ (in the notation of CHEVIE [26]), indexed by $1 \leq i \leq q^3 - \epsilon$ with $(q^2 + \epsilon q + 1) \not| i$ and $\chi_8(i) = \chi_8(qi) = \chi_8(q^2 i)$, each have degree $(q-\epsilon)^2(q+\epsilon)$ and are the only characters of $GL_3^\epsilon(q)$ of degree divisible by ℓ^{2d} when $\ell|(q^2-1)$. Inspection of the character table in CHEVIE reveals that $\chi_8(i)^\tau = \chi_8(-i)$ and no character in this series is invariant under τ. So, the characters we will be concerned with for this group are of the form $\chi_8(i) + \chi_8(-i)$ on $GL_3^\epsilon(q)$ and are indexed by $i \in I_{q^3-\epsilon}$.

Finally, when $\ell|(q-\epsilon)$, the irreducible characters θ of $Sp_4(q)$ with $\theta(1)_\ell = \ell^{2d}$ are those in the families (in the notation of CHEVIE) $\chi_5, \chi_{18}(i), \chi_{19}(i,j)$ when $\epsilon = 1$ and

$\chi_2, \chi_{15}(i,j), \chi_{18}(i)$ when $\epsilon = -1$. We note that χ_2 and χ_5 are the Weil characters ρ_2^2 and α_2, respectively, in the notation of [27] (see Table 4.2). Also, note that the indexing for the families $\chi_{15}(i,j)$ is $(i,j) \in I_{q-1}^{2*}$, for $\chi_{18}(i)$ is $i \in I_{q^2+1}$, and for $\chi_{19}(i,j)$ is $i, j \in I_{q+1}^{2*}$.

When $Q \in \mathrm{Syl}_\ell(G)$, all characters of $N_G(Q)$ have defect group Q (see for example [33, Corollary (15.39)]), since Q is an $\ell-$radical subgroup and $Q \in \mathrm{Syl}_\ell(N_G(Q))$. Hence in this case, $\mathrm{Irr}_0(N_G(Q)|Q) = \mathrm{Irr}_{\ell'}(N_G(Q))$.

7.3.2 $Q = Q_1$

Let $Q := Q_1$ with $\ell|(q-\epsilon)$. Then $N := N_G(Q) = (C_{q-\epsilon} : 2) \times Sp_4(q)$ and $C := C_G(Q) = C_{q-\epsilon} \times Sp_4(q)$ from Section 7.2. Note that a defect-zero character of N/Q or a height-zero character of N with defect group Q will be of the form $(\varphi, \theta) \in \mathrm{Irr}(C_{q-\epsilon} : 2) \times \mathrm{Irr}(Sp_4(q))$ with $\varphi(1)_\ell \theta(1)_\ell = |N/Q|_\ell = (q-\epsilon)_\ell^2 = \ell^{2d}$.

Now, with respect to the basis $\{e_1, f_1, e_2, e_3, f_2, f_3\}$ we may identify $Sp_4(q)$ with its image under the map

$$A \mapsto \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}$$

in $G = Sp_6(q)$ and $C_{q-\epsilon} : 2$ with the subgroup

$$\langle \mathrm{diag}(\zeta, \zeta^{-1}, I_4) \rangle \rtimes \left\langle \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_4 \end{pmatrix} \right\rangle,$$

where $\zeta = \widehat{\zeta_1}$ or $\widehat{\xi_1}$ is a primitive $(q-\epsilon)$ root of unity in $\mathbb{F}_{q^2}^\times$. (Note that for $\epsilon = -1$, this is an identification in the overgroup $Sp_6(q^2)$ rather than $Sp_6(q)$.) By the discussion in Section 7.3.1, we have $\frac{q-\epsilon-1}{2}$ characters of $C_{q-\epsilon} : 2$ of the form φ_i, with $i \in I_{q-\epsilon}$, which have degree 2, and 2 characters $1^{(1)}$ and $1^{(-1)}$ of degree 1. So, $\varphi(1)_\ell = 1$ for any $\varphi \in \mathrm{Irr}(C_{q-\epsilon} : 2)$. Hence it must be that $\theta(1)_\ell = \ell^{2d}$.

Therefore, if χ is nontrivial on $GL_1^\epsilon(q)$, Clifford theory and the discussion in Section 7.3.1 yield that χ is uniquely determined by a constituent (φ_i, ψ) of $\chi|_C$, where

$i \in I_{q-\epsilon}$ and $\psi \in \mathrm{Irr}(Sp_4(q))$ is one of the characters $\alpha_2, \chi_{18}(i)$(with $i \in I_{q^2+1}$), or $\chi_{19}(i,j)$(with $i,j \in I_{q+1}^{2*}$) when $\epsilon = 1$ and $\rho_2^2, \chi_{15}(i,j)$(with $(i,j) \in I_{q-1}^{2*}$), or $\chi_{18}(i)$(with $i \in I_{q^2+1}$) when $\epsilon = -1$. If χ is trivial on $GL_1^\epsilon(q)$ then $\chi|_C = (1, \psi)$ (with ψ again as above) is irreducible, and there are two choices $(1^{(1)}, \psi)$ and $(1^{(-1)}, \psi)$ for χ for each such choice of ψ.

Now, to be a character of N/Q, we require that Q be in the kernel. As $Q = \mathbf{O}_\ell(GL_1^\epsilon(q))$, we see that this means $\mathrm{dz}(N/Q)$ is comprised of the two characters with constituent $(1, \psi)$ on C and the characters with constituent (φ_i, ψ) on C with ψ as above and $i \in I_{q-\epsilon}$ such that $\ell^d | i$.

7.3.3 $Q = Q_2$

Let $Q := Q_2$. Recall that $N := N_G(Q) = (GL_2^\epsilon(q) : 2) \times Sp_2(q)$, and that $C := C_G(Q) = GL_2^\epsilon(q) \times Sp_2(q)$. With respect to the basis $\{e_1, e_2, f_1, f_2, e_3, f_3\}$, we can identify $GL_2^\epsilon(q) \times Sp_2(q)$ as the subgroup of matrices of the form $\mathrm{diag}(A, A^{-T}, B)$, with $A \in GL_2^\epsilon(q)$ and $B \in Sp_2(q)$. (Note that again when $\epsilon = -1$, this is an identification in $Sp_6(q^2)$.) We also identify the order-2 complement of $GL_2^\epsilon(q)$ in $GL_2^\epsilon(q) : 2$ with the group

$$\begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix},$$

which induces the automorphism $A \mapsto A^{-T}$ on $GL_2^\epsilon(q)$. (Note that viewing $GU_2(q) \leq Sp_6(q^2)$, this is the automorphism $(a_{ij}) = A \mapsto \overline{A} = (a_{ij}^q)$.) Let τ represent the automorphism of C which fixes $Sp_2(q)$ and yields the above action on $GL_2^\epsilon(q)$. Recall that we view $GL_2^\epsilon(q)$ as $C_{q-\epsilon} \times SL_2(q)$, so that τ actually acts as inversion on $C_{q-\epsilon}$ as in the case $Q = Q_1$ above, and fixes $SL_2(q)$ and $Sp_2(q)$. (We remark, however, that now $C_{q-\epsilon}$ is identified with the subgroup $\langle \mathrm{diag}(\zeta, \zeta, \zeta^{-1}, \zeta^{-1}, I_2) \rangle$ as opposed to $\langle \mathrm{diag}(\zeta, \zeta^{-1}, I_4) \rangle$ from the case $Q = Q_1$.)

Recall that by Clifford theory, $\chi \in \mathrm{Irr}(N)$ has degree $\chi(1) = 2\theta(1)$ or $\theta(1)$ for some

$\theta \in \mathrm{Irr}(C)$, and recall that again in this case, a defect-zero character of N/Q or a height-zero character of N with defect group Q will have degree satisfying $\chi(1)_\ell = \ell^{2d}$.

The characters of N satisfying this condition therefore have constituents on C of the form $((\varphi_i, \chi_*(j)), \chi_*(k))$, with $\varphi_i \in \mathrm{Irr}(C_{q-\epsilon})$ and $\chi_* \in \mathrm{Irr}(SL_2(q)) = \mathrm{Irr}(Sp_2(q))$ as in Section 7.3.1. If χ is nontrivial on $C_{q-\epsilon}$, then again it is uniquely determined by a choice of $i \in I_{q-\epsilon}$ and $(j,k) \in I_{q+\epsilon} \times I_{q+\epsilon}$. Here $\chi|_C = ((\varphi_i, \chi_*(j)), \chi_*(k)) + ((\varphi_{-i}, \chi_*(j)), \chi_*(k))$. If χ is trivial on $C_{q-\epsilon}$, then $\chi|_C$ is irreducible and there are again two choices of χ with $\chi|_C = ((1, \chi_*(j)), \chi_*(k))$, which we may write $(1^{(1)}, \chi_*(j), \chi_*(k))$ and $(1^{(-1)}, \chi_*(j), \chi_*(k))$.

This yields $\frac{q+\epsilon-1}{2}$ characters of the form $(1^{(\nu)}, \chi_*(j), \chi_*(j))$ for each $\nu = \pm 1$; $\frac{(q+\epsilon-1)(q+\epsilon-3)}{4}$ of the form $(1^{(\nu)}, \chi_*(j), \chi_*(k))$ (with $j \neq \pm k$) for each $\nu = \pm 1$; and $\frac{(q+\epsilon-1)^2(q-\epsilon-1)}{8}$ which have constituent on C of the form $(\varphi_i, \chi_*(j), \chi_*(k))$ (with the possibility $j = k$). Note that as $2 \not| (q \pm 1)$, characters of the form $(1^{(\nu)}, \chi_*(2j), \chi_*(k))$ or $(\varphi_i, \chi_*(2j), \chi_*(k))$ have the same indexing and number of characters of each type as above. (For example, the indexing and number of characters $(1^{(1)}, \chi_3(2i), \chi_3(i))$ is the same as for characters of the form $(1^{(1)}, \chi_3(i), \chi_3(i))$.)

Now, to be a character of N/Q, we require that Q be in the kernel. As $Q = \mathbf{O}_\ell(C_{q-\epsilon})$, we see that this means $\mathrm{dz}(N/Q)$ is comprised of the two characters with constituent $((1, \chi(j)), \chi(k))$ on C and the characters with constituent $((\varphi_i, \chi_*(j)), \chi_*(k))$ on C with $\ell^d | i \in I_{q-\epsilon}$ and $(j,k) \in I_{q+\epsilon} \times I_{q+\epsilon}$.

7.3.4 $Q = Q_3$

Let $Q := Q_3$ with $\ell | (q-\epsilon)$. Then $N_G(Q) = GL_3^\epsilon(q) : 2 = C_G(Q) : 2$ from Section 7.2. When $\epsilon = 1$, we identify the subgroup $GL_3(q)$ with its image in $G = Sp_6(q)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ with respect to the basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$. When $\epsilon = -1$, $GU_3(q)$ is conjugate in the overgroup $Sp_6(q^2)$ to the subgroup given by the image under the same map. In both cases, we identify the extension by C_2 with the group

$\left\langle \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \right\rangle$, which induces the automorphism $\tau\colon A \mapsto A^{-T}$ on $GL_3^\epsilon(q)$ under the above identification. (Note that viewing $GU_3(q) \leq Sp_6(q^2)$, this is the automorphism $(a_{ij}) = A \mapsto \overline{A} = (a_{ij}^q)$.)

As before, a character $\chi \in \mathrm{Irr}(N)$ has $\chi(1) = 2\theta(1)$ or $\theta(1)$ for some $\theta \in \mathrm{Irr}(C)$. Now, a defect-zero character of N/Q or a height-zero character of N with defect group Q will be a $\chi \in \mathrm{Irr}(N)$ satisfying $\chi(1)_\ell = |N/Q|_\ell$, which is $(q-\epsilon)_\ell^2 = \ell^{2d}$ when $\ell \neq 3$ and 3^{2d+1} when $\ell = 3$. Using the character table in CHEVIE [26], we see that when $\ell = 3$, there are no such characters for $GL_3^\epsilon(q)$, (and hence there are no such characters for N). Hence $\mathrm{dz}(N/Q)$ and $\mathrm{Irr}_0(N|Q)$ are empty in this case.

For the remainder of our discussion of $Q = Q_3$, we assume $\ell \neq 3$ and $\chi \in \mathrm{Irr}(N)$ with $\chi(1)_\ell = \ell^{2d}$. Then $\chi|_C = \chi_8(i) + \chi_8(-i)$ with $i \in I_{q^3-\epsilon}$, from the discussion in Section 7.3.1. To be a character of N/Q, χ must be trivial on Q, which under our identification is the subgroup $\mathbf{O}_\ell(Z(C))$, which consists of representatives of the conjugacy classes $C_1(k)$ for $m|k$ of $GL_3^\epsilon(q)$ in the notation of CHEVIE. Now, on the class $C_1(k)$ of $GL_3(q)$, the character $\chi_8(i)$ takes the value $(q-1)^2(q+1)\zeta_1^{ik}$, and on the class $C_1(k)$ of $GU_3(q)$, $\chi_8(i)$ takes the value $(q+1)^2(q-1)\xi_1^{ik}$. (Recall that ζ_1 and ξ_1 are the $(q-1)$st and $(q+1)$st roots of unity $\exp\left(\frac{2\pi\sqrt{-1}}{q-1}\right)$ and $\exp\left(\frac{2\pi\sqrt{-1}}{q+1}\right)$, respectively.) Hence we see that Q is in the kernel of $\chi_8(i)$ exactly when $\ell^d|i$. So $\mathrm{dz}(N/Q)$ is comprised of the $\frac{q(q+\epsilon)m}{6}$ characters of N with $\chi|_C = \chi_8(i) + \chi_8(-i)$, $i \in I_{q^3-\epsilon}$ with $\ell^d|i$.

7.3.5 $Q = Q_{1,1}$

Let $Q = Q_{1,1}$ with $\ell|(q-\epsilon)$. Then $N := N_G(Q) = (GL_1^\epsilon(q) : 2) \wr S_2 \times Sp_2(q)$ and $C := C_G(Q) = GL_1^\epsilon(q) \times GL_1^\epsilon(q) \times Sp_2(q)$ from Section 7.2. With respect to the basis $\{e_1, f_1, e_2, f_2, e_3, f_3\}$, we identify N with the image under the map

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A \in (C_{q-\epsilon} : 2) \wr S_2$ and $B \in Sp_2(q)$, again extending to the group $Sp_6(q^2)$ in the case $\epsilon = -1$. The two copies of $C_{q-\epsilon} : 2$ are viewed as in the case $Q = Q_1$, namely the subgroups

$$\langle \mathrm{diag}(\zeta, \zeta^{-1}, I_4)\rangle \rtimes \left\langle \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_4 \end{pmatrix} \right\rangle$$

and

$$\langle \mathrm{diag}(I_2, \zeta, \zeta^{-1}, I_2)\rangle \rtimes \left\langle \begin{pmatrix} I_2 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_2 \end{pmatrix} \right\rangle,$$

where ζ is a primitive $(q-\epsilon)$ root of unity in $\mathbb{F}_{q^2}^{\times}$. The S_2 factor is generated by the matrix $\begin{pmatrix} 0 & I_2 & \\ I_2 & 0 & \\ & & I_2 \end{pmatrix}$. Let $L := (GL_1^{\epsilon}(q) : 2) \times (GL_1^{\epsilon}(q) : 2) \times Sp_2(q)$ and let ω denote the action of S_2 on L, which fixes $Sp_2(q)$ and switches the two copies of $GL_1^{\epsilon}(q) : 2 = C_{q-\epsilon} : 2$, so that $\mathrm{diag}(\zeta, \zeta^{-1}, \zeta', (\zeta')^{-1}, X) \in C$ is mapped under ω to $\mathrm{diag}(\zeta', (\zeta')^{-1}, \zeta, \zeta^{-1}, X)$. Note that a character $(\varphi, \varphi', \theta) \in \mathrm{Irr}(L) = \mathrm{Irr}(C_{q-\epsilon} : 2) \times \mathrm{Irr}(C_{q-\epsilon} : 2) \times \mathrm{Irr}(Sp_2(q))$ is invariant under ω if and only if $\varphi = \varphi'$. So, the irreducible characters of N may be described as follows:

$$(\varphi_i, \varphi_j, \theta); \quad i \neq \pm j \in I_{q-\epsilon}; \quad \text{degree } 8\theta(1)$$

$$(\varphi_i, \varphi_i, \theta)^{(1)}, \quad (\varphi_i, \varphi_i, \theta)^{(-1)}; \quad i \in I_{q-\epsilon}; \quad \text{degree } 4\theta(1)$$

$$(\varphi_i, 1^{(1)}, \theta), \quad (\varphi_i, 1^{(-1)}, \theta); \quad i \in I_{q-\epsilon}; \quad \text{degree } 4\theta(1)$$

$$(1^{(1)}, 1^{(1)}, \theta)^{(1)}, \quad (1^{(1)}, 1^{(1)}, \theta)^{(-1)}, \quad (1^{(-1)}, 1^{(-1)}, \theta)^{(1)}, \quad (1^{(-1)}, 1^{(-1)}, \theta)^{(-1)}; \quad \text{degree } \theta(1)$$

$$(1^{(1)}, 1^{(-1)}, \theta); \quad \text{degree } 2\theta(1)$$

where in each case, $\theta \in \mathrm{Irr}(Sp_2(q))$, a character $(\varphi, \varphi', \theta)$ with $\varphi \neq \varphi'$ is $(\varphi, \varphi', \theta) + (\varphi', \varphi, \theta)$ on L, we have abused notation to denote by φ_i the character of $C_{q-\epsilon} : 2$ which restricts to $C_{q-\epsilon}$ as $\varphi_i + \varphi_{-i}$, and $(\varphi, \varphi, \theta)^{(\nu)}$ for $\nu \in \{\pm 1\}$ represent the two extensions of $(\varphi, \varphi, \theta) \in \mathrm{Irr}(L)$ to N.

