HEIGHT ZERO CHARACTERS IN PRINCIPAL BLOCKS

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Abstract. We classify finite groups whose principal blocks have at most five height-zero ordinary irreducible characters. This classification, together with the recently-shown principal block case of Héthelyi-Külshammer's conjecture, allows us to obtain a lower bound for the number of height-zero ordinary irreducible characters in the principal p-block of a finite group of order divisible by p.

1. Introduction

In order to better understand the relationship between complex and p-modular representations of a finite group G, Brauer partitioned the set of ordinary and p-Brauer irreducible characters of G into naturally defined subsets called p-blocks of G. Brauer's idea has developed into what is now known as block theory, a fundamental tool in the study of finite group representation theory. In a p-block B of a finite group, height-zero characters, which are ordinary characters in B whose degrees have minimal p-part, play an important role because of their direct involvement in several central problems in the area, notably Brauer's height zero conjecture [Bra63, Problem 12] and the Alperin-McKay conjecture [Alp75].

The principal p-block of a group G, which we will denote by $B_0(G)$, or sometimes just by B_0, is the one containing the trivial character $\mathbf{1}_G$ of G. Therefore, the height-zero characters in $B_0(G)$ are simply those characters with degree not divisible by p. The problem of determining the structure of the Sylow p-subgroups of finite groups whose principal p-block has a given number of irreducible characters can be seen as the modular analogue of the classical problem of classifying finite groups with a given number of conjugacy classes [VV85]. Thanks to recent contributions in [KS21] and [RSV21], those Sylow p-subgroups have been determined for principal p-blocks with up to five irreducible characters. This determination in turn has contributed to the positive solution of the Héthelyi-Külshammer conjecture for principal blocks in [HSF21]. The purpose of this paper is to generalize the results of the afore-mentioned papers from the perspective of height-zero characters.

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We write $k(B)$ to denote the number of ordinary irreducible characters in a block B and $k_0(B)$ to denote the number of height-zero characters in B. Our first main result classifies principal blocks with at most five height-zero irreducible characters.

Theorem 1.1. *Let G a finite group and p a prime. Let P be a Sylow p-subgroup and B_0 denote the principal p-block of G. We have:*

(A) *For $k \in \{2,3\}$, $k_0(B_0) = k$ if, and only if, P has order k.*
(B) *$k_0(B_0) = 4$ if, and only if, exactly one of the following happens:*
  (i) *$[P : P'] = 4$,*
  (ii) *$|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.*
(C) *$k_0(B_0) = 5$ if, and only if, exactly one of the following happens:*
  (i) *$|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1,4\}$,*
  (ii) *$|P| = 7$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2,3\}$.*

Theorem 1.1 suggests that the prime p is bounded from above in terms of the number of height-zero characters in the principal p-block. Our second main result confirms this, showing that $p \leqslant k_0(B_0)^2/4 + 1$.

Theorem 1.2. *Let G be a finite group of order divisible by a prime p and B_0 denote the principal p-block of G. Then*

$$k_0(B_0) \geqslant 2\sqrt{p-1}\,.$$

We remark that Theorem 1.1 extends [NST18, Theorems A and C], which treats the cases $(k_0(B_0), p) = (3,3)$ and $(k_0(B_0), p) = (4,2)$, while Theorem 1.2 improves [MM16], which proves a similar bound with $k_0(B_0)$ replaced by the number of all irreducible characters of degree not divisible by p of the group instead. Further, Theorems 1.1(B), 1.1(C), and 1.2 provide height-zero versions of the main results of [KS21], [RSV21], and [HSF21], respectively.

It can be shown that Theorems 1.1 and 1.2 above are implied by the statement of the Alperin-McKay conjecture and known results on the conjugacy class number of finite groups. In this sense, they provide further evidence for the validity of this conjecture. Moreover, Theorems 1.1 and 1.2 can be used to advance on the determination of the structure of the Sylow p-subgroups of finite groups with exactly six irreducible ordinary characters in its principal p-block. (This will be discussed in forthcoming work.)

Brauer's Problem 21 [Bra63] predicts that, for every positive integer k, there are finitely many isomorphism classes of groups which can occur as defect groups of blocks with k ordinary irreducible characters. This was shown by Külshammer and Robinson [KR96] to be a consequence of the Alperin-McKay conjecture and Zelmanov's solution of the restricted Burnside problem. In view of Theorems 1.1 and 1.2, it is reasonable to expect that the height-zero analogue of Brauer's prediction is true, at least for principal blocks. The following is another consequence of the Alperin-McKay conjecture, see Lemma 6.1, which we find rather interesting.

Conjecture 1.3. *For every positive integer k_0, there are finitely many isomorphism classes of (abelian) groups (of prime power order) which can occur as abelianizations of defect groups of principal blocks (of finite groups) with precisely k_0 height-zero irreducible characters.*

Conjecture 1.3 is equivalent to the statement that the index $[P : P']$ is bounded from above in terms of the number $k_0 := k_0(B_0(G))$, where $P \in \mathrm{Syl}_p(G)$. By Theorem 1.2, this is reduced to showing that $\mathrm{rk}(P/P')$ and $\log_p(\exp(P/P'))$ are both bounded in terms of k_0, where $\mathrm{rk}(P/P')$ and $\exp(P/P')$ are respectively the rank and the exponent of the abelian group P/P'. The problem of bounding $\log_p(\exp(P/P'))$ turns out to be related to recent advances on the study of fields of character values and Galois actions on characters, in the context of the Alperin-McKay-Navarro conjecture [Nav04, Conjecture B]. We will exploit this relationship in Section 6. In particular, in Theorem 6.2 we prove that $\exp(P/P')$ is bounded in terms of k_0 when $p = 2$.

The structure of this paper is as follows. In Section 2, we collect some previous results on blocks and normal subgroups as well as some proven consequences of the Alperin-McKay conjecture. In Section 3, we obtain a lower bound for the number of irreducible height-zero characters in principal blocks of almost simple groups. The proof of Theorem 1.1 is contained in Section 4. In Section 5, and relying on all the previous sections, we present a proof of Theorem 1.2. We finish our work by discussing Conjecture 1.3 and proving Theorem 6.2 in Section 6.

2. Preliminaries

We start by collecting some results on the interplay between block theory and the normal structure of a group. We refer the reader to [Nav98, Chapter 9] for first definitions and basic properties. Recall that if N is a normal subgroup of G and B and b are blocks of G and N respectively, then B covers b if there are $\chi \in \mathrm{Irr}(B)$ and $\theta \in \mathrm{Irr}(b)$ such that θ is an irreducible constituent of the restriction χ_N. For $\theta \in \mathrm{Irr}(N)$, we write $\mathrm{Irr}(G|\theta)$, respectively $\mathrm{Irr}(B|\theta)$, for the set of those characters of G, respectively B, containing θ as a constituent when restricted to N.

For a finite group G and a prime p, we denote by $B_0(G)$ the principal p-block of G whenever p is clear from, or irrelevant in, the context. It is clear that $B_0(G)$ covers $B_0(N)$. Recall that $\chi \in \mathrm{Irr}(G)$ belongs to $B_0(G)$ if, and only if,

$$\sum_{x \in G^0} \chi(x) \neq 0\,,$$

where G^0 is the set of p-regular elements in G. In particular, $\mathrm{Aut}(G)$ and $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ act on $\mathrm{Irr}(B_0(G))$, and also on the subset $\mathrm{Irr}_{p'}(B_0(G))$ of height-zero characters in $B_0(G)$. Here $\mathbb{Q}^{ab}$ is the smallest extension of $\mathbb{Q}$ containing all roots of unity.

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$.*

(i) $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$.
(ii) *For every $\theta \in \mathrm{Irr}(B_0(N))$, there exists $\chi \in \mathrm{Irr}(B_0(G)|\theta)$.*
(iii) *Suppose that $B \in \mathrm{Bl}(G)$ is the only block covering $b \in \mathrm{Bl}(N)$. Then for every $\theta \in \mathrm{Irr}(b)$, we have $\mathrm{Irr}(G|\theta) \subseteq \mathrm{Irr}(B)$.*

Proof. Part (i) follows as $B_0(G)$ dominates $B_0(G/N)$. Part (ii) is [Nav98, Theorem 9.4]. Part (iii) is [RSV21, Lemma 1.2], for instance. ∎

Note that if $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(B_0(G))$ satisfies that $N \subseteq \mathrm{Ker}(\chi)$, then it is not true in general that $\chi \in \mathrm{Irr}(B_0(G/N))$.

Lemma 2.2. *Let $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$.*

(i) *If $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to PN, then there is some $\chi \in \mathrm{Irr}(B_0(G)|\theta)$ of degree not divisible by p.*

(ii) *If $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to some character in $B_0(G)$ and $B_0(G)$ is the only block of G covering $B_0(N)$, then*

$$|\mathrm{Irr}_{p'}(B_0(G)|\theta)| = |\mathrm{Irr}_{p'}(G/N)|,$$

where $\mathrm{Irr}_{p'}(B_0(G)|\theta) := \mathrm{Irr}(B_0(G)) \cap \mathrm{Irr}_{p'}(G|\theta)$.