Now, a defect-zero character of N/Q or a height-zero character of N with defect group Q will be a $\chi \in \mathrm{Irr}(N)$ satisfying $\chi(1)_\ell = |N/Q|_\ell$, which is $(q-\epsilon)_\ell = \ell^d$, which means that $\theta(1)_\ell = \ell^d$. As established above, this means that $\theta = \chi_*(k)$ where $\chi_* = \chi_4$ when $\epsilon = 1$ and $\chi_* = \chi_3$ for $\epsilon = -1$. Hence the characters χ of N with $\chi(1)_\ell = \ell^d$ can be described as follows.

There is a unique such character of N whose restriction to C contains the constituent $(\varphi_i, \varphi_j, \chi_*(k))$ for $i \neq j \in I_{q-\epsilon}, k \in I_{q+\epsilon}$ and two whose restriction to C contains the constituent $(\varphi_i, \varphi_i, \chi_*(k))$ or $(\varphi_i, 1, \chi_*(k))$ for $i \in I_{q-\epsilon}, k \in I_{q+\epsilon}$. Finally, there are five such characters which have constituent $(1, 1, \chi_*(k))$ for each $k \in I_{q+\epsilon}$, so are trivial on $GL_1^\epsilon(q) \times GL_1^\epsilon(q)$. (These correspond to the five characters of $C_2 \wr S_2$, which we will later write as $(1^{(1)}, 1^{(-1)}), (1^{(1)}, 1^{(1)})^{(\lambda)}$, and $(1^{(-1)}, 1^{(-1)})^{(\lambda)}$, where $\lambda \in \{\pm 1\} = \mathrm{Irr}(C_2)$.)

Since $Q = \mathbf{O}_\ell\left((GL_1^\epsilon(q))^2\right)$, to be trivial on Q, the characters as listed above must satisfy that in addition, $\ell^d | i$ for all of the φ_i occurring in the restriction to C.

7.3.6 $Q = Q_{2,1}$

When $Q := Q_{2,1}$, we have $N := N_G(Q) = (GL_2^\epsilon(q):2) \times (GL_1^\epsilon(q):2)$ and $C := C_G(Q) = GL_2^\epsilon(q) \times GL_1^\epsilon(q)$. With respect to the basis $\{e_1, e_2, f_1, f_2, e_3, f_3\}$, the identification of $GL_2^\epsilon(q):2$ in G is the same as in the case $Q = Q_2$, and $GL_1^\epsilon(q):2 = C_{q-\epsilon}:2$ is identified like in the case $Q = Q_1$, with

$$C_{q-\epsilon}:2 = \langle \mathrm{diag}(I_4, \zeta, \zeta^{-1})\rangle \rtimes \left\langle \begin{pmatrix} I_4 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \right\rangle.$$

Here characters of N/Q of defect zero or height-zero characters of N with defect group Q will have $\chi(1)_\ell = \ell^d$. Hence χ must be of the form $((\varphi, \chi_*(j)), \varphi')$ where $(\varphi, \chi_*(j)) \in \mathrm{Irr}(GL_2^\epsilon(q):2)$ is as in the case $Q = Q_2$ and φ' is any member of $\mathrm{Irr}(C_{q-\epsilon}:2)$ as described in Section 7.3.1. This yields $\frac{(q+\epsilon-1)(q-\epsilon-1)^2}{8}$ characters of the form $((\varphi_i, \chi_*(j)), \varphi_k)$ (with the possibility $i = k$), $\frac{(q+\epsilon-1)(q-\epsilon-1)}{4}$ of each form

$\big((\varphi_i, \chi_*(j)), 1^{(\nu)}\big)$ and $\big((1^{(\nu)}, \chi_*(j), \varphi_i\big)$ for each $\nu \in \{\pm 1\}$ (where we have again abused notation by writing φ_i for the character of $C_{q-\epsilon} : 2$ which restricts to $C_{q-\epsilon}$ as $\varphi_i + \varphi_{-i}$), and $\frac{q+\epsilon-1}{2}$ of the form $(1^{(\nu)}, \chi_*(j), 1^{(\mu)})$ for each $(\nu, \mu) \in \{\pm 1\} \times \{\pm 1\}$.

To be trivial on Q, we again only further require $\ell^d | i$ for any φ_i occurring in the restriction to C.

7.3.7 $Q = Q_{1,1,1}$

Let $Q := Q_{1,1,1}$, with $\ell | (q - \epsilon)$, so $N := N_G(Q) = (GL_1^\epsilon(q) : 2) \wr S_3$ and $C := C_G(Q) = (GL_1^\epsilon(q))^3$. Also, write $L := (GL_1^\epsilon(q) : 2)^3$ to denote the normal subgroup of N with quotient S_3. With respect to the basis $\{e_1, f_1, e_2, f_2, e_3, f_3\}$, we identify C with the subgroup of elements of the form $\mathrm{diag}(a_1, a_1^{-1}, a_2, a_2^{-1}, a_3, a_3^{-1})$ with $a_i \in C_{q-\epsilon}$, and the order-two automorphisms on each $C_{q-\epsilon}$ acts as before, sending $a \mapsto a^{-1}$. Here the S_3 acts on L via $\mathrm{diag}(A_1, A_2, A_3)^\sigma = \mathrm{diag}(A_{\sigma^{-1}(1)}, A_{\sigma^{-1}(2)}, A_{\sigma^{-1}(3)})$ for $\sigma \in S_3$ and $A_i \in C_{q-\epsilon} : 2$.

Let $\theta = (\varphi, \varphi', \varphi'') \in \mathrm{Irr}(L) = \mathrm{Irr}(C_{q-\epsilon} : 2)^3$ be a constituent of $\chi \in \mathrm{Irr}(N)$ when restricted to L. Then θ is invariant under the S_3 action if and only if it is invariant under the A_3 action, if and only if $\varphi = \varphi' = \varphi''$. In this case, θ extends to a character of N and we get three such characters, corresponding to the three characters of the quotient $N/L = S_3$, by Gallagher's theorem, with degrees $\theta(1), \theta(1)$, and $2\theta(1)$. (This extension can be seen, for example, using [33, (11.31) and (6.20)].)

Moreover, θ has a stabilizer $T := N_\theta$ in N with $|T/L| = 2$ precisely when exactly two of φ, φ', and φ'' are the same. In this case, we get two extensions to T, and the character χ of N is determined by a constituent on T by Clifford correspondence [33, (6.11)]. Let ω denote a 3-cycle in S_3. Then the two characters of N with constituent θ on L have restriction to L as $\theta + \theta^\omega + \theta^{\omega^2}$ and have degree $3\theta(1)$.

Finally, if $\varphi \neq \varphi' \neq \varphi''$, then the irreducible character $\theta = (\varphi, \varphi', \varphi'')$ of L has stabilizer $N_\theta = L$. Hence such a character is uniquely determined by a constituent θ

on L, restricts to L as $\sum_{\rho\in S_3}\theta^\rho$, and has degree $6\theta(1)$.

From here, the number and indexing of each type of character can be seen easily from the description of characters of $C_{q-\epsilon}:2$.

The degree $\chi(1)$ of a defect-zero character of N/Q or hight-zero character of N with defect group Q must satisfy $\chi(1)_\ell = |N/Q|_\ell$, which is 1 when $\ell\neq 3$ and 3 when $\ell=3$. Since the characters of $C_{q-\epsilon}:2$ have degree 1 and 2, we see by the above discussion that if $\ell\neq 3$, then all characters of N satisfy this condition. If $\ell=3$, all except those characters whose restriction $\chi|_L$ to L has a constituent $(\varphi,\varphi,\varphi)$ satisfy this condition. Now, to be trivial on Q, we again just need to further require that $\ell^d|i$ for any φ_i appearing in the restriction to a copy of $C_{q-\epsilon}:2$.

7.3.8 $Q = P$

Now suppose $Q := P$ with $\ell = 3|(q-\epsilon)$. Write $N := N_G(Q)$ and $C := C_G(Q)$. Note that since $P\in \mathrm{Syl}_3(G)$, all characters of $N_G(P)$ have defect group P and $\mathrm{Irr}_0(N_G(P)|P) = \mathrm{Irr}_{3'}(N_G(P))$.

Write $P_1 := Q_{1,1,1}$, and note that $P = P_1 \rtimes C_3$. Then $C\leq C_G(P_1) = (C_{q-\epsilon})^3$, and since C must commute with the C_3-action, we see that in fact $C\cong C_{q-\epsilon}$ is the subgroup consisting of $(x,x,x)\in(C_{q-\epsilon})^3$ for $x\in C_{q-\epsilon}$.

Now, by [1, Theorem 2], P_1 is the unique maximal normal abelian subgroup in P. Hence, N must normalize P_1, so $N\leq N_G(P_1) = (GL_1^\epsilon(q):2)\wr S_3$.

Denote an element of $N_G(P_1)$ by $(X,Y,Z)\cdot h$ for $X,Y,Z\in C_{q-\epsilon}:2\leq Sp_2(q)$ and $h\in S_3$. Here as an element of G, (X,Y,Z) is given by $\mathrm{diag}(X,Y,Z)$, and we view $S_3\leq G$ with generators

$$\rho_1 := \begin{pmatrix} 0 & I_2 & 0\\ 0 & 0 & I_2\\ I_2 & 0 & 0\end{pmatrix}, \quad \text{and} \quad \rho_2 := \begin{pmatrix} 0 & I_2 & 0\\ I_2 & 0 & 0\\ 0 & 0 & I_2\end{pmatrix}.$$

Note that P/P_1 is also generated by ρ_1. We wish to determine the conditions on $(X,Y,Z)\cdot h$ which ensure that it is an element of N. That is, we must decide what conditions ensure that $(X, Y, Z) \cdot h$ sends ρ_1 to another element of P. Now, observing that

$$(X, Y, Z) \cdot h\rho_1 h^{-1} \cdot (X^{-1}, Y^{-1}, Z^{-1}) = (Z, Y, Z) \cdot \rho_1 \cdot (X^{-1}, Y^{-1}, Z^{-1})$$
$$= \begin{pmatrix} 0 & XY^{-1} & 0 \\ 0 & 0 & YZ^{-1} \\ ZX^{-1} & 0 & 0 \end{pmatrix},$$

we see that this is an element of P if and only if X, Y, and Z all belong to the same coset of $C_{q-1} : 2$ module C_{3^d}.

Hence, we see that N is as in [6, Formula (2.5)] and can be written as a semidirect product $K \rtimes S_3$, where $K \leq (C_{q-\epsilon} : 2)^3$ is comprised of elements (X, Y, Z) where $X, Y, Z \in C_{q-\epsilon} : 2$ belong to the same coset modulo C_{3^d}. Let $\varphi \in \mathrm{Irr}_0(N|P) = \mathrm{Irr}_{3'}(N)$. Since P_1 is normal in N, we know by Clifford theory that $\varphi|_{P_1}$ is the sum of N-conjugates of some (μ^i, μ^j, μ^k), with μ the character of C_{3^d} that sends a generator to a fixed primitive 3^d root of unity in $\mathbb{C}$ and $0 \leq i, j, k \leq 3^d - 1$. If the i, j, k are not all the same, then the S_3 action will cause the number of distinct conjugates in this decomposition to be a multiple of 3, and hence φ will have degree divisible by 3, contradicting the fact that φ has height zero. Hence an irreducible constituent of the restriction of φ to P_1 is of the form $\theta_i := (\mu^i, \mu^i, \mu^i)$ for some $0 \leq i \leq 3^d - 1$.

Now, we can write $K = (P_1 \times C_m) : 2$ (here $C_m \leq C$), and let $J := P_1 \times C_m$ be the index-2 subgroup. The extensions of θ_i to J are of the form $\theta_i \phi$ where $\phi \in \mathrm{Irr}(C_m)$, and each $\theta_i \phi$ is invariant under the S_3 action, so extends to $J \rtimes S_3$. (This can be seen, for example, from [33, (11.31) and (6.20)].)

Further, $\theta_i \phi$ restricts to C as φ_j for some $0 \leq j \leq q - \epsilon - 1$, and this restriction uniquely determines i and ϕ (indeed, $\varphi_j = \mu^i \phi$). From here, we see that if $\varphi|_C$ contains a nontrivial constituent, then $\varphi|_K$ is uniquely determined by a constituent φ_j for $j \in I_{q-\epsilon}$ of the restriction to C, and for each such choice of j there are 3 characters φ of N, by Gallagher's theorem. (By an abuse of notation, we will write $\varphi = \varphi_j \beta$ for these characters of N, with $\beta \in \mathrm{Irr}(S_3)$.) Moreover, there are 6 characters

$\varphi \in \mathrm{Irr}_0(N|P)$ with $C \leq \ker\varphi$, also by Gallagher's theorem. (These we will denote by $1^{(1)}\beta$ and $1^{(-1)}\beta$ for $\beta \in \mathrm{Irr}(S_3)$.)

Moreover, since $N_{S_3}(A_3)/A_3 = S_3/A_3 \cong C_2$, the fact that [6, Formula (2.5)] holds for Q yields $N/Q \cong (C_m : 2) \times C_2$. Certainly, any character of N/Q has defect zero since P is a Sylow 3-subgroup of G. We can view $C_m : 2$ as a quotient of

$$C_{q-\epsilon} : 2 = \langle \mathrm{diag}(a, a, a, a^{-1}, a^{-1}, a^{-1})\rangle \rtimes \left\langle \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \right\rangle,$$

with respect to the basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$, and as such, the characters of $C_m : 2$ are of the form $1^{(1)}, 1^{(-1)}$, and $\mathrm{Ind}_{C_{q-\epsilon}}^{C_{q-\epsilon}:2}\varphi_i$, where $3^d|i$, as before. For the C_2 factor, let $\langle\lambda\rangle = \mathrm{Irr}(C_2)$.

7.3.9 $Q = R$

Let $\ell = 3|(q-\epsilon)$ with $(q-\epsilon)_{3'} = m, (q-\epsilon)_3 = 3^d$, and let $Q = R$ be the group $Z \cdot E_{27} \leq GL_3^\epsilon(q)$ viewed as a subgroup of G as in Section 7.2. Then by Lemma 7.2.4, we see $N := N_G(R)$ has an index two subgroup N° satisfying $N/N^\circ \cong N_G(Z)/C_G(Z) = C_2$. Further, $R \lhd N^\circ$, and we have $N/R = (N^\circ/R).2$, with the order-2 automorphism given by the action of the map $\tau\colon A \mapsto (A^T)^{-1}$ on $GL_3^\epsilon(q)$. Also, $Sp_2(3) \cong N^\circ/(RZ(N^\circ)) \cong (N^\circ/R)/(RZ(N^\circ)/R)$, so N°/R contains a quotient group isomorphic to $Sp_2(3)$. Moreover, each linear character of $RZ(N^\circ)/R \cong Z(N^\circ)/(R \cap Z(N^\circ)) = Z(N^\circ)/\mathbf{O}_3(Z(N^\circ)) \cong C_m$ is extendable to a character of N°/R (again by Lemma 7.2.4). Hence by Gallagher's theorem, the characters of N°/R are exactly the characters $\theta\beta$ with $\theta \in \mathrm{Irr}(RZ(N^\circ)/R) = \mathrm{Irr}(C_m)$ and $\beta \in \mathrm{Irr}(N^\circ/(RZ(N^\circ)) = \mathrm{Irr}(Sp_2(3))$.

Since $|N/R| = 2m|Sp_2(3)|$, we have that a defect-zero character of N/R will have $\chi(1)_3 = 3$. Since N°/R has index 2 in N/R, the constituents of the restriction of χ to N°/R must satisfy this degree condition as well, so we require that β have degree divisible by 3. Since $Sp_2(3)$ has exactly one such character (namely, the Steinberg character, of degree 3), we will henceforth use β to denote this Steinberg character.

Note that β is invariant under the action of τ, and that as before, the principal character is the only character of C_m invariant under τ.

This yields $\frac{m-1}{2} + 2$ defect-zero characters of N/R, which we will denote by $1^{(1)}\beta, 1^{(-1)}\beta$, and $\varphi_i\beta$ for $i \in I_{q-\epsilon}$ with $3^d|i$, where by an abuse of notation, $\varphi_i\beta$ represents the defect-zero character whose restriction to $RZ(N^\circ)/R \cong C_m$ contains φ_i as a constituent and $1^{(1)}\beta, 1^{(-1)}\beta$ are the two extensions to N/R of defect-zero characters of N°/R trivial on $RZ(N^\circ)/R$.

7.3.10 $Q = Q^{(3)}$

Let $Q := Q^{(3)}$ with $\ell|(q^2 + \epsilon q + 1)$. Writing $N := N_G(Q)$ and $C := C_G(Q)$, we have $N = C : 6$, and $C = C_{q^3-\epsilon}$. Viewing Q as a subgroup of $GL_3^\epsilon(q)$ with the inclusion $A \mapsto \operatorname{diag}(A, A^{-T})$ in $Sp_6(q)$, with respect to the standard basis, C is also the centralizer of Q in $GL_3^\epsilon(q)$. Let $\tau \colon s \mapsto s^{-1}$, $\beta \colon s \mapsto s^{\epsilon q}$ so that $N/C = \langle \tau, \beta \rangle$ by Section 7.2. Let $\phi_i \in \operatorname{Irr}(C)$ for $0 \leq i < q^3 - \epsilon$ denote the character which maps $\widetilde{\zeta} \mapsto \zeta^i$, where $\widetilde{\zeta}$ is a fixed generator of C and $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{q^3-\epsilon}\right)$. Let $\chi \in \operatorname{Irr}(N)$ and let ϕ_i be a constituent of $\chi|_C$. Note that ϕ_i is invariant under the action of β if and only if $(q^2 + \epsilon q + 1)|i$. (Indeed, if $(q^2 + \epsilon q + 1)|i$, then ϕ_i is a character of $C_{q-\epsilon} = Z(GL_3^\epsilon(q))$, so is invariant under β. Conversely, if ϕ_i is invariant under the action of β, then $\zeta^i = \zeta^{\epsilon q i} = \zeta^{q^2 i}$, and hence ζ^i is a $(q-\epsilon)$th root of unity, meaning that $(q^2 + \epsilon q + 1)|i$.)

Now, since $\phi_i \neq \phi_{-i}$ for any $i \neq 0$, it follows that if $(q^2 + \epsilon q + 1) \not| i$, then ϕ_i has stabilizer C in N, and χ is uniquely determined by a constituent ϕ_i on C for $i \in I_{q^3-\epsilon}$, yielding $q(q^2-1)/6$ characters of this form. (This character restricts to C as the sum $\phi_i + \phi_{-i} + \phi_{qi} + \phi_{-qi} + \phi_{q^2 i} + \phi_{-q^2 i}$.)

If $i \neq 0$ and $(q^2+\epsilon q+1)|i$, then ϕ_i has $[N : \operatorname{stab}_N(\phi_i)] = 2$, and $[\operatorname{stab}_N(\phi_i) : C] = 3$. In this case, there are three choices of χ that restrict to C as the sum $\phi_i+\phi_{-i}$. That is, we obtain three characters χ for each choice of constituent ϕ_i for $i \in I_{q-\epsilon}$. (Note that there are $\frac{q-\epsilon-1}{2}$ such choices of i.) Finally, there are six characters χ with $C \leq \ker \chi$.

When $\ell \neq 3$, Q is a Sylow ℓ-subgroup of G and every character of N has degree prime to ℓ, and hence $\mathrm{Irr}_0(N|Q) = \mathrm{Irr}_{\ell'}(N) = \mathrm{Irr}(N)$.

When $\ell = 3$, recall that a defect-zero character of N/Q or a height-zero character of N with defect group Q will have $\chi(1)_3 = |N/Q|_3 = 3$. Let χ be such a character, with constituent ϕ_i on C. Then $|\mathrm{stab}_N(\phi_i)|$ cannot be divisible by 3, so ϕ_i must not be stabilized by β and we see $(q^2+\epsilon q+1) \not| i$. Hence the characters with $\chi(1)_3 = 3$ are exactly those with constituent ϕ_i on C with $(q^2+\epsilon q+1) \not| i$. To be trivial on Q, we just further require that $3^{d+1}|i$, which yields $m(n-1)/6$ defect-zero characters of N/Q, where $n = (q^2+\epsilon q+1)_{3'}$.

7.3.11 $\quad Q = Q^{(2)}$

Now let $\ell|(q^2+1)$ and let $Q := Q^{(2)}$ be a Sylow ℓ-subgroup of G. Write $C := C_G(Q) = C_{q^2+1} \times Sp_2(q)$ and $N := N_G(Q)$. Then again all characters of N have defect group Q, so $\mathrm{Irr}_0(N|Q)$ is exactly the set of characters of N with degree relatively prime to ℓ. However, $N = (C_{q^2+1} : 2^2) \times Sp_2(q)$, so every character of N satisfies this condition and $\mathrm{Irr}_0(N|Q) = \mathrm{Irr}(N)$.

Fix a generator $\widetilde{\xi_2}$ of C_{q^2+1} and let ϑ_i denote the character of C_{q^2+1} so that $\vartheta_i(\widetilde{\xi_2}) = \xi_2^i$, where $\xi_2 = \exp\left(\frac{2\pi\sqrt{-1}}{q^2+1}\right)$. Then since $\xi_2^i \neq \xi_2^{-i}$ or ξ_2^{iq} for $i \neq 0$, we see that $\mathrm{stab}_N(\vartheta_i) = C$. (Recall that N/C is generated by $\tau\colon s \mapsto s^{-1}$ and $\beta\colon s \mapsto s^q$ as in Section 7.2.)

Hence if $\chi \in \mathrm{Irr}(N)$ is nontrivial on C_{q^2+1}, then χ is of the form $\vartheta \times \theta$ where $\vartheta \in \mathrm{Irr}(C_{q^2+1} : 2^2)$ with $\vartheta|_{C_{q^2+1}} = \vartheta_i + \vartheta_{-i} + \vartheta_{qi} + \vartheta_{-qi}$ for some $i \in I_{q^2+1}$ and $\theta \in \mathrm{Irr}(Sp_2(q))$. (Note that there are $q^2/4$ such ϑ.) That is, χ is uniquely determined by a constituent $\vartheta_i \times \theta$ of $\chi|_C$, for $i \in I_{q^2+1}$ and $\theta \in \mathrm{Irr}(Sp_2(q))$. For each choice of $\theta \in \mathrm{Irr}(Sp_2(q))$, we also have 4 characters of N whose restriction to C_{q^2+1} is trivial.

7.4 The Maps

In this section, we describe maps which later will be used to show that $Sp_6(2^a)$ and $Sp_4(2^a)$ are "good" for the McKay, Alperin-McKay, Alperin weight, and blockwise Alperin weight conjectures. In Section 7.4.1, for radical subgroups Q with normalizer $N := N_G(Q)$, we describe disjoint sets $\underline{\mathrm{Irr}_0}(G|Q)$ and $\underline{\mathrm{Irr}_0}(N|Q)$ and bijections $\Omega_Q\colon \underline{\mathrm{Irr}_0}(G|Q) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q)$. In Section 7.5 below, we show that in fact $\mathrm{Irr}_0(G|Q) = \underline{\mathrm{Irr}_0}(G|Q)$ and $\mathrm{Irr}_0(N|Q) = \underline{\mathrm{Irr}_0}(N|Q)$, and that these are the required maps for the reduction of the Alperin-McKay conjecture in [69]. In Section 7.4.2 we also define maps $*_Q\colon \mathrm{IBr}_\ell(G|Q) \to \mathrm{dz}(N_G(Q)/Q)$, which we show in Section 7.5 are the required maps for the reduction of the (B)AWC in [70]. (We define the sets $\mathrm{IBr}_\ell(G|Q)$, $\underline{\mathrm{Irr}_0}(G|Q)$, and $\underline{\mathrm{Irr}_0}(N|Q)$ to be the sets of characters involved in the maps described here.) Also, in most cases, the characters for N here will be given by the description of an irreducible constituent on the centralizer $C := C_G(Q)$. That is, the maps we describe will be from a given set of characters of G to the set of characters of N with a given restriction to C. In these situations, the choice of bijection between these two sets does not matter, as long as the choice of image for a given family of characters is consistent throughout the choices of indexes J. In Section 7.4.3, we give similar maps for $Sp_4(q)$.

We note that we only define maps for ℓ-radical subgroups of positive defect. In the case $Q := \{1\}$, it is clear that the maps $\Omega_{\{1\}}$ and $*_{\{1\}}$ sending defect-zero characters of G (or their restriction to G°) to themselves will be the desired bijections.

7.4.1 The maps Ω_Q

As usual, N will denote $N_G(Q)$ for the ℓ-radical subgroup Q when the group Q we are discussing is evident, and ϵ is such that $\ell|(q^3 - \epsilon)$. Below are the maps $\Omega_Q\colon \underline{\mathrm{Irr}_0}(G|Q) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q)$ for each ℓ-radical conjugacy class representative Q.

First, let $\ell|(q^2 - 1)$. The bijections $\Omega_{Q_1}\colon \underline{\mathrm{Irr}_0}(G|Q_1) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q_1)$ are as follows:

$$\Omega_{Q_1}\colon \begin{cases} \{\chi_5,\chi_{11}\} \mapsto (1,\alpha_2) & \epsilon = 1 \\ \{\chi_4,\chi_9\} \mapsto (1,\rho_2^2) & \epsilon = -1 \end{cases}$$

$$\begin{cases} \chi_{17}(i) \mapsto (\varphi_i,\alpha_2) & \epsilon = 1 \\ \chi_{20}(i) \mapsto (\varphi_i,\rho_2^2) & \epsilon = -1 \end{cases} \quad i \in I_{q-\epsilon}$$

$$\begin{cases} \mathcal{E}_{23}(i,j) \mapsto (1,\chi_{19}(i,j)) & \epsilon = 1 \\ \mathcal{E}_{17}(i,j) \mapsto (1,\chi_{15}(i,j)) & \epsilon = -1 \end{cases} \quad (i,j) \in I_{q+\epsilon}^{2*}$$

$$\mathcal{E}_{24}(i) \mapsto (1,\chi_{18}(i)) \quad (\text{for } \epsilon = 1 \text{ or } -1), \quad i \in I_{q^2+1}$$

$$\begin{cases} \mathcal{E}_{28}(i,j,k) \mapsto (\varphi_i,\chi_{19}(j,k)) & \epsilon = 1 \\ \mathcal{E}_{26}(j,k,i)) \mapsto (\varphi_i,\chi_{15}(j,k)) & \epsilon = -1 \end{cases} \quad i \in I_{q-\epsilon}, (j,k) \in I_{q+\epsilon}^{2*}$$

$$\begin{cases} \mathcal{E}_{30}(i,j) \\ \mathcal{E}_{33}(j,i) \end{cases} \mapsto (\varphi_i,\chi_{18}(j)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \quad i \in I_{q-\epsilon}, j \in I_{q^2+1}.$$

We note that our notation means that every character lying in a Lusztig series indexed by a semisimple element of G^* in any of the families $g_{23}, g_{24}, g_{28}, g_{30}$ in the case $\ell|(q-1)$ and $g_{17}, g_{24}, g_{26}, g_{33}$ in the case $\ell(q+1)$ lies in $\underline{\mathrm{Irr}_0}(G|Q_1)$.

The bijection $\Omega_{Q_2}\colon \underline{\mathrm{Irr}_0}(G|Q_2) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q_2)$ is as follows:

$$\Omega_{Q_2}\colon \begin{cases} \mathcal{E}_9(i) \setminus \{\chi_{29}(i)\} \\ \mathcal{E}_8(i) \setminus \{\chi_{26}(i)\} \end{cases} \mapsto (1,\chi_*(2i),\chi_*(i)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \quad i \in I_{q+\epsilon}$$

$$\begin{cases} \mathcal{E}_{22}(i,j) \\ \mathcal{E}_{16}(i,j) \end{cases} \mapsto (1,\chi_*(2i),\chi_*(j)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \quad (i,j) \in I_{q+\epsilon}^2$$

$$\begin{cases} \mathcal{E}_{29}(i,j) \\ \mathcal{E}_{27}(i,j) \end{cases} \mapsto (\varphi_{2i_1},\chi_*(2i_2),\chi_*(j)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \quad \begin{matrix} i = i_1(q+\epsilon) + i_2(q-\epsilon) \in I_{q^2-1}, \\ j \in I_{q+\epsilon} \end{matrix}$$

Here recall that $\chi_* = \chi_4$ in the case $\ell|(q-1)$ and χ_3 in the case $\ell|(q+1)$. The exceptions of $\chi_{29}(i) \in \mathcal{E}_9(i)$ of degree $q(q-1)^3(q^2+q+1)(q^2+1)$ when $\ell|(q-1)$ and $\chi_{26}(i) \in \mathcal{E}_8(i)$ of degree $q(q+1)^3(q^2-q+1)(q^2+1)$ when $\ell|(q+1)$ are necessary, as they have defect zero. (Note that this leaves 2 elements of $\mathcal{E}_9(i), \mathcal{E}_8(i)$ to map to the two characters of N with constituent $(1,\chi_*(2i),\chi_*(i))$ on C.) All members of Lusztig series indexed by elements in the families g_{22}, g_{29} when $\ell|(q-1)$ and g_{16}, g_{27} when $\ell|(q+1)$ lie in $\underline{\mathrm{Irr}_0}(G|Q_2)$..

When $Q = Q_3$, there are no blocks with defect group Q when $\ell = 3|(q^2-1)$, but we have the additional radical subgroup $Q^{(3)}$ in this case, which does appear as a defect group. So, letting $\ell \neq 3$, the bijection $\Omega_{Q_3}\colon \underline{\mathrm{Irr}_0}(G|Q_3) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q_3)$ is as follows:

$$\begin{array}{c} \Omega_{Q_3} \\ (\ell \neq 3) \end{array} : \left\{ \begin{array}{l} \mathcal{E}_{31}(i) \\ \mathcal{E}_{34}(i) \end{array} \right. \mapsto \chi_8(i) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q^3-\epsilon}$$

When $\ell = 3$, the map $\Omega_{Q^{(3)}}$ is as follows:

$$\begin{array}{c} \Omega_{Q^{(3)}} \\ (\ell = 3) \end{array} : \left\{ \begin{array}{l} \mathcal{E}_{31}(i) \\ \mathcal{E}_{34}(i) \end{array} \right. \mapsto \phi_i \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q^3-\epsilon}$$

We note that when $\ell|(q^2-1)$, $\mathcal{E}_{31}(i)$ and $\mathcal{E}_{34}(i)$ contain only one character, as $C_{G^*}(g_{31}(i)) \cong C_{q^3-1}$ and $C_{G^*}(g_{34}(i)) \cong C_{q^3+1}$.