Proof. Part (i) is due to Murai [Mur94, Lemma 4.3]. We now prove part (ii). Let $\hat\theta \in \mathrm{Irr}(B_0(G))$ be an extension of θ. By Gallagher's theorem [Isa06, Corollary 6.17],

$$\mathrm{Irr}_{p'}(G|\theta) = \{\beta\hat\theta \mid \beta \in \mathrm{Irr}_{p'}(G/N)\}.$$

By hypothesis and Lemma 2.1(iii), $\mathrm{Irr}_{p'}(G|\theta) \subseteq \mathrm{Irr}(B_0(G))$. Putting these facts together, we see that $|\mathrm{Irr}_{p'}(G|\theta) \cap \mathrm{Irr}(B_0(G))| = |\mathrm{Irr}_{p'}(G|\theta)| = |\mathrm{Irr}_{p'}(G/N)|$. □

Lemma 2.3. *Let $M \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$. If $P\mathbf{C}_G(P) \subseteq M$, then $B_0(G)$ is the only block covering $B_0(M)$. In particular, $k(G/M) < k_0(B_0(G))$ as long as $P > 1$.*

Proof. The first statement is [RSV21, Lemma 1.3]. Recall that $k_0(B_0(M)) > 1$ if $P > 1$ by [Nav98, Problem 3.11]. Then the last part follows from Lemma 2.2(i) since G/M has order coprime to p. □

We will also make use of Alperin-Dade's theory of isomorphic principal blocks.

Theorem 2.4. *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $P \in \mathrm{Syl}_p(G)$ and assume that $G = N\mathbf{C}_G(P)$. Then restriction of characters defines a natural bijection between the irreducible characters of the principals blocks of G and N. In particular, $k_0(B_0(G)) = k_0(B_0(N))$.*

Proof. The case where G/N is solvable was proved in [Alp76] and the general case in [Dad77]. □

We end this section with some proven consequences of the Alperin-McKay conjecture, which posits that $k_0(B) = k_0(b)$, where for B a block of G and b is the Brauer first main correspondent of B [Nav98, Theorems 4.12 and 4.17]. Note that if B has defect group D, then b is a block of $\mathbf{N}_G(D)$ with defect group D. By Brauer's third main theorem [Nav98, Theorem 6.7], the Brauer first main correspondent of $B_0(G)$ is $B_0(\mathbf{N}_G(P))$.

Theorem 2.5. *If G is p-solvable and $P \in \mathrm{Syl}_p(G)$, then*

$$k_0(B_0(G)) = k_0(B_0(\mathbf{N}_G(P))) = k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P').$$

Proof. The first equality is the principal bock case of results by Dade [Dad79] and Okuyama-Wajima [OW80]. The second equality follows from Fong's theorem [Nav98, Theorem 10.20] and Itô's argument [Isa06, Theorem 6.15]. □

Lemma 2.6. *If the principal p-block $B_0(G)$ of a finite group G satisfies the Alperin-McKay conjecture, then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$ with equality if, and only if, $\sqrt{p-1} \in \mathbb{N}$ and $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is isomorphic to the Frobenius group $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.*

In particular, if G is p-solvable or all the non-abelian composition factors of G have cyclic Sylow p-subgroups, then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$.

Proof. The first part follows from [HSF21, §2.1]. The Alperin-McKay conjecture is known to be true for p-solvable groups by Theorem 2.5. The so-called inductive Alperin-McKay conditions are satisfied for all blocks with cyclic defect groups by Koshitani and Späth [Spa13, KS16], and thus the Alperin-McKay conjecture also holds true for groups in which all the non-abelian composition factors have cyclic Sylow p-subgroups. (Indeed, note that a simple group is involved in G if and only if it is involved in some composition factor of G, and hence any simple group involved in G has cyclic Sylow p-subgroups.) $\square$

Theorem 2.7. *Let G be a finite group with an abelian Sylow p-subgroup. Let $B_0(G)$ denote the principal p-block of G. Then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$ with equality if and only if $\sqrt{p-1} \in \mathbb{N}$ and $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is isomorphic to the Frobenius group $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.*

Proof. Note that, when P is abelian, $k_0(B_0(G)) = k(B_0(G))$, by the work of Kessar and Malle [KM13, Theorem 1.1] on the 'if part' of Brauer's height zero conjecture. The statement then follows by [HSF21, Theorems 1.1 and 1.3]. $\square$

3. Bounding height-zero characters in (almost) simple groups

To prove Theorem 1.1 and 1.2, we need to bound from below the number of height-zero characters in (almost) simple groups. That is the purpose of this section. We begin with the case of alternating and symmetric groups.

Proposition 3.1. *Let $p \geqslant 3$ be a prime and n be a positive integer. Then*

(i) *If $n \geqslant p+2$ then $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| \geqslant p$ and $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| \geqslant 2p$.*
(ii) *If $n = p$ or $p+1$ then $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| = (p+3)/2$ and $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| = p$*
(iii) *If $n \geqslant p^2$ then $|\mathrm{Irr}_{p'}(B_0(\mathsf{S}_n))| \geqslant p^2$, and thus, there are at least $p^2/2$ orbits of characters in $\mathrm{Irr}_{p'}(B_0(\mathsf{A}_n))$ under the action of S_n.*

Proof. Basics on the representation theory of symmetric and alternating groups can be found in [JK81, Ols93]. Let $\mathcal{P}(n)$ denote the set of all partitions of n. Irreducible ordinary characters of S_n are naturally labeled by partitions in $\mathcal{P}(n)$, and so for each such partition λ, we let $\chi^\lambda \in \mathrm{Irr}(\mathsf{S}_n)$ denote the corresponding character. For $q \in \mathbb{Z}^+$, the q-core of λ is the partition obtained from λ by successive removals of rim q-hooks until no q-hook is left.

A well-known result of Macdonald (see [Ols76, §2]) asserts that, if $\lambda \in \mathcal{P}(n)$ and the p-adic expansion of n is

$$a_0 + a_1 p + \cdots a_t p^t,$$

then the character χ^λ has p'-degree if and only if λ has precisely a_t hooks of length divisible by p^t and the character labeled by the p^t-core of λ has p'-degree. Moreover,

$$|\mathrm{Irr}_{p'}(\mathsf{S}_n)| = k(1, a_0)k(p, a_1)...k(p^t, a_t),$$

where, for $m, a \in \mathbb{N}$, $k(m, a)$ is the number of m-tuples of partitions of a.

When $n \geqslant p+2$ we have $|\mathrm{Irr}_{p'}(\mathsf{S}_n)| \geqslant 2k(p,1) = 2p$, and therefore $|\mathrm{Irr}_{p'}(\mathsf{A}_n)| \geqslant p$, proving part (i). For part (ii) we note that the p'-degree irreducible characters of S_p are labeled by hook-shape partitions of the form $(x, 1^{p-x})$ with $0 \leqslant x \leqslant p$, and exactly one of them, namely the one with $x = (p+1)/2$, is self-conjugate; also, the p'-degree irreducible characters of S_{p+1} are labeled by $(p+1)$, (1^{p+1}), and $(x, 2, 1^{p-x-1})$ with $2 \leqslant x \leqslant p-1$, and again exactly one of them is self-conjugate.

For part (iii), the assumptions on p and n imply that $n \geqslant 9$, and so $\mathsf{S}_n = \mathrm{Aut}(\mathsf{A}_n)$. Let $n = mp + r$ for some integers $m \geqslant 1$ and $0 \leqslant r < p$. Then [MO83, Theorem 1.10] implies that the number of height-zero characters in the principal block of S_n is the same as $k_0(B_0(\mathsf{S}_{mp}))$. By [Ols84, P. 44], this number is $\prod_{i\geqslant 0} k(p^{i+1}, b_i)$, where $m = \sum b_i p^i$ is the p-adic decomposition of m. Since $n \geqslant p^2$, we have $m \geqslant p$, and it follows that this number $\prod_{i\geqslant 0} k(p^{i+1}, b_i)$ is at least p^2, as desired. □

We next prove the key statement for (almost) simple groups needed for our main results.

Proposition 3.2. *Let S be a non-abelian simple group and $p \geqslant 5$ a prime dividing $|S|$. Assume that $P \in \mathrm{Syl}_p(S)$ is non-abelian. Then there are at least 6 characters in $\mathrm{Irr}_{p'}(B_0(S))$. Further, there are more than $2\sqrt{p-1}$ different $\mathrm{Aut}(S)$-orbits in $\mathrm{Irr}_{p'}(B_0(S))$.*

Proof. (I) First we note that the conclusion follows from Proposition 3.1(iii) for the alternating groups, since P is abelian for $n < p^2$ and $p^2/2 > \max\{6, 2\sqrt{p-1}\}$ for $p \geqslant 5$. For sporadic groups and the Tits group, the assumptions on p and P imply that either $p \in \{5, 7\}$ or $(S, p) = (J_4, 11)$ or $(M, 13)$. The GAP character table library [GAP] contains the character table and block distributions for S for the prime p in these cases. From this information, we can see that the statement holds.

(II) We now assume that S is a simple group of Lie type defined over $\mathbb{F}_q$, where q is a power of some prime q_0. First assume that $q_0 = p$. Let $\mathbf{G}$ be a simple algebraic group of adjoint type and F a Steinberg endomorphism on $\mathbf{G}$ such that $S \cong [G, G]$ where $G := \mathbf{G}^F$. By [Bru09, Lemma 5], the p'-degree irreducible characters of G are the same as semisimple characters, one for each conjugacy class of semisimple elements of ${\mathbf{G}^*}^{F^*}$, where $(\mathbf{G}^*, F^*)$ is the dual pair of $(\mathbf{G}, F)$. As the number of semisimple classes of ${\mathbf{G}^*}^{F^*}$ is at least q^r, where r is the rank of $\mathbf{G}$, by [Car85, Theorem 3.7.6], it follows that $|\mathrm{Irr}_{p'}(G)| \geqslant q^r$. Therefore, $|\mathrm{Irr}_{p'}(S)| \geqslant q^r/d$ where $d := |G/S|$ is the order of the group of diagonal automorphisms of S. By a result of Dagger and Humphreys (see [Cab18, Theorem 3.3]), S has precisely two p-blocks: the principal block and the defect-zero block containing only the Steinberg character (of degree $|S|_p$). Therefore, we have $k_0(B_0(S)) \geqslant q^r/d$. It is now easy to check that $q^r/d > 2\sqrt{p-1}|\mathrm{Out}(S)|$ for all S of Lie type in characteristic $p \geqslant 11$, using available information of $\mathrm{Out}(S)$, in [Atl85, p. xvi] for instance. Hence we are done unless $p \in \{5, 7\}$.

Now suppose $p \in \{5, 7\}$. In this case, we have $q^r/d \geqslant 6$ except if $S = \mathrm{PSL}_2(p)$, and we have $q^r/d \geqslant 5|\mathrm{Out}(S)|$ unless $S = \mathrm{PSL}_2(p)$; $\mathrm{PSL}_2(p^2)$; $\mathrm{PSL}_3^{\pm}(p)$; or $\mathrm{PSL}_4^{\pm}(5)$. However, if $S = \mathrm{PSL}_2(q)$, then $P \in \mathrm{Syl}_p(S)$ is abelian, and we are done in that case. So, assume $S = \mathrm{PSL}_n^{\pm}(p)$ with $(n, p) \in \{(3, 5), (3, 7), (4, 5)\}$. In these cases, we can see from the character table available in GAP that there are at least 5 distinct character values in $\mathrm{Irr}_{p'}(S)$, so that there are at least 5 $\mathrm{Aut}(S)$-orbits in $\mathrm{Irr}_{p'}(B_0(S))$, and we are again done.

(III) So, we may now assume that $p \nmid q$. Let $d := e_p(q)$ be the multiplicative order of q modulo p. If S is of exceptional type (including Suzuki and Ree groups and ${}^3\mathrm{D}_4(q)$), then the fact that P is non-abelian implies that $S = {}^2\mathrm{F}_4(q)$ or d is a regular number. If $S = {}^2\mathrm{F}_4(q)$, we see explicitly from [Mal90, Bemerkung 1] that the statement holds.

So, we assume that S is not of Suzuki or Ree type and that d is a regular number. In [RSV21, Lemma 3.7], it is shown in this case that there are at least 6 distinct $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}(B_0(S))$. In the proof of *loc. cit.*, it is in fact shown that there are at least 6 distinct p'-degree characters in $\mathrm{Irr}(B_0(S))$ lying in at least 5 $\mathrm{Aut}(S)$-orbits. (In fact, in most cases, there are at least 6 distinct such orbits.) Hence we are done in this case, since the assumption P is non-abelian also implies that $p < 11$.

(IV) We therefore assume for the remainder of the proof that S is of classical type. That is, S is of type A_n, ${}^2\mathrm{A}_n$, B_n, C_n, D_n, or ${}^2\mathrm{D}_n$. we may write $S = [G, G]$, where $G = \mathrm{PGL}_n(q)$, $\mathrm{PGU}_n(q)$, $\mathrm{SO}_{2n+1}(q)$, $\mathrm{PCSp}_{2n}(q)$, or $\mathrm{P}(\mathrm{CO}^{\pm}_{2n}(q))^0$, respectively.

Define e to be the smallest positive integer such that $p \mid (q^e - 1)$ when G is of type A, $p \mid (q^e - (-1)^e)$ when G is of type ${}^2\mathrm{A}$, or $p \mid (q^e \pm 1)$ when G is of type $\mathrm{B}, \mathrm{C}, \mathrm{D}$ or ${}^2\mathrm{D}$. Let $n = we + m$ where $0 \leqslant m < e$. The fact that P is non-abelian implies that $p \leqslant w$.

Let $\mathcal{W}$ denote the relative Weyl group of a Sylow d-torus of G. When $G = \mathrm{PGL}_n(q)$ or $\mathrm{PGU}_n(q)$, the group $\mathcal{W}$ is the wreath product $\mathsf{C}_e \wr \mathsf{S}_w$ and otherwise, it is a subgroup of index 1 or 2 of $\mathsf{C}_{2e} \wr \mathsf{S}_w$, see [BMM93, §3A]. In all cases, $\mathcal{W}$ has a factor group isomorphic to S_w. Note that p is good for $\mathbf{G}$. By generalized d-Harish-Chandra theory [BMM93, Theorems 3.2 and 5.24], there is a natural bijection between unipotent characters in the principal p-block of G and the irreducible characters of $\mathcal{W}$. Furthermore, by [Mal07, Corollary 6.6], the number of unipotent characters in $\mathrm{Irr}_{p'}(B_0(G))$ is at least the number of p'-degree irreducible characters of $\mathcal{W}$. Note that each unipotent character in $\mathrm{Irr}(B_0(G))$ restricts irreducibly to one in $\mathrm{Irr}(B_0(S))$.

Recall that $w \geqslant p \geqslant 5$, and thus $n \geqslant 5$. Assume for a moment that G is not $\mathrm{P}(\mathrm{CO}^{+}_{2n}(q))^0$ with n even. Then, by a result of Lusztig [Mal08, Theorem 2.5], every unipotent character of S is invariant under $\mathrm{Aut}(S)$. Therefore, the number of $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$ is at least the number of p'-degree irreducible characters of $\mathcal{W}$, which in turn is at least $|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$. Since $w \geqslant p$, and $p > 2\sqrt{p-1}$ for all $p \geqslant 5$, we are done by using Proposition 3.1(i) and (ii), except possibly if $w \in \{5, 6\}$ and $p = 5$.

(V) Now suppose $w \in \{5, 6\}$ and $p = 5$, and continue to assume G is not $\mathrm{P}(\mathrm{CO}^{+}_{2n}(q))^0$ with n even. Then part (IV) implies we have at least 5 $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$ by considering unipotent characters. We claim that $\mathrm{Irr}_{p'}(B_0(S))$ must contain at least 6 characters.

In the cases of type A, ${}^2\mathrm{A}$, and B, we may naturally view G as a central quotient of $H := \mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, and $\mathrm{SO}_{2n+1}(q)$. In the case of type C, S is a central quotient of $H := \mathrm{Sp}_{2n}(q)$. Therefore, in these cases by [Nav98, Theorem 9.9], $\mathrm{Irr}_{p'}(B_0(G))$ (respectively $\mathrm{Irr}_{p'}(B_0(S))$) can be identified with the members of $\mathrm{Irr}_{p'}(B_0(H))$ that are trivial on $\mathbf{Z}(H)$. Further, the two sets can be identified except in the case $5 \mid |\mathbf{Z}(H)|$ (i.e., when $5 \mid (q-1)$ and $H = \mathrm{GL}_n(q)$ or $5 \mid (q+1)$ and $H = \mathrm{GU}_n(q)$). In the case of $\mathrm{D}_n(q)$, and ${}^2\mathrm{D}_n(q)$, G is a central quotient of $\mathrm{SO}^{\epsilon}_{2n}(q)$, and hence a subquotient of $H := \mathrm{GO}^{\epsilon}_{2n}(q)$, and $\mathrm{Irr}_{p'}(B_0(G))$ may be identified with $\mathrm{Irr}_{p'}(B_0(\mathrm{SO}^{\epsilon}_{2n}(q)))$.

Now, by [MO83, Theorem (1.9)] and [Mal20, Theorem 5.17], there is a bijection between $\mathrm{Irr}_{p'}(B_0(H))$ and $\mathrm{Irr}_{p'}(B_0(H_{we}))$, where $H_{we} = \mathrm{GL}_{we}(q), \mathrm{GU}_{we}(q), \mathrm{SO}_{2we+1}(q)$, $\mathrm{Sp}_{2we}(q)$, or $\mathrm{GO}^{\epsilon}_{2we}(q)$. We further see from the formulas for $|\mathrm{Irr}_{p'}(B_0(H_{we}))|$ in [Mal20, Theorem 5.17] and [MO83, Proposition (2.13)] that this number is at least 10 in the type A, ^{2}A cases and at least 20 in the other cases. Hence we are done in case B and C. Further, in the case $H = \mathrm{GO}^{\epsilon}_{2n}(q)$, this yields more than 10 characters in $\mathrm{Irr}_{p'}(B_0(G))$ by restricting from H, and 6 characters in $\mathrm{Irr}_{p'}(B_0(S))$ further restricting to S.