The bijection $\Omega_{Q_{1,1}}\colon \underline{\mathrm{Irr}_0}(G|Q_{1,1}) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q_{1,1})$ is as follows:

$$\begin{array}{ll} \Omega_{Q_{1,1}}\colon & \left\{ \begin{array}{l} \mathcal{E}_7(i) \setminus \{\chi_{23}(i)\} \\ \mathcal{E}_6(i) \setminus \{\chi_{14}(i)\} \end{array} \right. \mapsto (1,1,\chi_*(i)) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q+\epsilon} \\ & \left\{ \begin{array}{l} \mathcal{E}_{20}(i,j) \\ \mathcal{E}_{20}(j,i) \end{array} \right. \mapsto (\varphi_i, 1, \chi_*(j)) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, j \in I_{q+\epsilon} \\ & \left\{ \begin{array}{l} \mathcal{E}_{18}(i,j) \\ \mathcal{E}_{21}(i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_i, \chi_*(j)) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, j \in I_{q+\epsilon} \\ & \left\{ \begin{array}{l} \mathcal{E}_{26}(i,j,k) \\ \mathcal{E}_{28}(k,i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_j, \chi_*(k)) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j) \in I^{2*}_{q-\epsilon}, k \in I_{q+\epsilon} \end{array}$$

Recall that $\chi_* = \chi_4$ when $\ell|(q-1)$ and χ_3 when $\ell|(q+1)$. Also, note that $\chi_{23}(i)$ has defect zero when $\ell|(q-1)$, and $\chi_{14}(i)$ has defect zero when $\ell|(q+1)$, so they have been excluded here.

The bijection $\Omega_{Q_{2,1}}\colon \underline{\mathrm{Irr}_0}(G|Q_{2,1}) \leftrightarrow \underline{\mathrm{Irr}_0}(N|Q_{2,1})$ is as follows:

$$\Omega_{Q_{2,1}}: \left\{\begin{array}{l} \mathcal{E}_{13}(i) \\ \mathcal{E}_{11}(i) \end{array}\right. \mapsto (1,\chi_*(2i),1) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q+\epsilon}$$

$$\left\{\begin{array}{l} \mathcal{E}_{21}(i,j) \\ \mathcal{E}_{18}(i,j) \end{array}\right. \mapsto (1,\chi_*(2i),\varphi_j) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q+\epsilon}, j\in I_{q-\epsilon}$$

$$\mathcal{E}_{19}(i) \mapsto (\varphi_{2i_1},\chi_*(2i_2),1) \quad \text{(for } \epsilon=1 \text{ or } -1), \quad i=i_1(q+\epsilon)+i_2(q-\epsilon)\in I_{q^2-1}$$

$$\left\{\begin{array}{l} \mathcal{E}_{27}(i,j) \\ \mathcal{E}_{29}(i,j) \end{array}\right. \mapsto (\varphi_{2i_1},\chi_*(2i_2),\varphi_j) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i=i_1(q+\epsilon)+i_2(q-\epsilon)\in I_{q^2-1}, j\in I_{q-\epsilon}$$

Again in this case, recall that $\chi_*=\chi_4$ when $\ell|(q-1)$ and χ_3 when $\ell|(q+1)$.

Now, when $Q=Q_{1,1,1}$, we must again distinguish between the cases $\ell\neq 3$ and $\ell=3$. First, suppose $\ell\neq 3$ so that $Q_{1,1,1}\in \mathrm{Syl}_\ell(G)$. The map $\Omega_{Q_{1,1,1}}$ in this case is:

$$\begin{array}{c}\Omega_{Q_{1,1,1}} \\ (\ell\neq 3)\end{array} : \left\{\begin{array}{l} \mathcal{E}_1\setminus\{\chi_5,\chi_{11}\} \\ \mathcal{E}_1\setminus\{\chi_4,\chi_9\} \end{array}\right. \mapsto (1,1,1) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array}$$

$$\left\{\begin{array}{l} \mathcal{E}_6(i)\setminus\{\chi_{17}(i)\} \\ \mathcal{E}_7(i)\setminus\{\chi_{20}(i)\} \end{array}\right. \mapsto (\varphi_i,1,1) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q-\epsilon}$$

$$\left\{\begin{array}{l} \mathcal{E}_8(i) \\ \mathcal{E}_9(i) \end{array}\right. \mapsto (\varphi_i,\varphi_i,\varphi_i) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q-\epsilon}$$

$$\left\{\begin{array}{l} \mathcal{E}_{11}(i) \\ \mathcal{E}_{13}(i) \end{array}\right. \mapsto (\varphi_i,\varphi_i,1) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q-\epsilon}$$

$$\left\{\begin{array}{l} \mathcal{E}_{17}(i,j) \\ \mathcal{E}_{23}(i,j) \end{array}\right. \mapsto (\varphi_i,\varphi_j,1) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad (i,j)\in I_{q-\epsilon}^{2*}$$

$$\left\{\begin{array}{l} \mathcal{E}_{16}(i,j) \\ \mathcal{E}_{22}(i,j) \end{array}\right. \mapsto (\varphi_i,\varphi_i,\varphi_j) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad (i,j)\in I_{q-\epsilon}^{2}$$

$$\left\{\begin{array}{l} \mathcal{E}_{25}(i,j,k) \\ \mathcal{E}_{32}(i,j,k) \end{array}\right. \mapsto (\varphi_i,\varphi_j,\varphi_k) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad (i,j,k)\in I_{q-\epsilon}^{3*}$$

Here recall that $\mathcal{E}_1$ is the set of unipotent characters and that the excluded characters $\{\chi_5,\chi_{11},\chi_{17}(i)\}$ when $\ell|(q-1)$ and $\{\chi_4,\chi_9,\chi_{20}(i)\}$ when $\ell|(q+1)$ lie in $\underline{\mathrm{Irr}_0}(G|Q_1)$.

Now suppose that $\ell = 3$. The map $\Omega_{Q_{1,1,1}}$ in this case is:

$$\begin{array}{ll}
\begin{array}{c}\Omega_{Q_{1,1,1}} \\ (\ell=3)\end{array} : & \left\{\begin{array}{l}\mathcal{E}_6(i)\setminus\{\chi_{17}(i)\} \\ \mathcal{E}_7(i)\setminus\{\chi_{20}(i)\}\end{array}\right. \mapsto (\varphi_i,1,1) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array} \quad i\in I_{q-\epsilon}, m\not| i \\
& \left\{\begin{array}{l}\mathcal{E}_{11}(i) \\ \mathcal{E}_{13}(i)\end{array}\right. \mapsto (\varphi_i,\varphi_i,1) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array} \quad i\in I_{q-\epsilon}, m\not| i \\
& \left\{\begin{array}{l}\mathcal{E}_{17}(i,j) \\ \mathcal{E}_{23}(i,j)\end{array}\right. \mapsto (\varphi_i,\varphi_j,1) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array} \quad (i,j)\in I^{2*}_{q-\epsilon}; \text{m does not divide one of i,j} \\
& \left\{\begin{array}{l}\mathcal{E}_{16}(i,j) \\ \mathcal{E}_{22}(i,j)\end{array}\right. \mapsto (\varphi_i,\varphi_i,\varphi_j) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array} \quad (i,j)\in I^{2}_{q-\epsilon}; i\not\equiv \pm j \mod m \\
& \left\{\begin{array}{l}\mathcal{E}_{25}(i,j,k) \\ \mathcal{E}_{32}(i,j,k)\end{array}\right. \mapsto (\varphi_i,\varphi_j,\varphi_k) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array} \quad \begin{array}{c}(i,j,k)\in I^{3*}_{q-\epsilon};\\ i\not\equiv\pm j\not\equiv\pm k\not\equiv\pm i \mod m\end{array}
\end{array}$$

Again, recall that $\chi_{17}(i)\in\mathcal{E}_6(i)$ and $\chi_{20}(i)\in\mathcal{E}_7(i)$ are in the sets $\underline{\mathrm{Irr}_0}(G|Q_1)$ in the respective cases $\epsilon = 1, -1$. In the case that $\ell = 3$, we have excluded the cases when m divides all indices, since then the $\mathcal{E}_i(J)$ given above actually lie in the principal block, so have defect group P. Similarly, if m divides i, then φ_i maps a $q-\epsilon$ root of unity to an ℓ^d root, so if m divides all indices, then the image ω^*_θ in $\overline{\mathbb{F}}_\ell^\times$ of the central character for the character θ of N is the same as that for the principal character 1_N. Hence they lie in the same block and θ has defect group P, which is also a Sylow ℓ-subgroup of N.

Moreover, in the cases of $\mathcal{E}_{16}(i,j), \mathcal{E}_{25}(i,j,k)$ (resp. $\mathcal{E}_{22}(i,j), \mathcal{E}_{32}(i,j,k)$) (for $3|(q-1)$, respectively $3|(q+1)$), we must also exclude any case where the indices are all equivalent (but nonzero) modulo m, as then these series lie in the block $B_8(k)$ (respectively $B_9(k)$) for some $k\in I_{q-\epsilon}$ with $3^d|k$, which also has defect group P.

Now, when $\ell = 3$, let $P\in\mathrm{Syl}_3(G)$. Then $C = C_G(P)\cong C_{q-\epsilon}$ is the subgroup $Z(GL_3^\epsilon(q))$ viewed as a subgroup of G in the usual way. Hence the notation for the constituents of a character of N restricted to C are 1 and φ_i as before. The map Ω_P is as follows:

$$\Omega_P\colon \left\{\begin{array}{l}\{\chi_1,\chi_3,\chi_4,\chi_9,\chi_{10},\chi_{12}\}\\ \{\chi_1,\chi_2,\chi_5,\chi_8,\chi_{11},\chi_{12}\}\end{array}\right. \mapsto (1) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array}$$

$$\left\{\begin{array}{l}\mathcal{E}_8(i)\\ \mathcal{E}_9(i)\end{array}\right. \mapsto (\varphi_{3i}) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array}\quad , \quad i\in I_{q-\epsilon}$$

We will see that the 3-radical subgroup R when $\ell = 3|(q^2-1)$ does not appear as a defect group for any block of G, which is why we have no map Ω_R. (See part (2) of the proof of Proposition 7.5.4.)

Suppose now that $3 \neq \ell|(q^4+q^2+1)$ so that $Q^{(3)} \in \mathrm{Syl}_\ell(G)$ is the unique (up to conjugacy) radical subgroup. Let $\ell|(q^2+\epsilon q+1)$. The map $\Omega_{Q^{(3)}}$ in this case is:

$$\begin{array}{c}\Omega_{Q^{(3)}}\\ (\ell\neq 3)\end{array} : \left\{\begin{array}{l}\{\chi_1,\chi_3,\chi_4,\chi_9,\chi_{10},\chi_{12}\}\\ \{\chi_1,\chi_2,\chi_5,\chi_8,\chi_{11},\chi_{12}\}\end{array}\right. \mapsto (1) \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array}$$

$$\left\{\begin{array}{l}\mathcal{E}_8(i)\\ \mathcal{E}_9(i)\end{array}\right. \mapsto \phi_{(q^2+\epsilon q+1)i} \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array}\quad i\in I_{q-\epsilon}$$

$$\left\{\begin{array}{l}\mathcal{E}_{31}(i)\\ \mathcal{E}_{34}(i)\end{array}\right. \mapsto \phi_i \quad \begin{array}{c}\epsilon=1\\ \epsilon=-1\end{array}\quad i\in I_{q^3-\epsilon}$$

Now let $\ell|(q^2+1)$. Then $Q := Q^{(2)} \in \mathrm{Syl}_\ell(G)$ and $\mathrm{Irr}_0(G|Q) = \mathrm{Irr}_{\ell'}(G)$. Let b_0 and b_1 be the unipotent ℓ-blocks of G, as in [76], and let $\mathcal{U}(b)$ denote the unipotent characters in the block b. The map $\Omega_{Q^{(2)}}$ is as follows:

$$\begin{aligned}
\Omega_{Q^{(2)}}\colon & \mathcal{U}(b_0) \mapsto (1,1)\\
& \mathcal{U}(b_1) \mapsto (1,\chi_2)\\
& \mathcal{E}_6(i)\setminus\{\chi_{15}(i),\chi_{16}(i)\} \mapsto (1,\chi_3(i)) \qquad i\in I_{q-1}\\
& \mathcal{E}_7(i)\setminus\{\chi_{21}(i),\chi_{22}(i)\} \mapsto (1,\chi_4(i)) \qquad i\in I_{q+1}\\
& \chi_{55}(i) \mapsto (\vartheta_i,1) \qquad i\in I_{q^2+1}\\
& \chi_{56}(i) \mapsto (\vartheta_i,\chi_2) \qquad i\in I_{q^2+1}\\
& \chi_{62}(j,i) \mapsto (\vartheta_i,\chi_3(j)) \qquad i\in I_{q^2+1};\quad j\in I_{q-1}\\
& \chi_{65}(i,j) \mapsto (\vartheta_i,\chi_4(j)) \qquad i\in I_{q^2+1};\quad j\in I_{q+1}
\end{aligned}$$

We remark that here we have used the notation of CHEVIE [26] for the characters of $Sp_2(q)$.

We also remark that the fact that in all of the above maps, the number of characters of N with the same constituents on $C_G(Q)$ matches the number of $\chi\in\mathrm{Irr}(G)$ that we have mapped to them, follows from the discussion in Section 7.3 and the Bonnafé-Rouquier correspondence together with the knowledge of $C_{G^*}(s)$ and its unipotent blocks for semisimple $s\in G^*$. The indexing sets for the $\mathcal{E}_i(J)$ are evident from [47] - note that they match the indexing sets for the images under Ω_Q, as described in Section 7.3.