Now consider the case $H = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$. Write $H' := \mathrm{SL}_n(q)$, respectively $\mathrm{SU}_n(q)$, so that $S = H'/\mathbf{Z}(H')$. Characters of H are partitioned into so-called Lusztig series $\mathcal{E}(H, s)$, indexed by semisimple elements $s \in H^*$, where in this case the dual group H^* is isomorphic to H. In particular, $\mathrm{Irr}(B_0(H))$ lies in the union of $\mathcal{E}(H, s)$ where s has order a power of p, by [CE04, Theorem 9.12]. If $\chi \in \mathrm{Irr}(B_0(H))$ is nonunipotent but lies above a unipotent character, then χ is the tensor product of a unipotent character with a linear character of H. But linear characters of H are in natural bijection with characters of $\mathbf{Z}(H^*)$, and it follows that $\chi \in \mathcal{E}(H, z)$, where $z \in \mathbf{Z}(H^*)$ is nontrival with order a power of 5 (see, for example, [CE04, Proposition 8.26]). This is a contradiction, and we are done unless $5 \mid |\mathbf{Z}(H^*)|$. In the latter case, $e = 1$ and $w = n \in \{5, 6\}$. Here the principal block of H is the unique block containing unipotent characters. Then $\mathrm{Irr}(B_0(H))$ consists of all series $\mathcal{E}(H, s)$ where $s \in H^* \cong H$ has order a power of 5 by [CE04, Theorem 9.12]. First suppose that $n = 6$. Then there is a semisimple element s of $H \cong H^*$ that lies in H', has order a power of 5, and has $\mathbf{C}_{H^*}(s) \cong \mathrm{GL}_5(q) \times \mathsf{C}_{q-1}$, respectively $\mathrm{GU}_5(q) \times \mathsf{C}_{q+1}$. Then the members of $\mathcal{E}(H, s)$ are trivial on $\mathbf{Z}(H)$ (see, for example, [SFT21, Proposition 2.6]) and restrict to non-unipotent characters of H', and hence S. Since $\mathbf{C}_{H^*}(s)$ is of index prime to 5 in H^*, there is a so-called semisimple character in this series of degree $[H : \mathbf{C}_{H^*}(s)]_{q_0'}$, and hence height-zero, and we are done. Now, consider the case $n = 5$. Then $|\mathbf{Z}(H')| = 5$. In this case, every member of $\mathrm{Irr}_{5'}(B_0(H))$ restricts to one of the five unipotent characters in $\mathrm{Irr}_{5'}(B_0(H'))$. However, consider the element $s \in H'$ of order 5 whose eigenvalues are $\{\zeta, \zeta^2, \zeta^3, \zeta^4, 1\}$, where $\zeta \in \mathbb{F}_{q^2}^{\times}$ has order 5. We have $\mathbf{C}_{H^*}(s) \cong \mathsf{C}_{q-1}^5$, respectively C_{q+1}^5, so that $[H^* : \mathbf{C}_{H^*}(s)] = 5$. Let $\chi \in \mathcal{E}(H, s)$ be the semisimple element, so that $\chi(1)_5 = 5$. Since $s \in H'$, we have χ is trivial on the center. Further, sz is $H = H^*$-conjugate to s, where $z = \zeta \cdot I_5 \in \mathbf{Z}(H')$. It follows that the restriction of χ to H' is not irreducible, and hence splits into 5 non-unipotent characters in $\mathrm{Irr}_{5'}(B_0(H'))$. Then $|\mathrm{Irr}_{5'}(B_0(S))| \geqslant 6$, as claimed.

(VI) So lastly, suppose $G = \mathrm{P}(\mathrm{CO}^+_{2n}(q))^0$ with $n \geqslant 6$ even. (Recall that $n \geqslant p \geqslant 5$.) Then every unipotent character of S is still invariant under the field automorphisms. The graph automorphism of order 2 fixes all unipotent characters labeled by non-degenerate symbols, but interchanges the two unipotent characters in all pairs labeled by the same degenerate symbol of defect 0 and rank n (see [Mal08, Theorem 2.5] and also [Car85, p. 471] for the parametrization of unipotent characters of type D groups). In this case it is sufficient to show that $|\mathrm{Irr}_{p'}(\mathcal{W})| > \max\{12, 4\sqrt{p-1}\}$.

Recall that $\mathcal{W}$ is a subgroup of index 1 or 2 in $X := \mathsf{C}_{2e} \wr \mathsf{S}_w$. Fix $\theta \in \mathrm{Irr}(\mathsf{C}_{2e})$. The character $\psi := \theta \times \cdots \times \theta \in \mathrm{Irr}(B)$ of the base subgroup B of X is X-invariant and hence extendible to X, by [Mat95, Lemma 1.3]. It follows that the irreducible characters of X that lie over ψ are in bijective correspondence with irreducible characters of S_w by Gallagher's theorem (see [Isa06, Corollary 6.17]), and therefore the number of those characters of p'-degree is exactly equal to $|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$. According to [BMM93, p. 51], irreducible characters of X are labeled by $2e$-tuples of partitions $(a_i \mapsto w_i)$ with $\sum w_i = w$. When $\mathcal{W}$ is a subgroup of index 2 in X, those characters of X that split when restricted to $\mathcal{W}$ are described in *loc. cit.* In particular, the previously considered characters of X lying over ψ all restrict irreducibly to $\mathcal{W}$. Letting θ be arbitrary in $\mathrm{Irr}(\mathsf{C}_{2e})$, we deduce that the number of irreducible p'-degree characters of $\mathcal{W}$ is at least $2e|\mathrm{Irr}_{p'}(\mathsf{S}_w)|$, which in turn is at least $2p$ by Proposition 3.1. Note again that $p > 2\sqrt{p-1}$ for all $p \geqslant 5$. We see then that we are done unless $p = 5$, $w \in \{5, 6\}$, and $e = 1$.

In the latter case, we have shown that $\mathrm{Irr}_{p'}(B_0(S))$ contains at least 5 $\mathrm{Aut}(S)$-orbits, so it again suffices to show that $\mathrm{Irr}_{p'}(B_0(S))$ contains 6 elements. The exact same argument in (V) in the case $H = \mathrm{GO}^{\epsilon}_{2n}(q)$ applies here, and we are done. □

4. Principal blocks with at most 5 height-zero characters

The aim of this section is to prove Theorem 1.1. We begin by recording some divisibility results for small primes.

Lemma 4.1. *Let p be a prime and G a finite group. Let B be a p-block of positive defect of G.*

(i) *If $p = 2$ then $2 \mid k_0(B)$.*
(ii) *If $p = 3$ then $3 \mid k_0(B)$.*
(iii) *If $p = 2$ and the defect d of B is at least 2, then $4 \mid k_0(B)$. Furthermore, if B has no character of height 1, then $k_0(B) \equiv 2^d (\mathrm{mod}\, 8)$.*

Proof. This follows from [Lan81, Corollaries 1.3 and 1.6] (see also [NST18, Lemma 2.2] and [RSV21, Theorems 1.6 and 1.7]). □

Theorem 4.2. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) *$k_0(B_0(G)) = 2$,*
(ii) *$k(B_0(G)) = 2$,*
(iii) *P is cyclic of order 2.*

Proof. The fact that $k(B_0(G)) = 2$ is equivalent to $|P| = 2$ is well-known, see [Bra82]. Assume that $k_0(B_0(G)) = 2$. If $p = 2$ then $|P| = 2$ by Lemma 4.1(iii), as wanted, and $p = 3$ cannot happen by Lemma 4.1(ii). Now, if $p \geqslant 5$, [GRSS20, Theorem A] implies that G is p-solvable. Therefore, by Lemma 2.6, we have $k_0(B_0(G)) \geqslant 2\sqrt{p-1} \geqslant 4$, a contradiction. □

Notice that a group G satisfying the equivalent conditions in Theorem 4.2 is always solvable (by Feit-Thompson's odd-order theorem). While Theorem 4.2 on principal blocks with 2 height-zero characters easily follows from results already appearing in the literature, the following result on blocks with 3 height-zero characters is much more difficult to prove; in fact, the proof is already nontrivial when one considers just 3-blocks, see the remark before [NST18, Theorem C].

Theorem 4.3. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) $k_0(B_0(G)) = 3$,
(ii) $k(B_0(G)) = 3$,
(iii) *P is cyclic of order 3.*

Proof. The fact that $k(B_0(G)) = 3$ implies $|P| = 3$ follows from the main result of [Bel90] (we refer the reader to [KS21, Theorem 3.1] for an independent proof of this result). Moreover, if $|P| = 3$ then [Nav98, Theorem 11.1] implies that $k_0(B_0(G)) = k(B_0(G)) = 3$. Therefore, it remains to prove that (i) implies (iii). So assume that $k_0(B_0(G)) = 3$.

By Lemma 4.1(i), we may assume that $p \geqslant 3$, and as the statement we need to prove is precisely [NST18, Theorem C] when $p = 3$, we may assume furthermore that $p \geqslant 5$. Our aim is now to show that if $P > 1$ then $k_0(B_0) \geqslant 4$.

Notice that if G is p-solvable, then $k_0(B_0(\mathbf{N}_G(P))) = k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P')$ by Theorem 2.5. That number can be seen to be greater than or equal to 4 by looking at [VV85, Table 1]. We may thus assume that G is not p-solvable.

We consider a chief series $1 = G_0 < G_1 < \cdots < G_n = G$ of G with $G_j \trianglelefteq G$ for every $0 \leqslant j \leqslant n$. Let k be maximal such that p divides $[G_{k+1} : G_k]$. Since $k_0(B_0) \geqslant k_0(B_0(G/G_k))$, in order to show that $k_0(B_0) \geqslant 4$ we may assume that $G_{k+1} = 1$, and thus $N := G_{k+1}$ is a minimal normal subgroup of G of order divisible by p with $[G : N]$ not divisible by p. If N is abelian, then G is p-solvable. Hence N is semisimple with, say t, simple chief factors isomorphic to the simple non-abelian group S (of order divisible by p).

Write $M = N\mathbf{C}_G(P)$. Since $P \in \mathrm{Syl}_p(N)$, by the Frattini argument, $G = N\mathbf{N}_G(P)$ so that $M \trianglelefteq G$. By Lemma 2.3 we have that $k(G/M) < k_0(B_0)$. If $k(G/M) \geqslant 3$, then we are done. Hence we may assume that $[G : M] \leqslant 2$. Again by Lemma 2.3, for every $\eta \in \mathrm{Irr}_{p'}(B_0(M))$ we have that $\mathrm{Irr}(G|\eta) = \mathrm{Irr}_{p'}(G|\eta) \subseteq \mathrm{Irr}_{p'}(B_0)$. In particular, we would be done if $k_0(B_0(M)) \geqslant 4$, and thus we may assume $G = M$.

By Theorem 2.4 we have that $k_0(B_0) = k_0(B_0(N)) = k_0(B_0(S))^t$. If $t > 1$, then $k_0(B_0) \geqslant 4$ by [Nav98, Problem 3.11]. Then $t = 1$ and we may assume $G = S$ is a simple non-abelian group of order divisible by $p \geqslant 5$.

By Proposition 3.2, we may assume that P is abelian. Then $k_0(B_0) = k(B_0)$ by the main result of [KM13], and $k(B_0) \geqslant 2\sqrt{p-1} \geqslant 4$ by [HSF21, Theorem 1.1]. □

We remark that Theorems 4.2 and 4.3 prove Theorem 1.1(A).

In order to prove parts (B) and (C) of Theorem 1.1, we make use of the classification of Sylow p-subgroups of finite groups with precisely four or five ordinary irreducible characters in the principal p-block worked out in [KS21, RSV21]. We record this classification in the following two results.

Theorem 4.4. *Let G be a finite group and p a prime. Let B_0 denote the principal p-block of G. Then $k(B_0) = 4$ if, and only if, exactly one of the following happens:*

(i) $|P| = 4$,
(ii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

Proof. The 'if' implication is clear by Lemma 4.1 when $p = 2$ and [Nav98, Theorem 11.1] when $p = 5$. Assume that $k(B_0(G)) = 4$. By [KS21], then $P \in \{\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4, \mathsf{C}_5\}$. Moreover, if $|P| = 5$, then $k(B_0(G)) = 4$ forces $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$ by [Nav98, Theorem 11.1]. □

Theorem 4.5. *Let G be a finite group and p a prime. Let B_0 denote the principal p-block of G. Then $k(B_0) = 5$ if, and only if, precisely one of the following happens:*

(i) $P = \mathsf{D}_8$,
(ii) $P = \mathsf{Q}_8$ *and* $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$,
(iii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$,
(iv) $|P| = 7$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

Proof. The 'if' implication follows from results of Brauer [Nav98, Theorem 11.1] when $|P| = p$ and of Brauer [Bra66, Theorem 7B] and Olsson [Ols75, Theorem 3.13] when $P \in \{\mathsf{D}_8, \mathsf{Q}_8\}$. For the reverse implication, notice that, by the discussion above, it suffices to show that $P \in \{\mathsf{C}_5, \mathsf{C}_7, \mathsf{D}_8, \mathsf{Q}_8\}$. That is the main result of [RSV21]. □

Next we prove part (B) of Theorem 1.1. Recall that if $\chi \in \mathrm{Irr}(G)$, then $\det(\chi)$ is a linear character of G uniquely determined by χ (see [Isa06, Problem 2.3]). The determinantal order $o(\chi) = |G/\mathrm{Ker}(\det(\chi))|$ of χ is related to character extension properties.

Theorem 4.6. *Let p be a prime and G a finite group. Let P be a Sylow p-subgroup of G. Then $k_0(B_0(G)) = 4$ if, and only if, exactly one of the following happens:*

(i) $[P : P'] = 4$,
(ii) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

Proof. By [NST18] the statement holds if $p = 2$, so we may assume p is odd.

If $|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$ then $k_0(B_0) = 4$ by [Nav98, Theorem 11.1], and the 'if' implication holds.

Suppose that $k_0(B_0) = 4$. We want to prove the 'only if' implication. We may further assume that $p \geqslant 5$ by Lemma 4.1(ii). By [Nav98, Theorem 11.1] it is enough to show that if $k_0(B_0) = 4$ and $p \geqslant 5$, then $|P| = 5$. Let G be a counterexample of minimal order to such a statement.

Step 1. G is not p-solvable.

Write $K := \mathbf{O}_{p'}(\mathbf{N}_G(P))$. Assume, to the contrary, that G is p-solvable. Then by Theorem 2.5, we have that $k(\mathbf{N}_G(P)/KP') = 4$. Inspecting [VV85, Table 1], we see that $\mathbf{N}_G(P)/KP' \cong \mathsf{D}_{10}$. In particular, $[P : P'] = 5$, implying $|P| = 5$ and thus contradicting G being a counterexample.

Step 2. $\mathbf{O}_{p'}(G) = 1$.

Notice that $k_0(B_0(G/\mathbf{O}_{p'}(G))) = 4$ by [Nav98, Theorem 9.9(c)], so $\mathbf{O}_{p'}(G) = 1$ by the minimality of G as a counterexample.

Step 3. Let $1 \neq N$ be a minimal normal subgroup of G. Then p does not divide $[G : N]$.

Assume otherwise, so that $1 < k_0(B_0(G/N)) \leqslant 4$. The fact that $p \geqslant 5$ implies $k_0(B_0(G/N)) = 4$. By the minimality of G as a counterexample, $p = 5$ and $[PN : N] = 5$.

The fact that $k_0(B_0(G/N)) = k_0(B_0)$ in particular means that every $\chi \in \mathrm{Irr}_{p'}(B_0)$ lies over $\mathbf{1}_N$. By Lemma 2.2(i) we conclude that no $\mathbf{1}_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))$ extends to PN. By Step 2, the group N has order divisible by p and there are 2 cases.

Case (a). Suppose that N is an elementary abelian p-group, so $N \subseteq P$. Then P acts on N necessarily fixing some non-trivial element of N. Hence, there exists some $1_N \neq \theta \in \mathrm{Irr}_{p'}(B_0(N))$ that is P-invariant. By [Isa06, Theorem 11.22], θ extends to P, and we get a contradiction.

Case (b). Suppose that N is semisimple with t chief factors isomorphic to S. By [GRSS20, Proposition 2.1] there is some $\mathbf{1}_S \neq \theta \in \mathrm{Irr}_{p'}(B_0(S))$ invariant under the action of a Sylow p-subgroup of $\mathrm{Aut}(S)$. Let $\mathbf{1}_N \neq \psi$ be equal to the direct product of t copies of θ in N. Then $\psi \in \mathrm{Irr}_{p'}(B_0(N))$ is P-invariant and $o(\psi) = 1$ because N is perfect. By [Isa06, Corollary 8.16], ψ extends to PN, again yielding a contradiction.

Step 4. By Steps 1 and 3, we have that N is semisimple with t chief factors isomorphic to S, a simple non-abelian group of order divisible by p. Let $M = N\mathbf{C}_G(P)$. Then $M = G$.

By the Frattini argument, $G = N\mathbf{N}_G(P)$, and hence $M \trianglelefteq G$. Notice that the elements in $\mathrm{Irr}_{p'}(B_0(M))$ are the irreducible constituents of χ_M for every $\chi \in \mathrm{Irr}_{p'}(B_0)$.

Suppose that $M < G$. Then by Lemma 2.3 we have that $1 < k(G/M) < 4$. This leaves two possibilities.

First assume $k(G/M) = 2$, and so $[G : M] = 2$. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma\}$ where $M \subseteq \mathrm{Ker}(\alpha)$. If $\beta_M = \gamma_M$, then $k_0(B_0(M)) = 2$, which is absurd as $p \geqslant 5$. Otherwise $k_0(B_0(M)) = 5$. By Theorem 2.4, we have that $5 = k(B_0(S))^t$. This forces $t = 1$, $P \subseteq S$ and $k_0(B_0(S)) = 5$. By [GRSS20, Proposition 2.1] some $\mathbf{1}_S \neq \theta \in \mathrm{Irr}_{p'}(B_0(S))$ is $\mathrm{Aut}(S)$-invariant. By Theorem 2.4, let $\varphi \in \mathrm{Irr}_{p'}(B_0(M))$ be such that $\varphi_S = \theta$. For every $g \in G$, $\varphi^g \in \mathrm{Irr}_{p'}(B_0(M))$ extends $\theta^g = \theta$. By Theorem 2.4, φ is G-invariant. Consequently, φ has 2 extensions in $\mathrm{Irr}_{p'}(B_0)$, those must be β and γ by Lemma 2.1. Then $\beta_M = \gamma_M$, a contradiction.