7.4.2 The maps $*_Q$

Let $\ell|(q^2-1)$. We now define bijections $*_Q\colon \mathrm{IBr}_\ell(G|Q)\leftrightarrow \mathrm{dz}(N/Q)$ for each ℓ-radical Q of $G=Sp_6(q)$. (We will see in Section 7.5 that when $\ell \not|(q^2-1)$, it is not necessary to define a bijection here.) In this section, we will abuse notation by denoting by simply B the irreducible ℓ-Brauer characters $\mathrm{IBr}_\ell(B)$ in a block B of G. The bijections $*_{Q_1}\colon \mathrm{IBr}_\ell(G|Q_1)\leftrightarrow \mathrm{dz}(N/Q_1)$ are as follows:

$$*_{Q_1}\colon \left\{ B_1 \mapsto \begin{array}{ll} (1,\alpha_2) & \epsilon=1 \\ (1,\rho_2^2) & \epsilon=-1 \end{array} \right.$$

$$\left\{ \begin{array}{ll} B_6(i)^{(1)} \mapsto (\varphi_i,\alpha_2) & \epsilon=1 \\ B_7(i)^{(1)} \mapsto (\varphi_i,\rho_2^2) & \epsilon=-1 \end{array} \right. \quad i\in I_{q-\epsilon}, \ell^d|i$$

$$\left\{ \begin{array}{ll} B_{23}(i,j) \mapsto (1,\chi_{19}(i,j)) & \epsilon=1 \\ B_{17}(i,j) \mapsto (1,\chi_{15}(i,j)) & \epsilon=-1 \end{array} \right. \quad (i,j)\in I^{2*}_{q+\epsilon}$$

$$B_{24}(i) \mapsto (1,\chi_{18}(i)) \quad (\text{for } \epsilon=1 \text{ or } -1) \quad i\in I_{q^2+1}$$

$$\left\{ \begin{array}{ll} B_{28}(i,j,k) \mapsto (\varphi_i,\chi_{19}(j,k)) & \epsilon=1 \\ B_{26}(j,k,i)) \mapsto (\varphi_i,\chi_{15}(j,k)) & \epsilon=-1 \end{array} \right. \quad i\in I_{q-\epsilon}, \ell^d|i; \quad (j,k)\in I^{2*}_{q+\epsilon}$$

$$\left\{ \begin{array}{l} B_{30}(i,j) \\ B_{33}(j,i) \end{array} \right. \mapsto (\varphi_i,\chi_{18}(j)) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q-\epsilon}, \ell^d|i; \quad j\in I_{q^2+1}.$$

The bijection $*_{Q_2}\colon \mathrm{IBr}_\ell(G|Q_2) \leftrightarrow \mathrm{dz}(N/Q_2)$ is as follows:

$$*_{Q_2}\colon \left\{ \begin{array}{l} B_9(i)^{(0)} \\ B_8(i)^{(0)} \end{array} \right. \mapsto (1,\chi_*(2i),\chi_*(i)) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array}, \quad i\in I_{q+\epsilon}$$

$$\left\{ \begin{array}{l} B_{22}(i,j) \\ B_{16}(i,j) \end{array} \right. \mapsto (1,\chi_*(2i),\chi_*(j)) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array}, \quad i\neq j\in I_{q+\epsilon}$$

$$\left\{ \begin{array}{l} B_{29}(i,j) \\ B_{27}(i,j) \end{array} \right. \mapsto (\varphi_{2i_1},\chi_*(2i_2),\chi_*(j)) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad \begin{array}{r} i=i_1(q+\epsilon)+i_2(q-\epsilon)\in I_{q^2-1}, \\ \ell^d|i; \quad j\in I_{q+\epsilon} \end{array}$$

Recall that when $Q=Q_3$, there are no defect-zero characters of N/Q when $\ell = 3|(q^2-1)$. So, letting $\ell\neq 3$, the bijection $*_{Q_3}\colon \mathrm{IBr}_\ell(G|Q_3) \leftrightarrow \mathrm{dz}(N/Q_3)$ is as follows:

$$\begin{array}{c} *_{Q_3} \\ (\ell\neq 3) \end{array} : \left\{ \begin{array}{l} B_{31}(i) \\ B_{34}(i) \end{array} \right. \mapsto \chi_8(i) \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q^3-\epsilon}, \ell^d|i$$

When $\ell=3$, we have the additional cyclic 3-radical subgroup $Q^{(3)}$. The map $*_{Q^{(3)}}$ is as follows:

$$\begin{array}{c} *_{Q^{(3)}} \\ (\ell=3) \end{array} : \left\{ \begin{array}{l} B_{31}(i) \\ B_{34}(i) \end{array} \right. \mapsto \phi_i \quad \begin{array}{c} \epsilon=1 \\ \epsilon=-1 \end{array} \quad i\in I_{q^3-\epsilon}, 3^{d+1}|i$$

The bijection $*_{Q_{1,1}}\colon \mathrm{IBr}_\ell(G|Q_{1,1}) \leftrightarrow \mathrm{dz}(N/Q_{1,1})$ is as follows:

$$*_{Q_{1,1}}: \begin{cases} B_7(i)^{(0)} \\ B_6(i)^{(0)} \end{cases} \mapsto (1,1,\chi_*(i)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q+\epsilon}$$

$$\begin{cases} B_{20}(i,j) \\ B_{20}(j,i) \end{cases} \mapsto (\varphi_i, 1, \chi_*(j)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q-\epsilon}, \ell^d | i; \quad j \in I_{q+\epsilon}$$

$$\begin{cases} B_{18}(i,j) \\ B_{21}(i,j) \end{cases} \mapsto (\varphi_i, \varphi_i, \chi_*(j)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q-\epsilon}, \ell^d | i; \quad j \in I_{q+\epsilon}$$

$$\begin{cases} B_{26}(i,j,k) \\ B_{28}(k,i,j) \end{cases} \mapsto (\varphi_i, \varphi_j, \chi_*(k)) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad (i,j) \in I^{2*}_{q-\epsilon}, \ell^d | i,j; \quad k \in I_{q+\epsilon}$$

The bijection $*_{Q_{2,1}} : \mathrm{IBr}_\ell(G|Q_{2,1}) \leftrightarrow \mathrm{dz}(N/Q_{2,1})$ is as follows:

$$*_{Q_{2,1}}: \begin{cases} B_{13}(i) \\ B_{11}(i) \end{cases} \mapsto (1, \chi_*(2i), 1) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q+\epsilon}$$

$$\begin{cases} B_{21}(i,j) \\ B_{18}(i,j) \end{cases} \mapsto (1, \chi_*(2i), \varphi_j) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q+\epsilon}; \quad j \in I_{q-\epsilon}, \ell^d | j$$

$$B_{19}(i) \mapsto (\varphi_{2i_1}, \chi_*(2i_2), 1) \qquad (\text{for } \epsilon = 1 \text{ or } -1), \quad i = i_1(q+\epsilon) + i_2(q-\epsilon) \in I_{q^2-1}, \ell^d | i$$

$$\begin{cases} B_{27}(i,j) \\ B_{29}(i,j) \end{cases} \mapsto (\varphi_{2i_1}, \chi_*(2i_2), \varphi_j) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad \begin{matrix} i = i_1(q+\epsilon) + i_2(q-\epsilon) \in I_{q^2-1}; \\ j \in I_{q-\epsilon}; \quad \ell^d | i,j \end{matrix}$$

Suppose $\ell \neq 3$ so that $Q_{1,1,1} \in \mathrm{Syl}_\ell(G)$. The map $*_{Q_{1,1,1}}$ in this case is:

$$\underset{(\ell\neq 3)}{*_{Q_{1,1,1}}} : B_0 \mapsto (1,1,1)$$

$$\begin{cases} B_6(i)^{(0)} \\ B_7(i)^{(0)} \end{cases} \mapsto (\varphi_i, 1, 1) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q-\epsilon}, \ell^d | i$$

$$\begin{cases} B_8(i) \\ B_9(i) \end{cases} \mapsto (\varphi_i, \varphi_i, \varphi_i) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q-\epsilon}, \ell^d | i$$

$$\begin{cases} B_{11}(i) \\ B_{13}(i) \end{cases} \mapsto (\varphi_i, \varphi_i, 1) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad i \in I_{q-\epsilon}, \ell^d | i$$

$$\begin{cases} B_{17}(i,j) \\ B_{23}(i,j) \end{cases} \mapsto (\varphi_i, \varphi_j, 1) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad (i,j) \in I_{q-\epsilon}^{2*}, \ell^d | i,j$$

$$\begin{cases} B_{16}(i,j) \\ B_{22}(i,j) \end{cases} \mapsto (\varphi_i, \varphi_i, \varphi_j) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad (i,j) \in I_{q-\epsilon}^{2}, \ell^d | i,j$$

$$\begin{cases} B_{25}(i,j,k) \\ B_{32}(i,j,k) \end{cases} \mapsto (\varphi_i, \varphi_j, \varphi_k) \quad \begin{matrix} \epsilon = 1 \\ \epsilon = -1 \end{matrix} \qquad (i,j,k) \in I_{q-\epsilon}^{3*}; \quad \ell^d | i,j,k$$

Now suppose that $\ell = 3$. In this case, we will distribute B_0 between the three sets $\mathrm{IBr}_3(G|Q_{1,1,1})$, $\mathrm{IBr}_3(G|P)$, and $\mathrm{IBr}_3(G|R)$. Of the 10 Brauer characters in B_0, we require that four of these belong to $\mathrm{IBr}_3(G|Q_{1,1,1})$ (to map to the characters $(1^{(1)}, 1^{(1)}, 1^{(-1)})^{(1)}$, $(1^{(1)}, 1^{(1)}, 1^{(-1)})^{(-1)}$, $(1^{(1)}, 1^{(-1)}, 1^{(-1)})^{(1)}$, and $(1^{(1)}, 1^{(-1)}, 1^{(-1)})^{(-1)}$ of $N_G(Q_{1,1,1})/Q_{1,1,1}$), another four belong to $\mathrm{IBr}_3(G|P)$ (to map to the characters $(1^{(1)}, 1)$, $(1^{(-1)}, 1)$, $(1^{(1)}, \lambda)$, and $(1^{(-1)}, \lambda)$ of $N_G(P)/P$), and the final two belong to $\mathrm{IBr}_3(G|R)$ (to map to the characters $1^{(1)}\beta$ and $1^{(-1)}\beta$ of $N_G(R)/R$).

In fact, the choice of this partition is arbitrary, as long as the number of characters assigned to each subgroup is correct, so we will simply write $B_0 = B_0(Q_{1,1,1}) \cup B_0(P) \cup B_0(R)$ for an appropriate partition. Similarly, of the three Brauer characters of the block $B_8(i)$ with $3^d | i$ (resp. $B_9(i)$) when $\ell | (q-1)$ (resp. $\ell | (q+1)$), we require that two of these are members of $\mathrm{IBr}_3(G|P)$ and the other is a member of $\mathrm{IBr}_3(G|R)$. Again, the partition is arbitrary, and we will write $B_8(i) = B_8(i,P) \cup B_8(i,R)$ (resp. $B_9(i) = B_9(i,P) \cup B_9(i,R)$) for an appropriate partition. Below are the corresponding maps.

$$\underset{(\ell=3)}{*_{Q_{1,1,1}}} : B_0(Q_{1,1,1}) \mapsto (1^{(\nu)}, 1^{(\nu)}, 1^{(\mu)}), \quad \nu \neq \mu \in \{\pm 1\}$$

$$\left\{ \begin{array}{l} B_6(i)^{(0)} \\ B_7(i)^{(0)} \end{array} \right. \mapsto (\varphi_i, 1, 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, 3^d | i$$

$$\left\{ \begin{array}{l} B_{11}(i) \\ B_{13}(i) \end{array} \right. \mapsto (\varphi_i, \varphi_i, 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, 3^d | i$$

$$\left\{ \begin{array}{l} B_{17}(i,j) \\ B_{23}(i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_j, 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j) \in I^{2*}_{q-\epsilon}, 3^d | i, j$$

$$\left\{ \begin{array}{l} B_{16}(i,j) \\ B_{22}(i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_i, \varphi_j) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j) \in I^{2}_{q-\epsilon}, 3^d | i, j$$

$$\left\{ \begin{array}{l} B_{25}(i,j,k) \\ B_{32}(i,j,k) \end{array} \right. \mapsto (\varphi_i, \varphi_j, \varphi_k) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j,k) \in I^{3*}_{q-\epsilon}, \quad 3^d | i, j, k$$

Note that for the image of $B_0(Q_{1,1,1})$, we have used the notation for the constituent when restricted to L rather than C. To describe the maps $*_P$ and $*_R$, we use the notation of characters of N described in Section 7.3.

$$\underset{(\ell=3)}{*_P} : B_0(P) \mapsto \{(1^{(\nu)}, \lambda^{\mu})\colon \nu, \mu \in \{\pm 1\}\}$$

$$\left\{ \begin{array}{l} B_8(i,P) \\ B_9(i,P) \end{array} \right. \mapsto \{(\varphi_{3i}, \lambda^{\mu})\colon \mu \in \{\pm 1\}\} \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, 3^d | i$$

$$\underset{(\ell=3)}{*_R} : B_0(R) \mapsto \{1^{(\mu)}\beta\colon \mu \in \{\pm 1\}\}$$

$$\left\{ \begin{array}{l} B_8(i,R) \\ B_9(i,R) \end{array} \right. \mapsto \varphi_{3i}\beta \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, 3^d | i$$

7.4.3 The Maps for $Sp_4(2^a)$

Here we will use the notation of [75] for blocks of $H = Sp_4(q)$, with q even. As in the case of $Sp_6(q)$, we will use maps $*_Q$ for the (B)AWC and maps Ω_Q for the (Alperin-)McKay reductions. The images are again given by a constituent on the centralizer $C_H(Q)$. We will also write simply B for the irreducible Brauer characters $\mathrm{IBr}_\ell(B)$ in a block B of H when defining our maps $*_Q$.

First let $\ell|(q^2-1)$. The maps $*_Q$ are as follows:

$$
\begin{array}{ll}
*_{Q_1}: & \left\{\begin{array}{l} B_9(i) \\ B_7(i) \end{array}\right. \mapsto (\chi_*(i), 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \quad i \in I_{q+\epsilon} \\
 & \left\{\begin{array}{l} B_{17}(j,i) \\ B_{17}(i,j) \end{array}\right. \mapsto (\chi_*(i), \varphi_j) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \quad i \in I_{q+\epsilon}; \quad j \in I_{q-\epsilon}, \ell^d|j
\end{array}
$$

$$
\begin{array}{ll}
*_{Q_2}: & \left\{\begin{array}{l} B_{13}(i) \\ B_{11}(i) \end{array}\right. \mapsto (\chi_*(2i), 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \quad i \in I_{q+\epsilon} \\
 & \left\{\begin{array}{l} B_{16}(i) \\ B_{16}(i) \end{array}\right. \mapsto (\chi_*(2i_1), \varphi_{2i_2}) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \quad i = i_1(q-\epsilon) + i_2(q+\epsilon) \in I_{q^2-1}, \ell^d|i
\end{array}
$$

Recall that $N_H(Q_1) \cong (GL_1^\epsilon(q) : 2) \times Sp_2(q)$ and $N_H(Q_2) \cong GL_2^\epsilon(q) : 2 \cong (SL_2^\epsilon(q) \times C_{q-\epsilon}) : 2$ with $[N_H(Q) : C_H(Q)] = 2$ in either case, and that $\chi_* \in \mathrm{Irr}(SL_2(q))$ is χ_4 and χ_3 in the cases $\epsilon = 1$ and -1, respectively. Also, recall that $N_H(Q_{1,1}) \cong (GL_1^\epsilon(q) : 2) \wr S_2$ and $C_H(Q_{1,1}) \cong (GL_1^\epsilon(q))^2$.

$$
\begin{aligned}
*_{Q_{1,1}}\colon & B_0 \mapsto (1,1) \\
& \left\{ \begin{array}{l} B_7(i) \\ B_9(i) \end{array} \right. \mapsto (\varphi_i, 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, \ell^d | i \\
& \left\{ \begin{array}{l} B_{11}(i) \\ B_{13}(i) \end{array} \right. \mapsto (\varphi_i, \varphi_i) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon}, \ell^d | i \\
& \left\{ \begin{array}{l} B_{15}(i,j) \\ B_{19}(i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_j) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j) \in I^{2*}_{q-\epsilon}, \ell^d | i, j
\end{aligned}
$$

Continuing to let $\ell|(q^2-1)$, the maps Ω_Q are as follows:

$$
\begin{aligned}
\Omega_{Q_1}\colon & \left\{ \begin{array}{l} \{\chi_9(i), \chi_{10}(i)\} \\ \{\chi_7(i), \chi_8(i)\} \end{array} \right. \mapsto (\chi_*(i), 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q+\epsilon} \\
& \left\{ \begin{array}{l} \chi_{17}(j,i) \\ \chi_{17}(i,j) \end{array} \right. \mapsto (\chi_*(i), \varphi_j) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q+\epsilon}; \quad j \in I_{q-\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{Q_2}\colon & \left\{ \begin{array}{l} \{\chi_{13}(i), \chi_{14}(i)\} \\ \{\chi_{11}(i), \chi_{12}(i)\} \end{array} \right. \mapsto (\chi_*(2i), 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q+\epsilon} \\
& \left\{ \begin{array}{l} \chi_{16}(i) \\ \chi_{16}(i) \end{array} \right. \mapsto (\chi_*(2i_1), \varphi_{2i_2}) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i = i_1(q-\epsilon) + i_2(q+\epsilon) \in I_{q^2-1}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{Q_{1,1}}\colon & \{\chi_1, \chi_3, \chi_4, \chi_5, \chi_6\} \mapsto (1,1) \\
& \left\{ \begin{array}{l} \{\chi_7(i), \chi_8(i)\} \\ \{\chi_9(i), \chi_{10}(i)\} \end{array} \right. \mapsto (\varphi_i, 1) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon} \\
& \left\{ \begin{array}{l} \{\chi_{11}(i), \chi_{12}(i)\} \\ \{\chi_{13}(i), \chi_{14}(i)\} \end{array} \right. \mapsto (\varphi_i, \varphi_i) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad i \in I_{q-\epsilon} \\
& \left\{ \begin{array}{l} \chi_{15}(i,j) \\ \chi_{19}(i,j) \end{array} \right. \mapsto (\varphi_i, \varphi_j) \quad \begin{array}{c} \epsilon = 1 \\ \epsilon = -1 \end{array} \qquad (i,j) \in I^{2*}_{q-\epsilon}
\end{aligned}
$$

Now let $\ell | (q^2+1)$, so $Q = Q^{(2)} \in \mathrm{Syl}_\ell(H)$ and $N_H(Q) \cong C_{q^2+1} : 2^2$ with $C_H(Q) \cong C_{q^2+1}$. In this case we only define a map Ω_Q:

$$\begin{aligned}\Omega_{Q^{(2)}} \colon \{\chi_1, \chi_2, \chi_5, \chi_6\} &\mapsto (1)\\ \chi_{18}(i) &\mapsto (\vartheta_i) \qquad i \in I_{q^2+1}\end{aligned}$$

7.5 $Sp_6(2^a)$ and $Sp_4(2^a)$ are Good for the Conjectures

In this section, we prove Theorem 1.2.1. We begin with a discussion regarding the automorphisms of $G = Sp_6(2^a)$ and $H = Sp_4(2^a)$ before proving a few propositions, which describe some properties of the maps and sets defined in Section 7.4.