Secondly assume that $k(G/M) = 3$. Then every nontrivial $\theta \in \mathrm{Irr}_{p'}(B_0(M))$ lies under the same member of $\mathrm{Irr}_{p'}(B_0)$. Hence $|\{\psi(1) \mid \psi \in \mathrm{Irr}_{p'}(B_0(M))\}| \leqslant 2$. By the main result of [GRSS20] we get that M is p-solvable, and hence so is G, contradicting Step 1.

Final step. We have $G = N\mathbf{C}_G(P)$, where N is semisimple with t chief factors isomorphic to S. By Theorem 2.4, $4 = k_0(B_0) = k_0(B_0(N)) = k_0(B_0(S))^t$. As $p \geqslant 5$, this forces $t = 1$, $P \subseteq S$, and $k_0(B_0(S)) = 4$. By Proposition 3.2, P is abelian. Then $k_0(B_0) = k(B_0) = 4$ by [KM13]. Then Theorem 4.4 implies that $|P| = 5$, the final contradiction. $\square$

Finally, we classify groups with 5 height-zero characters in the principal block, thus completing the proof of Theorem 1.1.

Theorem 4.7. *Let G be a finite group and p a prime. Let $P \in \mathrm{Syl}_p(G)$ and let B_0 denote the principal p-block of G. Then $k_0(B_0) = 5$ if, and only if, precisely one of the following happens:*

(i) $|P| = 5$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$.
(ii) $|P| = 7$ *and* $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

Proof. First we remark that the 'if part' follows by [Nav98, Theorem 11.1].

Assume that $k_0(B_0) = 5$. By Lemma 4.1, p cannot be 2 or 3, and hence $p \geqslant 5$. By [Nav98, Theorem 11.1], it suffices to show that if $k_0(B_0) = 5$ and $p \geqslant 5$, then $|P| \in \{5, 7\}$. Assume that G is a counterexample of minimal order to such a statement. By the main result of [KM13] and Theorem 4.5, we have that P is not abelian. Also we can see that G is not p-solvable and $\mathbf{O}_{p'}(G) = 1$, proceeding as in the proof of the case $k_0(B_0) = 4$. (Some arguments will be similar to ones used in the proof of Theorem 4.6 so here we will just sketch those.)

Let N be a minimal normal subgroup of G, with $N \neq 1$. We first show that p does not divide the index $[G : N]$.

Assume otherwise, so that $1 < k_0(B_0(G/N)) \leqslant 5$. As $p \geqslant 5$, then $4 \leqslant k_0(B_0(G/N)) \leqslant 5$. In the case where $k_0(B_0(G/N)) = 5$, we obtain a contradiction from Lemma 2.2(i) as we can always find some $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ that extends to PN (note that by minimality of G as a counterexample PN/N is cyclic and we can proceed as in the proof of the case $k_0(B_0) = 4$).

Hence $k_0(B_0(G/N)) = 4$. By Theorem 4.6, we have that $[PN : N] = 5$. Notice that in this case $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$, where χ is the only member of $\mathrm{Irr}_{p'}(B_0)$ not belonging to $\mathrm{Irr}_{p'}(B_0(G/N))$. We distinguish the cases where N is abelian and semisimple.

Case (a). Suppose that N is abelian, then N is an elementary abelian p-group and $N \leqslant P$. Since P is not abelian, and as P/N is cyclic of order 5, then $P \cap \mathbf{C}_G(N) = N$. Hence $N \in \mathrm{Syl}_p(\mathbf{C}_G(N))$. Since $\mathbf{O}_{p'}(G) = 1$, that implies $\mathbf{C}_G(N) = N$. Let $1_N \neq \theta \in \mathrm{Irr}(N)$ be P-invariant. Since P/N is cyclic, θ extends to P by [Isa06, Theorem 11.22]. Take $Q/N \in \mathrm{Syl}_q(G_\theta/N)$ with $q \neq p$. Then θ extends to Q by [Isa06, Corollary 8.16]. By [Isa06, Corollary 11.31] θ extends to G_θ. By the Fong-Reynolds correspondence [Nav98, Theorem 9.14],

$$|\mathrm{Irr}_{p'}(B_0|\theta)| = |\mathrm{Irr}_{p'}(B_0(G_\theta)|\theta)|\,.$$

Recall that $|\mathrm{Irr}_{p'}(B_0|\theta)| = |\{\chi\}| = 1$ under our assumptions, as χ is the only member of $\mathrm{Irr}_{p'}(B_0)$ possibly lying over a nontrivial character of N, and by Lemma 2.2(i) some $\mathrm{Irr}_{p'}(B_0)$ lies over θ. Let $b_0 = B_0(N)$. By [Nav98, Corollary 9.21], we have that $b_0^{G_\theta} = B_0(G_\theta)$ is the only block of G_θ covering b_0. Let $\eta \in \mathrm{Irr}(G_\theta)$ be an extension of θ. In particular, η lies in $B_0(G_\theta)$. By Lemma 2.2(ii)

$$|\mathrm{Irr}_{p'}(B_0(G_\theta)|\theta)| = |\mathrm{Irr}_{p'}(G_\theta/\mathbf{C}_G(N)))| \geqslant 2\,,$$

a contradiction.

Case (b). Suppose that N is semisimple with t chief factors isomorphic to S. By [GRSS20, Proposition 2.1] there are $\mathbf{1}_S \neq \alpha, \beta \in \mathrm{Irr}_{p'}(B_0(S))$ invariant under the action of a Sylow p-subgroup of $\mathrm{Aut}(S)$ with $\alpha(1) \neq \beta(1)$. Let $\mathbf{1}_N \neq \psi$ be equal to the direct product of t copies of α in N and $\mathbf{1}_N \neq \varphi$ be equal to the direct product of t copies of β in N. Then $\psi, \varphi \in \mathrm{Irr}_{p'}(B_0(N))$ are P-invariant. Alos $o(\psi) = 1 = o(\varphi)$ because N is perfect. By [Isa06, Corollary 8.16] both ψ and φ extend to PN, yielding a contradiction by Lemma 2.2(i).

We have shown that p does not divide $[G : N]$. In particular, N is semisimple with, say t, chief factors isomorphic to the non-abelian simple group S (of order divisible by p). Take $M = N\mathbf{C}_G(P) \trianglelefteq G$. Then $1 \leqslant k(G/M) < 5$ by Lemma 2.3. We show that $G = M$ by analyzing the different values $1 < k(G/M) < 5$. Before proceeding with the analysis, we make the following observation. By [GRSS20, Proposition 2.1] some $\mathbf{1}_S \neq \varphi \in \mathrm{Irr}_{p'}(B_0(S))$ is $\mathrm{Aut}(S)$-invariant. In particular, if θ is the direct product of t copies of φ, then $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ is G-invariant. By Theorem 2.4, let $\psi \in \mathrm{Irr}_{p'}(B_0(M))$ be such that $\psi_S = \theta$. Then $\mathbf{1}_M \neq \psi$ is a G-invariant member of $\mathrm{Irr}_{p'}(B_0(M))$.

If $k(G/M) = 2$, then $[G : M] = 2$. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$ where $M \subseteq \mathrm{Ker}(\alpha)$. Since ψ extends to G, we may assume that β and γ are the two extensions of ψ. In particular, χ_M must decompose as the sum of two distinct members of $\mathrm{Irr}_{p'}(B_0(M))$. In particular, $|\mathrm{Irr}_{p'}(B_0(M))| = k_0(B_0(M)) = 4$ and by Theorem 4.6 we obtain $|P| = 5$, a contradiction.

If $k(G/M) = 3$, then G/M is isomorphic to C_3 or S_3. Write $\mathrm{Irr}_{p'}(B_0) = \{\mathbf{1}_G, \alpha, \beta, \gamma, \chi\}$, where α and β contain M in their respective kernels. Recall that $\mathbf{1}_M \neq \psi \in \mathrm{Irr}_{p'}(B_0(M))$ is G-invariant. Notice that $|\mathrm{Irr}_{p'}(B_0|\psi)| = |\mathrm{Irr}(G|\psi)| \geqslant 3$, which is impossible.

If $k(G/M) = 4$, then every nontrivial $\eta \in \mathrm{Irr}_{p'}(B_0(M))$ lies under the same member of $\mathrm{Irr}_{p'}(B_0)$. Hence $|\{\eta(1) \mid \eta \in \mathrm{Irr}_{p'}(B_0(M))\}| \leqslant 2$. By the main result of [GRSS20] we conclude that M is p-solvable, then so is G, a contradiction.