Let Q be an ℓ-radical subgroup of $G = Sp_6(2^a)$, where $\ell \neq 2$ is a divisor of $|G|$. Let σ_2 be the field automorphism of G induced by the Frobenius map $F_2 \colon x \mapsto x^2$. That is, $(a_{ij})^{\sigma_2} = (a_{ij}^2)$ for (a_{ij}) some matrix in G. Then $\mathrm{Aut}(G) = \langle G, \sigma_2 \rangle$. Let Q be an ℓ-radical subgroup of G. If Q is generated by diagonal matrices and matrices with entries in $\mathbb{F}_2$, then $Q^{\sigma_2} = Q$, and we will write $\sigma := \sigma_2$. Otherwise, Q is conjugate in $\underline{G} := Sp_6(\overline{\mathbb{F}}_q)$ to a group D of this form. Moreover, the G-conjugacy class of Q is determined by D. If $Q = \langle x \rangle$ is cyclic, then x is conjugate in $\underline{G}$ to a generator, y for D. But y is also conjugate in $\underline{G}$ to y^{σ_2}, so x is conjugate to x^{σ_2} in $\underline{G}$. But since two semisimple elements of G are conjugate whenever they are conjugate in $\underline{G}$ (see, for example, the description in [47] of conjugacy classes of G), we see that Q is conjugate in G to Q^{σ_2}. If Q is abelian but not cyclic, we can view Q as a subgroup of the product of lower-rank symplectic groups (e.g. $Q_{2,1} \leq Sp_4(q) \times Sp_2(q)$), and a similar argument on the direct factors shows that Q is G-conjugate to Q^{σ_2}. Finally, if Q is nonabelian, then $\ell = 3$ and Q must be either R or P, in which case Q^{σ_2} must be G-conjugate to Q since Q is the unique (up to G-conjugacy) ℓ-radical subgroup of its isomorphism type.

Hence in any case, we know that there is some $\sigma \in \mathrm{Aut}(G)$ (obtained by multiplying σ_2 by an inner automorphism) so that $Q^\sigma = Q$ and $\mathrm{Aut}(G) = \langle G, \sigma\rangle$. For the remainder of this section, given the ℓ-radical subgroup Q, σ will denote this automorphism.

Now let $H = Sp_4(2^a)$. Then $\mathrm{Out}(H)$ is still cyclic, generated by a graph automorphism γ_2. Now, the action of γ_2 switches the fundamental roots of the root system of type B_2, and the action on the elements of H can be seen from [15, Proposition 12.3.3]. We see that γ_2 satisfies $\gamma_2^2 = \sigma_2$. We may then replace γ_2 with some γ which fixes a Sylow ℓ-subgroup and satisfies $\gamma^2 = \sigma$.

Our first two propositions show that the maps defined in Section 7.4 commute with the automorphism groups of G and H.

Proposition 7.5.1. *Let $G = Sp_6(q)$, with $q = 2^a$, $\ell \neq 2$ a prime dividing $|G|$, and $Q \leq G$ a nontrivial ℓ-radical subgroup. Then the maps Ω_Q and $*_Q$ (for $\ell|(q^2-1)$) described in Section 7.4 are* $\mathrm{Aut}(G)$*-equivariant.*

Proof. Let $\chi \in \underline{\mathrm{Irr}_0}(G|Q)$ (resp. $\chi \in \mathrm{IBr}_\ell(G|Q)$) as defined in Section 7.4. Since $\mathrm{Out}(G) = \mathrm{Aut}(G)/G \cong C_a$ is cyclic, it suffices to show that $(\Omega_Q(\chi))^\sigma = \Omega_Q(\chi^\sigma)$ (resp. $(\chi^{*_Q})^\sigma = (\chi^\sigma)^{*_Q}$) for a generator, σ, of $\mathrm{Out}(G)$. In particular, let σ be the automorphism of G described above and note that we can write $\sigma = y\sigma_2$ for some $y \in G$.

As usual, let $N := N_G(Q)$ and $C := C_G(Q)$ denote the normalizer and centralizer of the ℓ-radical subgroup Q.

(1) Note that σ fixes the unipotent classes of G. Now, a semisimple class of G is determined by its eigenvalues (possibly in an extension field of $\mathbb{F}_q$) on the action of the natural module $\langle e_1, e_2, e_3, f_1, f_2, f_3\rangle_{\mathbb{F}_q}$ of G. Hence, as the action of σ on semisimple classes of G is to square the eigenvalues, we see that σ sends the class $C_i(j_1, j_2, j_3)$ of G (in the notation of CHEVIE [26], with the possibility of some of the indices j_k being null) to the class $C_i(2j_1, 2j_2, 2j_3)$ (which we mention is also equal to the class

$C_i(-2j_1, -2j_2, -2j_3))$.

Let $\theta \in \mathrm{Irr}(G)$. Then $\theta^\sigma(g) = \theta(g^{\sigma^{-1}})$ for $g \in G$. From the observations in the above paragraph and careful inspection of the character values for irreducible characters of G in CHEVIE and [47], we see that the character $\chi_i(j_1, j_2, j_3)$ (again in the notation of CHEVIE) is mapped under σ to $\chi_i(2j_1, 2j_2, 2j_3)$. That is, σ preserves the family of a character, and in the notation of Section 7.4, $\mathcal{E}_i(J)^\sigma = \mathcal{E}_i(2J)$, where $2J = (2j_1, .., 2j_k)$ for an indexing set $J = (j_1, ..., j_k)$.

Now, as discussed in Section 7.1, the set $\mathcal{E}_i(J)$ forms a basic set for the block $B_i(J)$, so by writing $\varphi \in B_i(J)$ as a linear combination of the $\widehat{\theta}$ for $\theta \in \mathcal{E}_i(J)$, we see that $B_i(J)^\sigma = B_i(2J)$, with the character families preserved. Also, note that both Brauer and ordinary characters of unipotent blocks of G are fixed under σ.

(2) Now, by similar argument to part (1), the action of σ on the irreducible ordinary characters of $Sp_4(q), SL_2(q)$, and $GL_3^{\pm}(q)$ that we require in the descriptions of $\mathrm{dz}(N/Q)$ and $\underline{\mathrm{Irr}_0}(N|Q)$ is analogous to the action on $\mathrm{Irr}(G)$. That is, these characters are indexed in a similar fashion $\{\chi_i(j_1, j_2, j_3)\}$ in CHEVIE, and we have $\chi_i(j_1, j_2, j_3)^\sigma = \chi_i(2j_1, 2j_2, 2j_3)$.

(3) From the description of the action of σ on semisimple and unipotent classes of G, we see that σ squares the elements of $GL_1^\epsilon(q)$ and commutes with τ. (Recall that τ is the involutory automorphism $\zeta \mapsto \zeta^{-1}$ on $GL_1^\epsilon(q) \cong C_{q-\epsilon}$.) Hence when it occurs in N, the character of $GL_1^\epsilon(q) : 2$ with φ_i as a constituent on $GL_1^\epsilon(q)$ is mapped under σ to the character with φ_{2i} as a constituent. (When $i = 0$, the choice ± 1 of extension is fixed as well. That is, σ fixes $1^{(1)}$ and $1^{(2)}$.)

Similarly, the action of σ on $\mathrm{Irr}(C_{q^3-\epsilon} : 6)$ (resp. $\mathrm{Irr}(C_{q^2+1} : 2^2)$) is to send the character with ϕ_i (resp. ϑ_i) as a constituent on C_{q^3-1} (resp. C_{q^2+1}) to the one with ϕ_{2i} (resp. ϑ_{2i}) as a constituent and fix the characters with $i = 0$, since σ squares elements of $C_{q^3-\epsilon}$ or C_{q^2+1} and commutes with the action of the generators of the order-6 or 4 complement.

(4) The observations in (1) – (3) imply that $(\Omega_Q(\chi))^\sigma = \Omega_Q(\chi^\sigma)$ (resp. $(\chi^{*_Q})^\sigma =$

$(\chi^\sigma)^{*_Q})$ for our choice of generator σ, except possibly when $Q = P$ or R and $\ell = 3|(q^2-1)$.

Now, when $Q = P$, the discussion on height-zero characters of $N_G(P)$, combined with (3) and the fact that σ commutes with the action of the S_3-subgroup of N yields that the character $\varphi_i\beta$ of N (in the notation from Section 7.3.8) is mapped to $\varphi_{2i}\beta$ under σ. Hence again in this case, the maps are equivariant.

Finally, let $Q = R$. Since Ω_R is trivial, we need only consider the map $*_R$, and therefore the members of $\mathrm{dz}(N_G(R)/R)$. By Section 7.3.9, this set is comprised of the characters $\varphi_i\beta$ with $3^d|i$, where $\beta \in \mathrm{Irr}(Sp_2(3))$ is the Steinberg character and (recall the abuse of notation) $\varphi_i = \varphi_{-i}$ is the character whose restriction to $C_{q-\epsilon}$ contains φ_i as a constituent. Now, β, $1^{(1)}$, and $1^{(-1)}$ are fixed by σ, and $\varphi_i^\sigma = \varphi_{2i}$ as before, so we see that $*_R$ is again equivariant. □

Proposition 7.5.2. *Let $H = Sp_4(q)$, with $q = 2^a$, $\ell \neq 2$ a prime dividing $|H|$, and $Q \leq H$ a nontrivial ℓ-radical subgroup. Then the maps Ω_Q and $*_Q$ (for $\ell|(q^2-1)$) described in Section 7.4 are* $\mathrm{Aut}(H)$*-equivariant.*

Proof. Again, it suffices to show that $*_Q$ and Ω_Q commute with the generator γ of $\mathrm{Out}(H)$. We will use the notation of classes and characters from CHEVIE, [26]. From comparing notation of CHEVIE, [21], and [15], we deduce that the action of γ on the unipotent classes of H is to switch C_2 and C_3 and fix the other unipotent classes. Moreover, $C_7(i)^\gamma = C_{11}(i)$ and $C_{11}(i)^\gamma = C_7(2i)$. Similarly, $C_9(i)^\gamma = C_{13}(i)$ and $C_{13}(i)^\gamma = C_9(2i)$. Hence γ switches Q_1 and Q_2. Also, $C_{15}(i,j)^\gamma = C_{15}(i+j, i-j)$, $C_{19}(i,j)^\gamma = C_{19}(i+j, i-j)$, and $Q_{1,1}$ is stabilized by γ. Moreover, $C_{17}(i,j) = C_{16}(i(q+1)+j(q-1))$, $C_{16}(i)^\gamma = C_{17}(i \mod (q-1), j \mod (q+1))$, and $C_{18}(i)^\gamma = C_{18}((q+1)i)$.

From this, using the character table for H in CHEVIE [26], we can see the action of γ on the relevant characters (and blocks) of H. Namely, $B_7(i)^\gamma = B_{11}(i)$, $B_{11}(i)^\gamma = B_7(2i)$, $B_9(i)^\gamma = B_{13}(i)$, $B_{13}(i)^\gamma = B_9(2i)$, $B_{15}(i,j)^\gamma = B_{15}(i+j, i-j)$, $B_{19}(i,j)^\gamma = B_{19}(i+j, i-j)$, and $\chi_{18}(i)^\gamma = \chi_{18}((q+1)i)$. Also, B_0 is fixed, except that χ_3 and χ_4

are switched.

Let φ_i for $i \in I_{q-\epsilon}$ be as usual. Considering the action of γ on elements of $N_H(Q_1)$ and $N_H(Q_2)$, we see that the characters $(\varphi_i, \chi_*(j))$ of $C_H(Q_1)$ are mapped under γ to the corresponding character $(\varphi_i, \chi_*(j))$ in $C_H(Q_2)$. Applying γ again yields $(\varphi_{2i}, \chi_*(2j))$ in $C_H(Q_1)$. Moreover, for $\nu \in \{\pm 1\}$, $(1^{(\nu)}, \chi_*(j)) \in \mathrm{Irr}(N_H(Q_1))$ is mapped to the corresponding character $(1^{(\nu)}, \chi_*(j)) \in \mathrm{Irr}(N_H(Q_2))$, which is then mapped to $(1^{(\nu)}, \chi_*(2j)) \in \mathrm{Irr}(N_H(Q_1))$.

Inspecting the values of the characters of $N_H(Q_{1,1})/C_H(Q_{1,1}) \cong C_2 \wr S_2$, we see that they are fixed under γ, aside from $(1^{(1)}, 1^{(1)})^{(-1)}$ and $(1^{(-1)}, 1^{(-1)})^{(1)}$, which are switched. So, choosing $\{\chi_3, \chi_4\} \mapsto \{(1^{(1)}, 1^{(1)})^{(-1)}, (1^{(-1)}, 1^{(-1)})^{(1)}\}$, we see that this is consistent with our maps.

Also, the characters θ of $N_H(Q_{1,1})$ which are nontrivial on $C_H(Q_{1,1})$ satisfy that if (φ_i, φ_j) is a constituent of $\theta|_{C_H(Q_{1,1})}$, then $(\varphi_{i+j}, \varphi_{i-j})$ is a constituent of $\theta^\gamma|_{C_H(Q_{1,1})}$, where $i, j \in I_{q-\epsilon} \cup \{0\}$, and $\varphi_0 := 1_{C_{q-\epsilon}}$. Moreover, in the case $i = 0$, the action on $C_2 \wr S_2$ yields that the choice of extension is fixed under γ (i.e. $(\varphi_i, 1^{(\nu)})^\gamma = (\varphi_i, \varphi_i)^{(\nu)}$ where $\nu \in \{\pm 1\}$).

Finally, $\vartheta_i \in \mathrm{Irr}(C_G(Q^{(2)})$ is mapped under γ to $\vartheta_{(q+1)i}$, and when $i = 0$ the choice of extension to $N_G(Q^{(2)})$ is fixed by γ.

Altogether, these discussions yield that $(\chi^{*_Q})^\gamma = (\chi^\gamma)^{*_Q}$ for each $\chi \in \mathrm{IBr}_\ell(H|Q)$, as desired, and similar for Ω_Q.

□

We now show that our maps send a block in G to its Brauer correspondent in $N_G(Q)$.

Proposition 7.5.3. *Let $G = Sp_6(2^a)$ or $Sp_4(2^a)$, ℓ an odd prime dividing $|G|$, and Q a nontrivial ℓ-radical subgroup of G. Let the sets $\underline{\mathrm{Irr}_0}(G|Q), \underline{\mathrm{Irr}_0}(N_G(Q)|Q)$, and $\mathrm{IBr}_\ell(G|Q)$ and the maps Ω_Q, $*_Q$ be as described in Section 7.4. Then*

- *If $\chi \in \underline{\mathrm{Irr}_0}(G|Q)$ with $B \in \mathrm{Bl}(G|\chi)$ and $b \in \mathrm{Bl}(N_G(Q)|\Omega_Q(\chi))$, then $b^G = B$.*

- *If* $\chi \in \mathrm{IBr}_\ell(G|Q)$ *with* $B \in \mathrm{Bl}(G|\chi)$ *and* $b \in \mathrm{Bl}(N_G(Q)|\chi^{*Q})$, *then* $b^G = B$.