Finally, if $G = M$, then by Theorem 2.4 we have that $k_0(B_0(S))^t = 5$. Hence $t = 1$ and $k_0(B_0) = 5$. By Proposition 3.2, P must be abelian, a contradiction. □

5. Bounding height-zero characters in principal blocks

In this section we prove Theorem 1.2. We begin with a technical result due to G. Navarro.

Lemma 5.1 (Navarro)**.** *Let $S_1 \times \cdots \times S_t = N \trianglelefteq G$, where $\{S_1, ..., S_t\}$ are transitively permuted by conjugation of G: $S_i = S_1^{x_i}$ for some $x_i \in G$. Let $\theta := \theta_1 \in \mathrm{Irr}(S_1)$ such that $\mathbf{Z}(S_1) \subseteq \mathrm{Ker}(\theta)$ and that there exists $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)))$ with $\alpha_{S_1} = e\theta$ for some $e \in \mathbb{N}$. Set $\psi := \theta_1 \times \cdots \times \theta_t$ where $\theta_i := \theta_1^{x_i}$. Then there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ such that $\chi_N = a\psi$ for some $e^t \geqslant a \in \mathbb{N}$.*

Proof. This is the content of [Mar21, Lemma 4.4]. □

Lemma 5.1 is useful when one wants to produce characters in $\mathrm{Irr}_{p'}(B_0(G))$ that lie above certain characters of a non-abelian minimal normal subgroup of G. In such a situation, the existence of θ and α satisfying the hypothesis of Lemma 5.1 is presented in the following, which is [GRSS20, Proposition 2.1].

Lemma 5.2. *Let S be a non-abelian simple group of order divisible by a prime $p \geqslant 5$. Then there exist $1_S \neq \theta \in \mathrm{Irr}_{p'}(S)$ and $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathrm{Aut}(S)))$ such that $\alpha_S \in \{\theta, 2\theta\}$. Further, when S is not $\mathrm{P}\Omega_8^+(q)$, one may choose α so that it extends θ.*

We can now prove Theorem 1.2 in the case of non-abelian Sylow.

Theorem 5.3. *Let G be a finite group and p a prime. Assume that the Sylow p-subgroups of G are non-abelian. Then $k_0(B_0(G)) > 2\sqrt{p-1}$.*

Proof. First, if $p \leqslant 7$ then it is sufficient to assume that $k_0(B_0(G)) \leqslant 4$. However, by Theorem 1.1, in such case, P is abelian or $k_0(B_0(B)) = 4$ and $p = 2$, and thus we are done by Theorem 2.7. Therefore, we may and will assume from now on that $p \geqslant 11$.

We adapt some arguments in the proof of [HSF21, Theorem 1.1]. Let G be a counterexample with minimal order. In particular, $\mathbf{O}_{p'}(G)$ is trivial, $P \in \mathrm{Syl}_p(G)$ is non-abelian, and $k_0(B_0(G)) \leqslant 2\sqrt{p-1}$. Let $1 \neq N$ be a minimal normal subgroup of G. We claim that p does not divide $[G:N]$.

Assume, to the contrary, that $p \mid [G:N]$. Then $PN/N \in \mathrm{Syl}_p(G/N)$ must be abelian, by the fact $k_0(B_0(G)) \geqslant k_0(B_0(G/N))$ and the minimality of G. It then follows from Theorem 2.7 that $k_0(B_0(G/N)) \geqslant 2\sqrt{p-1}$. Altogether, we deduce that

$$k_0(B_0(G)) = k_0(B_0(G/N)) = 2\sqrt{p-1}.$$

Assume that N is abelian, which means that N is actually an elementary abelian p-group, because $\mathbf{O}_{p'}(G) = 1$. Let $\mathbf{1}_N \neq \theta \in \mathrm{Irr}(N)$ be P-invariant. Theorem 2.7 implies that $P/N \in \mathrm{Syl}_p(G/N)$ is of order p, and it follows that θ extends to P. By Lemma 2.2(i), we deduce that there exists some $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ that lies over θ. We now have $k_0(B_0(G)) > k_0(B_0(G/N))$, violating the conclusion of the previous paragraph.

We may assume that N is non-abelian. Suppose that S is a simple direct factor of N, and notice that p divides the order of S, because $\mathbf{O}_{p'}(G) = 1$. By Lemma 5.2, there exist $\theta \in \mathrm{Irr}_{p'}(S)$ and $\alpha \in \mathrm{Irr}_{p'}(B_0(\mathrm{Aut}(S)))$ such that $\alpha_S \in \{\theta, 2\theta\}$. Lemma 5.1 then implies that there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ such that $N \nsubseteq \mathrm{Ker}(\chi)$, again violating the equality $k_0(B_0(G)) = k_0(B_0(G/N))$. The claim $p \nmid [G:N]$ is now fully proved.

Recall that $p \mid |N|$. By Lemma 2.6, we are done if N is abelian, so let us assume that N is not, and furthermore, as above let S be a (non-abelian) simple factor of N. By Proposition 3.2, there are more than $2\sqrt{p-1}$ different $\mathbf{N}_G(S)$-orbits on $\mathrm{Irr}_{p'}(B_0(S))$. If two characters $\eta, \theta \in \mathrm{Irr}_{p'}(B_0(S))$ are not conjugate under the action of $\mathbf{N}_G(S)$ then the characters $\eta \times \cdots \times \eta$ and $\theta \times \cdots \times \theta$ of N are not conjugate under the action of G. We deduce that there are more than $2\sqrt{p-1}$ different G-orbits on $\mathrm{Irr}_{p'}(B_0(N))$. It immediately follows that $k_0(B_0(G)) > 2\sqrt{p-1}$ since there is a character in $\mathrm{Irr}_{p'}(B_0(G))$ lying over characters in each such G-orbit, by Lemma 2.2(i). □

The following result covers Theorem 1.2 in the introduction. The equivalence of (i) and (iv) was already shown in [HSF21, Theorem 1.3].

Theorem 5.4. *Let G be a finite group and p a prime such that $p \mid |G|$. Then $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$. Moreover, for $P \in \mathrm{Syl}_p(G)$, the following are equivalent:*

(i) $k(B_0(G)) = 2\sqrt{p-1}$.
(ii) $k_0(B_0(G)) = 2\sqrt{p-1}$.
(iii) $k_0(B_0(\mathbf{N}_G(P))) = 2\sqrt{p-1}$.
(iv) $\sqrt{p-1} \in \mathbb{N}$ *and* $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ *is isomorphic to the Frobenius group* $\mathsf{C}_p \rtimes \mathsf{C}_{\sqrt{p-1}}$.

Proof. The first statement follows from Theorem 2.7 (which is a consequence of [HSF21, Theorem 1.1] and [KM13, Theorem 1.1]) and Theorem 5.3. In fact, these results also imply the equivalence of (i) and (ii). The fact that (i) is equivalent to (iv) is precisely [HSF21, Theorem 1.3], and the equivalence of (iii) and (iv) follows by Lemma 2.6. □

We remark that the second statement of Theorem 5.4 is consistent with both Brauer's height zero conjecture and the Alperin-McKay conjecture for principal blocks. We have learned that the unproven half of Brauer's height zero conjecture for principal blocks has been confirmed very recently by Malle and Navarro [MN21]. However, note that our proofs are independent of this result.

6. On Conjecture 1.3

We end the paper with some discussion on Conjecture 1.3. It asserts that, if one fixes the number of height-zero characters in the principal p-block of a finite group, then $[P : P']$ is bounded, where P is a Sylow p-subgroup of the group. The conjecture therefore may be viewed as the analogue of Brauer's Problem 21 [Bra63] and famous Landau's theorem [L1903] for height-zero characters in principal blocks.

Lemma 6.1. *Conjecture 1.3 follows from the Alperin-McKay conjecture.*

Proof. Fix a positive integer k_0 and let G be a finite group with precisely k_0 height-zero characters in the principal p-block of G. Assume that the Alperin-McKay conjecture holds for principal blocks. As explained in Theorem 2.5, we then have

$$k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P') = k_0.$$

By Landau's theorem (see [L1903]), it follows that the order of the quotient group $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))P'$ must be bounded, and thus $[P : P']$ is bounded as well. $\square$

Recall that $p \leqslant k_0^2/4 + 1$ by Theorem 1.2, where $k_0 := k_0(B_0(G))$. Moreover,

$$[P : P'] \leqslant p^{\log_p(\exp(P/P'))\cdot \mathrm{rk}(P/P')}.$$

Conjecture 1.3 is therefore reduced to showing that $\log_p(\exp(P/P'))$ and $\mathrm{rk}(P/P')$ are both bounded in terms of k_0. We recall that $\mathrm{rk}(P/P') = \log_p([P : \Phi(P)])$, where $\Phi(P)$ is the Frattini subgroup of P. The problem of bounding $\mathrm{rk}(P/P')$ in terms of k_0 seems highly nontrivial to us at the moment. On the other hand, the problem of determining $\log_p(\exp(P/P'))$ appears to be related to the Alperin-McKay-Navarro conjecture. We take advantage of recent advances [NT19, NT21] on the study of fields of values of characters of degree not divisible by p to prove that $\exp(P/P')$ is bounded in terms of k_0 when $p = 2$ in Theorem 6.2 below.