Proof. Let $N := N_G(Q)$ and $C := C_G(Q)$. As $b \in \mathrm{Bl}(N_G(Q))$, b^G is defined and $b^G = B$ if and only if $\lambda_B(\mathcal{K}^+) = \lambda_b\left((\mathcal{K} \cap C)^+\right)$ for all conjugacy classes $\mathcal{K}$ of G (see, for example, [33, Lemma 15.44]). Let $\chi \in \mathrm{Irr}(G|B)$. The central character ω_χ for G can be computed in CHEVIE, and the values of $\varphi \in \mathrm{Irr}(N|b)$ on C can be computed by their descriptions and using the character tables for $Sp_4(q), SL_2(q)$, and $GL_3^{\pm}(q)$ available in CHEVIE. Hence it remains only to determine the fusion of classes of C into G in order to compute $\omega_\varphi\left((\mathcal{K} \cap C)^+\right) = \frac{1}{\varphi(1)} \sum_{\mathcal{C} \subseteq \mathcal{K}} \varphi(g)|\mathcal{C}|$, where $g \in \mathcal{C}$ and the sum is taken over classes $\mathcal{C}$ of C which lie in $\mathcal{K}$, and compare the image of this under $*$ with $\omega_\chi(\mathcal{K}^+)^*$.

We present here the complete discussion for $*_R$ when $G = Sp_6(q)$, $\ell = 3|(q-\epsilon)$. The other situations are similar, though quite tedious.

When $Q = R$, we have $C = C_{q-\epsilon} = Z(GL_3^\epsilon(q))$, embedded in G in the usual way. The set $\mathrm{IBr}_3(G|R)$ consists of two Brauer characters in a unipotent block and one Brauer character in each set $B_8(i)$ if $\epsilon = 1$ or $B_9(i)$ if $\epsilon = -1$ with $i \in I_{q-\epsilon}$ divisible by 3^d. Choosing $\chi = 1_G$ for $B = B_0$, $\chi = \chi_{27}(i)$ for B the block containing $B_8(i)$, and $\chi = \chi_{30}(i)$ for B the block containing $B_9(i)$, we have $\omega_{1_G}(\mathcal{K}^+)^* = 0 = \omega_{\chi_{27}(i)}(\mathcal{K}^+)^*$ when $\epsilon = 1$ for every nontrivial conjugacy class $\mathcal{K} \neq C_{25}(j)$ of G (in the notation of CHEVIE) for any $j \in I_{q-1}$ and $\omega_{1_G}(\mathcal{K}^+)^* = 0 = \omega_{\chi_{30}(i)}(\mathcal{K}^+)^*$ when $\epsilon = -1$ for every nontrivial conjugacy class $\mathcal{K} \neq C_{28}(j)$ for any $j \in I_{q+1}$.

Now, let ζ generate $C \cong C_{q-\epsilon}$, so ζ^i is identified in G with the semisimple element with eigenvalues $\tilde{\zeta}$ and $\tilde{\zeta}^{-1}$, each of multiplicity 3, where $\tilde{\zeta}$ is a fixed primitive $(q-\epsilon)$ root of unity in $\overline{\mathbb{F}_q}^\times$. Then $\{\zeta^i, \zeta^{-i}\} = C_{25}(i) \cap C$ if $\epsilon = 1$ and $= C_{28}(i) \cap C$ if $\epsilon = -1$.

Let $\overline{\zeta} = \exp\left(\frac{2\pi\sqrt{-1}}{q-\epsilon}\right)$ in $\mathbb{C}^\times$ and let $\chi := \chi_{27}(i)$ or $\chi_{30}(i)$ and $\mathcal{K} = C_{25}(j)$ or $C_{28}(j)$, in the cases $\epsilon = 1, -1$, respectively. Then $\omega_\chi(\mathcal{K}^+)^* = (\overline{\zeta}^{3ij} + \overline{\zeta}^{-3ij})^*$ from CHEVIE, since $(q-\epsilon)^* = 3^* = 0$. But the value of $\varphi := \varphi_{3i}\beta$ on ζ^j is $\overline{\zeta}^{3ij}$, so $\omega_\varphi(\mathcal{K} \cap C)^+) = \frac{(\overline{\zeta}^{3ij} + \overline{\zeta}^{-3ij})2}{2} = \overline{\zeta}^{3ij} + \overline{\zeta}^{-3ij}$. Hence we have $b^G = B$ in this case.

Moreover, $\omega_{1_G}(\mathcal{K})^* = 2$, so $b^G = B$ in this case as well, since if $\varphi = 1^{(\nu)}\beta$ for $\nu \in \{\pm 1\}$, then $\omega_\varphi(\mathcal{K} \cap C)^+) = 2$, completing the proof for $Q = R$. $\square$

The next proposition shows that the sets defined in Section 7.4.1 are in fact the height-zero characters of $Sp_6(2^a)$.

Proposition 7.5.4. *Let $G = Sp_6(2^a)$ or $Sp_4(2^a)$ and $Q \leq G$ a nontrivial ℓ-radical subgroup with ℓ an odd prime dividing $|G|$. The sets $\underline{\mathrm{Irr}_0}(G|Q)$ and $\underline{\mathrm{Irr}_0}(N_G(Q)|Q)$ defined in Section 7.4 are exactly the sets $\mathrm{Irr}_0(G|Q)$ and $\mathrm{Irr}_0(N_G(Q)|Q)$ of height-zero characters of G and $N_G(Q)$, respectively, with defect group Q.*

Proof. (1) Let $N := N_G(Q)$, $\varphi \in \underline{\mathrm{Irr}_0}(N|Q)$, and $\chi \in \underline{\mathrm{Irr}_0}(G|Q)$ such that $\Omega_Q(\chi) = \varphi$. Let $b \in \mathrm{Bl}(N|\varphi)$, so that b^G is the block B containing χ, by Proposition 7.5.3. Let D_b and D_B denote defect groups for b and B, respectively, so we may assume $D_b \leq D_B$. Then as Q is ℓ-radical, we know that $Q \leq D_b \leq D_B$ (see, for example, [33, Corollary 15.39]). Now, since $|G|_\ell/|D_B|$ must be the highest power of ℓ dividing the degree of every member of $\mathrm{Irr}(B)$, inspection of the character degrees in B yields that $|D_B| = |Q|$, so in fact $Q = D_b = D_B$. Hence by inspection of the degrees of characters in our constructed sets, we see that $\underline{\mathrm{Irr}_0}(G|Q) \subseteq \mathrm{Irr}_0(G|Q)$ and $\underline{\mathrm{Irr}_0}(N|Q) \subseteq \mathrm{Irr}_0(N|Q)$.

(2) Moreover, we have constructed the set $\underline{\mathrm{Irr}_0}(G|Q)$ to contain all characters $\chi' \in \mathrm{Irr}(B)$ whose degrees satisfy $\chi'(1)_\ell = [G:Q]_\ell$. (That is to say, given any block in $\mathrm{Bl}(G)$, if we included in $\underline{\mathrm{Irr}_0}(G|Q)$ one irreducible ordinary character of the block whose degree satisfies this condition, then we included *all* such members of the block.) Further, every block $B' \in \mathrm{Bl}(G)$ of positive defect intersects the set $\underline{\mathrm{Irr}_0}(G|Q')$ for some ℓ-radical subgroup Q', so we see that in fact $\underline{\mathrm{Irr}_0}(G|Q) = \mathrm{Irr}_0(G|Q)$. Note that when $\ell = 3$, this means R does not occur as a defect group for any block of $G = Sp_6(2^a)$.

(3) Now, except in the case $G = Sp_6(q)$ with $\ell = 3$ and $Q = Q_{1,1,1}$ or P, from the discussion in Section 7.3 we see that in fact every character θ of N with $\theta(1)_\ell =$

$|N|_\ell/|Q|$ has been included in $\underline{\mathrm{Irr}_0}(N|Q)$, so $\underline{\mathrm{Irr}_0}(N|Q) = \mathrm{Irr}_0(N|Q)$. Hence we are left with the case $\ell = 3$ and $Q = Q_{1,1,1}$ or P. However, by the discussion after the description of the map $\Omega_{Q_{1,1,1}}$ in the case $\ell = 3$, we see that $\underline{\mathrm{Irr}_0}(N|Q_{1,1,1}) = \mathrm{Irr}_0(N|Q_{1,1,1})$ in this case as well. Finally, for $Q = P$, we have already described $\mathrm{Irr}_0(N|P)$ in Section 7.3.8.

□

We note that Proposition 7.5.4 is consistent with Brauer's height-zero conjecture, which says that an ℓ-block B of a finite group has an abelian defect group if and only if every irreducible ordinary character in B has height zero. It is also consistent with a consequence of [36, Theorem 7.14], which implies that the defect group for a block which is not quasi-isolated (i.e. satisfies the conditions for Bonnafé-Rouquier's theorem) is isomorphic to the defect group of its Bonnafé-Rouquier correspondent.

We are now prepared to show that $Sp_6(q)$ and $Sp_4(q)$ are (B)AWC-good.

Theorem 7.5.5. *Let $G = Sp_6(2^a)$ with $a \geq 1$ or $Sp_4(2^a)$ with $a \geq 2$. Then G is "good" for the Alperin weight and blockwise Alperin weight conjectures for all primes $\ell \neq 2$.*

Proof. 1) Let $\ell \neq 2$ be a prime dividing $|G|$. Since $\mathrm{Out}(G)$ is cyclic, we know G is BAWC-good for any prime ℓ such that a Sylow ℓ-subgroup of G is cyclic, by [70, Proposition 6.2]. Hence, G is BAWC-good for ℓ as long as $\ell \nmid (q^2 - 1)$. Moreover, considerations in GAP show that the statement is true for $\ell = 3$ when $G = Sp_6(2)$. (The main tools here were the PrimeBlocks command, the Brauer character table for the double cover $2.Sp_6(2)$ in the Character Table Library [11], as well as the faithful permutation representation of $2.Sp_6(2)$ on 240 points given in the online ATLAS [77].) Henceforth, we shall assume $\ell|(q^2 - 1)$ and $a \geq 2$.

2) As $a \geq 2$, the Schur multiplier of G is trivial, so G is its own Schur cover, so in the notation of [53, Section 3], we may assume S is just G itself. Furthermore,

[70, Lemma 6.1] implies that it suffices to show that G is AWC-good for ℓ in the sense of [53] and that the maps used satisfy condition 4.1(ii)(3) of [70]. For the trivial group $Q = \{1\}$, the map $*_{\{1\}}\colon \{\widehat{\chi}|\chi \in \mathrm{dz}(G)\} \to \mathrm{dz}(G)$, $\widehat{\chi} \mapsto \chi$ sending the restriction of defect-zero characters to G° to the original defect-zero character satisfies the conditions trivially. Hence, it suffices to show that our sets $\mathrm{IBr}_\ell(G|Q)$ and maps $*_Q$ defined in Section 7.4.2 satisfy the conditions of [53, Section 3] and that for $\chi \in \mathrm{IBr}_\ell(G|Q)$, χ is a member of the induced block b^G, where $b \in \mathrm{Bl}(N_G(Q)|\chi^{*_Q})$. By Proposition 7.5.3, the latter condition is satisfied.

3) Since $Z(G) = 1$ and our sets $\mathrm{IBr}_\ell(G|Q)$ depend only on the conjugacy class of Q, we know that our sets satisfy [53, Condition 3.1.a]. Our sets $\mathrm{IBr}_\ell(G|Q)$ are certainly disjoint, since distinct Lusztig series or blocks are disjoint, and the union of all of these with the set $\{\widehat{\chi}|\chi \in \mathrm{dz}(G)\}$ is all of $\mathrm{IBr}_\ell(G)$, by Chapter 4 and the results of [76] and [75], so our sets also satisfy [53, Condition 3.1.b]. Moreover, by Propositions 7.5.1 and 7.5.2, our maps and sets also satisfy the final partition condition and bijection condition, [53, Conditions 3.1.c, 3.2.a].

4) Let Q be an ℓ-radical subgroup, and fix $\theta \in \mathrm{IBr}_\ell(G|Q)$. Identify G with $\mathrm{Inn}(G)$, so that we can write $G \lhd \mathrm{Aut}(G)$. Write $X := \mathrm{Aut}(G)_\theta$ and let $B := X_Q$ be the subgroup of $\mathrm{Aut}(G)$ stabilizing both Q and θ. Then certainly, $G \lhd X$, $Z(G) \leq Z(X)$, θ is X-invariant, and B is exactly the set of automorphisms of G induced by the conjugation action of $N_X(Q)$ on G. Moreover, $C_X(G)$ is trivial and since X/G is cyclic, so is the Schur multiplier $H^2(X/G, \overline{\mathbb{F}}_\ell^\times)$. Hence the normally embedded conditions [53, Conditions 3.3.a-d] are trivially satisfied, completing the proof. $\square$

Before proving the corresponding statement for the (Alperin-)McKay conjecture, we need the following lemma.

Lemma 7.5.6. *Let ℓ be a prime, S be a simple group with universal ℓ' covering group G, and Q be an ℓ-radical subgroup satisfying Conditions (i) and (ii) of [69, Definition 7.2] with $M_Q = N_G(Q)$. Let $\chi \in \mathrm{Irr}_0(G|Q)$ be such that $\mathrm{Aut}(S)_\chi/S$ is* *cyclic and let* $\eta \in \mathrm{Aut}(G)_\chi$ *with* $\mathrm{Aut}(S)_\chi = \langle S, \eta\rangle$. *Then there are* $\tilde{\chi} \in \mathrm{Irr}(A(\chi))$ *and* $\widetilde{\Omega_Q(\chi)} \in \mathrm{Irr}(N_{A(\chi)}(Q))$, *where* $A(\chi) := \langle G, \eta\rangle$, *such that:*

1. $\tilde{\chi}|_G = \chi$
2. $\widetilde{\Omega_Q(\chi)}|_{N_G(Q)} = \Omega_Q(\chi)$
3. $\tilde{b}^{A_{\ell'}} = \tilde{B}$, *where* $\tilde{b}$ *is the block of* $N_{A_{\ell'}}(Q)$ *containing* $\widetilde{\Omega_Q(\chi)}|_{N_{A_{\ell'}}(Q)}$, $\tilde{B}$ *is the block of* $A_{\ell'}$ *containing* $\widetilde{\chi}|_{A_{\ell'}}$, *and* $G \leq A_{\ell'} \leq A(\chi)$ *so that* $A_{\ell'}/G$ *is the Hall* ℓ'*-subgroup of* $A(\chi)/G$.

Proof. First, note that χ extends to $A(\chi)$ since $A(\chi)/G$ is cyclic and χ is invariant under $A(\chi)$. Let $\varphi := \Omega_Q(\chi)$. Since the map Ω_Q is $\mathrm{Aut}(G)_Q$-equivariant, we have $\varphi = \Omega_Q(\chi^\alpha) = \varphi^\alpha$ for any $\alpha \in N_{A(\chi)}(Q)$, so φ is invariant under $N_{A(\chi)}(Q)$ and therefore extends to some $\widetilde{\varphi} \in \mathrm{Irr}\left(N_{A(\chi)}(Q)\right)$ since $N_{A(\chi)}(Q)/N_G(Q)$ is cyclic. Let $\widetilde{b}$ be the block of $N_{A_{\ell'}}(Q)$ containing the restriction $\widetilde{\varphi}|_{N_{A_{\ell'}}(Q)}$ and let B be the block of G containing χ. Then $\tilde{b}^{A_{\ell'}}$ is defined, by [33, Lemma 15.44], and we claim that $\tilde{b}^{A_{\ell'}}$ covers B, so that by [52, Theorem 9.4], we can choose an extension $\tilde{\chi}$ of χ to $A(\chi)$ so that $\tilde{\chi}|_{A_{\ell'}}$ is contained in $\tilde{b}^{A_{\ell'}}$.

To prove the claim, first note that by [52, Theorem 9.5], $\tilde{b}^{A_{\ell'}}$ covers B if and only if the central functions satisfy $\lambda_{\tilde{b}^{A_{\ell'}}}(\mathcal{K}^+) = \lambda_B(\mathcal{K}^+)$ for all classes $\mathcal{K}$ of $A_{\ell'}$ contained in G. Let b be the block of $N_G(Q)$ containing φ, so that $b^G = B$ by Condition (ii) of [69, Definition 7.2] and $\lambda_{\tilde{b}}$ covers λ_b by [52, Theorem 9.2]. Let $\mathcal{K}_1, \ldots, \mathcal{K}_k$ be the classes of G so that $\mathcal{K} = \bigcup_i \mathcal{K}_i$. Notice that $\mathcal{K}_i \cap N_{A_{\ell'}}(Q) = \mathcal{K}_i \cap N_G(Q)$ can be viewed as a union of classes of $N_{A_{\ell'}}(Q)$ contained in $N_G(Q)$ and $\bigcup_i(\mathcal{K}_i \cap N_{A_{\ell'}}(Q)) = \mathcal{K} \cap N_{A_{\ell'}}(Q)$, so

$$\begin{aligned}\lambda_B(\mathcal{K}^+) &= \sum_i \lambda_B(\mathcal{K}_i^+) = \sum_i \lambda_{b^G}(\mathcal{K}_i^+) = \sum_i \lambda_b\left((\mathcal{K}_i \cap N_G(Q))^+\right)\\ &= \sum_i \lambda_{\tilde{b}}\left((\mathcal{K}_i \cap N_{A_{\ell'}}(Q))^+\right) = \lambda_{\tilde{b}}\left((\mathcal{K} \cap N_{A_{\ell'}}(Q))^+\right) = \lambda_{\tilde{b}^{A_{\ell'}}}(\mathcal{K}^+),\end{aligned}$$

which proves the claim. $\square$

Theorem 7.5.7. *Let $G = Sp_6(2^a)$ or $Sp_4(2^a)$ with $a \geq 2$. Then G is "good" for the McKay and Alperin-McKay conjectures for all primes $\ell \neq 2$.*

Proof. 1) Again, notice that G is its own Schur cover, so $G = S$ in the notation of either [69, Definition 7.2] or [34, Section 10]. Also, note that reasoning similar to part (4) of the proof of Theorem 7.5.5 implies that G satisfies conditions (5)-(8) of the definition of McKay-good in [34, Section 10]. Hence, if G is "good" for the Alperin-McKay conjecture (i.e. satisfies the inductive-AM-condition described in [69, Definition 7.2]), then G satisfies conditions (1)-(4) of the definition of McKay-good, so is also "good" for the McKay conjecture. (Indeed, in the case Q is a Sylow ℓ-subgroup of G, the set $\mathrm{Irr}_{\ell'}(G)$ is exactly the set of height-zero characters of G with defect group Q.) Again, when $Q = \{1\}$, the map sending defect-zero characters to themselves satisfies the conditions trivially.

2) Let $Q \neq 1$ be an ℓ-radical subgroup of G which occurs as a defect group for some ℓ-block of G. Hence by replacing with a conjugate subgroup, we may assume that Q is one of the groups described in Section 7.2 aside from R. The group $M_Q := N_G(Q)$ satisfies condition (i) of [69, Definition 7.2]). Moreover, Propositions 7.5.1, 7.5.2, 7.5.3, and 7.5.4 imply that the map Ω_Q from Section 7.4 satisfies condition (ii) of [69, Definition 7.2]. (Again note that $Z(G)$ is trivial.)

3) Now, let $A := \mathrm{Aut}(G)$ and let $\chi \in \mathrm{Irr}_0(G|Q)$. Write $A_\chi := \mathrm{stab}_A(\chi)$ and write $A_{Q,\chi}$ for the subgroup $N_{A_\chi}(Q)$ of elements of A which stabilize both Q and χ. Write $\chi' := \Omega_Q(\chi)$ and let $\widetilde{\chi}$ and $\widetilde{\chi'}$ be the extensions of χ to A_χ and χ' to $A_{Q,\chi}$ as in Lemma 7.5.6, since A/G is cyclic. Say $\mathcal{P}$ and $\mathcal{P}'$, respectively, are the representations affording these extensions. Then certainly, these representations satisfy the first three subconditions of condition (iii) of [69, Definition 7.2] and it suffices to show that they satisfy the final subcondition. (Note that here $\mathrm{rep}\colon S \to G$ is simply the identity map.)

4) Let x be an ℓ-regular element of $M_Q A_{Q,\chi} = A_{Q,\chi}$ with $Q \in \mathrm{Syl}_\ell(C_G(x))$. If

$x \in M_Q$, we are done by [69, Proposition 7.4]. So, suppose that $x \notin M_Q$. That is, $x \in N_{A_\chi}(Q) \setminus N_G(Q)$. Since x is an ℓ'-element, we see that in fact $x \in N_{A_{\ell'}}(Q)$, where $G \leq A_{\ell'} \leq A_\chi$ is as in Lemma 7.5.6. Let $\mathcal{K}$ be the conjugacy class of x in $A_{\ell'}$. Since $Q \in \mathrm{Syl}_\ell(C_G(x))$, we also have $Q \in \mathrm{Syl}_\ell(C_{A_{\ell'}}(x))$, since $[A_{\ell'} : G]$ is prime to ℓ. Hence $\mathcal{K} \cap C_{A_{\ell'}}(Q)$ is the class of $N_{A_{\ell'}}(Q)$, containing x (see, for example, [52, Lemma 4.16]). Let $\tilde{B}$ and $\tilde{b}$ be the blocks of $A_{\ell'}$ and $N_{A_{\ell'}}(Q)$ containing $\widetilde{\chi}$ and $\widetilde{\chi'}$, respectively, so that $\tilde{b}^{A_{\ell'}} = \tilde{B}$. Then we have $\lambda_{\tilde{B}}(\mathcal{K}^+) = \lambda_{\tilde{b}}\left((\mathcal{K} \cap C_{A_{\ell'}}(Q))^+\right)$, which implies that

$$\left(\frac{|A_{\ell'}|\widetilde{\chi}(x)}{|C_{A_{\ell'}}(x)|\chi(1)}\right)^* = \left(\frac{|N_{A_{\ell'}}(Q)|\widetilde{\chi'}(x)}{|C_{N_{A_{\ell'}}(Q)}(x)|\chi'(1)}\right)^*.$$

Moreover, except possibly in the case $G = Sp_4(q)$ and $Q = Q_1$ or Q_2, we can choose η as in Lemma 7.5.6 to stabilize Q, by the discussion preceding Proposition 7.5.1, and therefore $[A_{\ell'} : G] = [N_{A_{\ell'}}(Q) : N_G(Q)]$ and $[C_{A_{\ell'}}(x) : C_G(x)] = [C_{N_{A_{\ell'}}(Q)}(x) : C_{N_G(Q)}(x)]$. However, note that if $Q = Q_1$ or Q_2 when $G = Sp_4(q)$, then $\gamma \notin A_\chi$ (see the proof of Proposition 7.5.2), but $\gamma^2 = \sigma$ fixes Q, so the same is true in this case. This yields

$$\left(\frac{|G|\widetilde{\chi}(x)}{|C_G(x)|\chi(1)}\right)^* = \left(\frac{|N_G(Q)|\widetilde{\chi'}(x)}{|C_{N_G(Q)}(x)|\chi'(1)}\right)^*.$$

Now, since $\chi \in \mathrm{Irr}_0(G|Q)$ and $\chi' \in \mathrm{Irr}_0(N_G(Q)|Q)$, we see $[G : Q]_\ell = \chi(1)_\ell$ and $[N_G(Q) : Q]_\ell = \chi'(1)_\ell$, so

$$\left(\frac{|G|_{\ell'}\widetilde{\chi}(x)}{|C_G(x)|_{\ell'}\chi(1)_{\ell'}}\right)^* = \left(\frac{|N_G(Q)|_{\ell'}\widetilde{\chi'}(x)}{|C_{N_G(Q)}(x)|_{\ell'}\chi'(1)_{\ell'}}\right)^*,$$

and

$$([G : N_G(Q)]_{\ell'}\widetilde{\chi}(x)\chi'(1)_{\ell'})^* = \left([C_G(x) : C_{N_G(Q)}(x)]_{\ell'}\widetilde{\chi'}(x)\chi(1)_{\ell'}\right)^*.$$

Now, note that $C_{N_G(Q)}(x) = C_G(x) \cap N_G(Q) = N_{C_G(x)}(Q)$, so by Sylow's theorems, $[C_G(x) : C_{N_G(Q)}(x)]_{\ell'} = [C_G(x) : C_{N_G(Q)}(x)] \equiv 1 \mod \ell$, and hence $\epsilon_\chi^*\chi(1)_{\ell'}^*\widetilde{\chi}(x)^* = ([G : N_G(Q)]_{\ell'}\widetilde{\chi}(x)\chi'(1)_{\ell'})^* = \left(\widetilde{\chi'}(x)\chi(1)_{\ell'}\right)^*$, where ϵ_χ is as in [69, Definition 7.2], and finally $\epsilon_\chi^*\mathrm{Tr}(\mathcal{P}(x))^* = \mathrm{Tr}(\mathcal{P}'(x))^*$, as desired.

□

To finish the proof of Theorem 1.2.1, we need to prove that when $q = 2$, the groups $Sp_6(2)$ and $Sp_4(2)'$ are AM-good.

Theorem 7.5.8. *Let $S = Sp_6(2)$ or $Sp_4(2)' \cong A_6$. Then S is "good" for the Alperin-McKay conjecture for all primes.*

Proof. Let $G := 6.A_6$ be the universal covering group of $S := A_6$ and ℓ a prime dividing $|A_6|$. We can construct G in GAP using the generators given in the online ATLAS [77] for the faithful permutation representation of G on 432 letters. Using the PrimeBlocks function to calculate the sizes of the defect groups and calculating the Sylow subgroups of centralizers of ℓ'-elements, we see that the only noncentral defect group of G are Sylow ℓ-subgroups. Fix $P \in \mathrm{Syl}_\ell(G)$. Again using the PrimeBlocks function, the knowledge of the action of the outer automorphism group of A_6 on the conjugacy classes of A_6, and the character information for $6.A_6, 6.A_6.2_1$, and $6.A_6.2_2$ in the GAP Character Table Library [11], we see that we can construct bijections satisfying conditions (i) and (ii) of the Inductive AM-condition [69, Definition 7.2], with $M_P := N_G(P)$. Further, by [69, Proposition 4.2], for $\chi \in \mathrm{Irr}_0(G|P)$, there exist $\mathcal{P}, \mathcal{P}'$ satisfying the first three requirements of condition (iii), so it remains to show that they fulfill the final requirement, [69, (7.4)].

Now, if $\ell = 3$ or 2, then calculating with the automorphism group in GAP yields that the centralizer $C_{\mathrm{Aut}(S)}(PZ(G)/Z(G))$ is an ℓ-group, so this final requirement is satisfied by [69, Proposition 7.4].

If $\ell = 5$, then $|C_{\mathrm{Aut}(S)}(PZ(G)/Z(G))| = 10$ and this centralizer is cyclic. Let g be the order-2 element in $C_{\mathrm{Aut}(S)}(PZ(G)/Z(G))$. Now, $\langle S, g\rangle$ has order 720, and comparing the character table with those of $A_6.2_1, A_6.2_2$, and $A_6.2_3$, we see that $\langle S, g\rangle = A_6.2_2$. Moreover, the height-zero characters (in the notation of the GAP Character Table Library) of $G = 6.A_6$ which are fixed under g are $\chi_1, \chi_4, \chi_5, \chi_6, \chi_{10}$, and χ_{11} of degrees $1, 8, 8, 9, 8$, and 8, respectively, and hence all other characters satisfy the final condition again by [69, Proposition 7.4].

Our constructed bijections map these characters to characters of $N_G(P)$ with degree $1, 2, 2, 1, 2$, and 2, respectively, and we see that for these characters, $\epsilon_\chi \equiv -1$ mod 5, except in the case $\chi = \chi_1 = 1_G$, in which case $\epsilon_{1_G} = 1$. (Here ϵ_χ is as defined following [69, (7.4)].) Further, $\mathrm{Aut}(S)_\chi = \mathrm{Aut}(S)$ for $\chi = \chi_1$ or χ_6 and $\mathrm{Aut}(S)_\chi = \langle S, g\rangle$ for the four characters of degree 8 under consideration. Also, χ_1, χ_4, χ_5, and χ_6 lie in the principal block of G, and can be viewed as characters of $S = G/Z(G)$. Similarly, the characters of $N_G(P)$ that they map to lie in the principal block of $N_G(P)$ and can be viewed as characters of $N_G(P)/Z(G)$. Considering the character tables for $\mathrm{Aut}(S)_\chi$ and $\mathrm{Aut}(S)_{P,\chi}$, we see that these characters lift to characters of $\mathrm{Aut}(S)_\chi$ and $(N_G(P)/Z(G))\,\mathrm{Aut}(S)_{P,\chi}$ satisfying the final condition of [69, Definition 7.2].

The remaining two characters of G and $N_G(P)$ under consideration are trivial on the elements of $Z(G)$ of order 3 and are nontrivial on the element $z \in Z(G)$ of order 2. Moreover, the values of χ_4 and χ_{10} are identical on $2'$-elements and satisfy $\chi_4(x) = -\chi_{10}(x)$ when 2 divides $|x|$. The same is true for χ_5 compared with χ_{11}, and similarly for the corresponding pairs of characters of $N_G(P)$. Hence if $\mathrm{rep}\colon S \to G$ is the $Z(G)$-section used for condition (iii) of [69, Definition 7.2] for the character $\chi = \chi_4$, respectively χ_5, then replacing rep with

$$\mathrm{rep}'\colon y \mapsto \begin{cases} \mathrm{rep}(y) & \text{if } 2 \nmid |y| \\ \mathrm{rep}(y)\cdot z & \text{if } 2 \mid |y| \end{cases}$$

yields that condition (iii) of [69, Definition 7.2] is satisfied when $\chi = \chi_{10}$, respectively χ_{11}, using the same extensions as in the case $\chi = \chi_4$, respectively χ_5.

Now let $G := 2.Sp_6(2)$ be the universal covering group of $S := Sp_6(2)$ and let ℓ be a prime dividing $|G|$. Then $\mathrm{Aut}(G) \cong \mathrm{Aut}(S) \cong S$, and in this case, the inductive AM-condition [69, Definition 7.2] is satisfied as long as the usual Alperin-McKay conjecture is satisfied. The following considerations in GAP similar to the case A_6 above and the situation for the BAWC yield that we can construct the desired bijections.

Let $P \in \mathrm{Syl}_\ell(G)$. Using the PrimeBlocks function to calculate the sizes of the defect groups and calculating the Sylow subgroups of centralizers of ℓ'-elements, we see that the only noncentral defect group of G are Sylow ℓ-subgroups when $\ell \neq 3$.

For $\ell = 7$, each of $N_G(P)$ and G have 2 blocks with defect group P, and in each case, both blocks have 7 height-zero characters. For $\ell = 2$, each of $N_G(P)$ and G have one bock with defect group P, and these blocks have 16 characters. This verifies the Alperin-McKay conjecture in these cases.

When $\ell = 5$, $N_G(P)$ and G both have 5 blocks with defect group P. In each case, all but one of these blocks has 5 height-zero characters, and the last has 4. The blocks with 4 characters are both nontrivial on $Z(G)$. For each of G and $N_G(P)$, one of the blocks with 5 height-zero characters is nontrivial on $Z(G)$, and the height-zero characters of the remaining blocks with defect group P are trivial on $Z(G)$. Inspection of the central character values available yields that the blocks with 4 height-zero characters are in Brauer correspondence, so the Alperin-McKay is satisfied in this case.

In the case $\ell = 3$, calculating the sizes of the defect groups and studying the defects of the blocks of the normalizers of the Sylow subgroups of centralizers of ℓ'-elements yields that we have two noncentral defect groups, namely the Sylow subgroup P and a cyclic defect group Q_1 of size 3. (Indeed, for the other Sylow subgroups D for centralizers of ℓ'-elements, $N_G(D)$ has no defect group D, but Brauer's first main gives a bijection between $\mathrm{Bl}(N_G(D)|D)$ and $\mathrm{Bl}(G|D)$.)

Now, G and $N_G(P)$ both have two blocks with defect group P, and each block in each case has 9 height-zero characters. $N_G(Q_1)$ and G have two blocks with defect group Q_1, and in each case, each block as 3 height-zero characters, and the Alperin-McKay conjecture is satisfied.

$\square$

Theorems 7.5.5, 7.5.7, and 7.5.8 complete the proof of the main theorem, Theorem

1.2.1.

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