We first need to introduce some notation. The field of values of $\chi \in \mathrm{Irr}(G)$ is $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) \mid g \in G)$. Notice that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{\exp(G)}$, where for an integer m, we write $\mathbb{Q}_m := \mathbb{Q}(e^{2\pi i/m})$. We define $c(\chi)$ as the smallest positive integer c such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_c$. The number $c(\chi)$ has been referred to as the *Feit number* of χ in connection with a conjecture by W. Feit [Nav18, §3.3] and as the *conductor* of χ [NT21]. We recall that χ is p-rational if p does not divide $c(\chi)$. Moreover, in [HMM21, §2], $c_p(\chi)$ the *p-rationality level* of χ is defined as $\log_p(c(\chi)_p)$, where n_p is the p-part of the integer n. The p-rationality level of χ measures how p-rational χ is. Indeed, χ is p-rational if, and only if, $c_p(\chi) = 0$.

The Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ acts on the set of irreducible characters of any finite group G preserving character degrees. It also acts on the set of height-zero characters of principal blocks of finite groups as discussed in Section 2. For a positive integer e, let σ_e denote the automorphism of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ that fixes roots of unity of order not divisible by p and sends p-power roots of unity ξ to ξ^{1+p^e}. By [NT19, Theorem B], we know that if e is any positive integer such that all of the height-zero characters in the principal p-block of G are fixed by σ_e, then $\log_p(\exp(P/P'))$ is at most e.

Theorem 6.2. *Let $p = 2$ and $P \in \mathrm{Syl}_p(G)$. Then $\exp(P/P')$ is bounded in terms of $k_0 := k_0(B_0(G))$. In fact,*

$$\exp(P/P') \leqslant 2(k_0 - 1)$$

whenever P is nontrivial.

Proof. Let B_0 denote the principal p-block of G and set

$$e(G) := \max_{\chi \in \mathrm{Irr}_{p'}(B_0)} \{\log_p(c(\chi)_p)\}.$$

So this $e(G)$ is the largest p-rationality level of a character in $\mathrm{Irr}_{p'}(B_0)$. First suppose that $e(G) = 0$. Then all the characters in $\mathrm{Irr}_{p'}(B_0)$ are p-rational and therefore σ_1-invariant. [NT19, Theorem B] then implies that $\exp(P/P') \leqslant p = 2$, and the theorem follows since $k_0 \geqslant 2$ when $P > 1$ by Theorem 1.2.

So let $e(G) \geqslant 1$. Then all the characters in $\mathrm{Irr}_{p'}(B_0)$ are $\sigma_{e(G)}$-invariant, and therefore by [NT19, Theorem B] we have

$$\log_p(\exp(P/P')) \leqslant e(G).$$

Let $\psi \in \mathrm{Irr}_{p'}(B_0)$ be such that $c(\psi)_p = p^{e(G)}$; that is, choose $\psi \in \mathrm{Irr}_{p'}(B_0)$ with maximal p-rationality level. By [NT21, Theorem A1], we have $\mathbb{Q}_{p^{e(G)}} \subseteq \mathbb{Q}(\psi)$ and it follows that

$$[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant [\mathbb{Q}_{p^{e(G)}} : \mathbb{Q}] = (p-1)p^{e(G)-1} = p^{e(G)-1}.$$

On the other hand, any Galois conjugate of ψ belongs in $\mathrm{Irr}_{p'}(B_0(G))$. As the number of those conjugates is exactly $[\mathbb{Q}(\psi) : \mathbb{Q}]$ and ψ is nontrivial, we deduce that

$$k_0 - 1 \geqslant [\mathbb{Q}(\psi) : \mathbb{Q}].$$

The last three displayed inequalities imply that

$$\exp(P/P') \leqslant 2(k_0 - 1),$$

and this concludes the proof. ∎

The proof of Theorem 6.2 in fact shows that $\exp(P/P')/2 + 1$ is bounded above by the number of characters in $\mathrm{Irr}_{p'}(B_0)$ with maximal p-rationality level.

One might naturally ask what happens when p is odd. The p-odd analogue of [NT21, Theorem A1] is not true in general. Navarro and Tiep proposed in [NT21, Conjecture B3 and Theorem B1] that, if $\chi \in \mathrm{Irr}_{p'}(G)$ with $c(\chi)_p = p^a$, then $[\mathbb{Q}_{p^a} : (\mathbb{Q}(\chi) \cap \mathbb{Q}_{p^a})]$ is not divisible by p. If that turns out to be true, one may follow the same arguments as in the proof of Theorem 6.2 to show that

$$[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant p^{e(G)-1},$$

whenever ψ is a character in $\mathrm{Irr}_{p'}(B_0(G))$ with maximal p-rationality level. It would follow then that $e(G)$, and hence $\exp(P/P')$, is bounded in terms of the number k_0 of height-zero irreducible characters in $B_0(G)$. Note that the bound $[\mathbb{Q}(\psi) : \mathbb{Q}] \geqslant p^{e(G)-1}$ does not directly imply that p is bounded in terms of k_0 since $e(G)$ could be 1. Therefore we do need Theorem 1.2 for this argument to work.

References

[Alp75] J. L. Alperin, *The main problem of block theory*, Proc. Conf. Finite Groups, Academic Press, New York (1975), 341–356.

[Alp76] J. L. Alperin, Isomorphic blocks, *J. Algebra* **43** (1976), 694–698.

[Bel90] V. A. Belonogov, Finite groups with a small principal p-block, In: Group-theoretic investigations (in Russian), *Akad. Nauk SSSR Ural. Otdel.*, Sverdlovsk (1990), 8–30.

[Bra82] J. Brandt, A lower bound for the number of irreducible characters in a block, *J. Algebra* **2** (1982), 509–515.

[Bra63] R. Brauer, *Representations of finite groups*, Lectures on Modern Mathematics, Vol. I (1963), 133–175.

[Bra66] R. Brauer, Some applications of the theory of blocks of characters of finite groups III, *J. Algebra* **3** (1966), 225–255.

[BMM93] M. Broué, G. Malle, and J. Michel, Generic blocks of finite reductive groups, *Astérisque* **212** (1993), 7–92.

[Bru09] O. Brunat, On the inductive McKay condition in the defining characteristic, *Math. Z.* **263** (2009), 411–424.

[Cab18] M. Cabanes, Local methods for blocks of finite simple groups. Local representation theory and simple groups, 179–265, *EMS Ser. Lect. Math., Eur. Math. Soc.*, Zürich, 2018.

[CE04] M. Cabanes and M. Enguehard, *Representation theory of finite reductive groups*, New Mathematical Monographs **1**, Cambridge University Press, Cambridge, 2004.

[Car85] R. W. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Wiley and Sons, New York et al, 1985, 544 pp.

[Atl85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[Dad77] E. C. Dade, Remarks on isomorphic blocks, *J. Algebra* **45** (1977), 254–258.

[Dad79] E. C. Dade, *A correspondence of characters,* The Santa Cruz Conference on Finite Groups, Proc. Sympos. Pure Math., vol. 37, Univ. California, Santa Cruz, CA, 1979, Amer. Math. Soc., Providence, RI (1980), pp. 401–403

[GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*, 2020. http://www.gap-system.org

[GRSS20] E. Giannelli, N. Rizo, B. Sambale, and A. A. Schaeffer Fry, Groups with few p'-characters degrees in the principal block, *Proc. Amer. Math. Soc.* **148** (2020), 4597–4614.

[HK00] L. Héthelyi and B. Külshammer, On the number of conjugacy classes of a finite solvable group, *Bull. London Math. Soc.* **32** (2000), 668–672.

[HMM21] N. N. Hung, G. Malle, and A. Maróti, On almost p-rational characters of p'-degree, submitted. arXiv:2104.02994

[HSF21] N. N. Hung and A. A. Schaeffer Fry, On Héthelyi-Külshammer's conjecture for principal blocks, submitted. arXiv:2102.09077

[Isa06] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[IN01] I. M. Isaacs and G. Navarro, Characters of p'-degree of p-solvable groups, *J. Algebra* **246** (2001), 394–413.

[JK81] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, Vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[KM13] R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, *Ann. of Math.* **178** (2013), 321–384.

[KS21] S. Koshitani and T. Sakurai, The principal p-block with small number of characters, *Bull. London Math. Soc.*, to appear. arXiv:2001.09970v2.

[KS16] S. Koshitani and B. Späth, The inductive Alperin-McKay and Blockwise Alperin Weight conditions for blocks with cyclic defect groups and odd primes, *J. Group Theory* **19** (2016), 777–813.

[KR96] B. Külshammer and G. R. Robinson, Alperin-McKay implies Brauer's problem 21, *J. Algebra* **180** (1996), 208–210.

[L1903] E. Landau, Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante, *Math. Ann.* **56** (1903), 671–676.

[Lan81] P. Landrock, On the number of irreducible characters in a 2-block, *J. Algebra* **68** (1981), 426–442.

[Mal90] G. Malle, Die unipotenten charaktere von ${}^2F_4(q^2)$. *Comm. Algebra* **18** (1990), 2361–2381.

[Mal07] G. Malle, Height 0 characters of finite groups of Lie type, *Represent. Theory* **11** (2007), 192–220.

[Mal08] G. Malle, Extensions of unipotent characters and the inductive McKay condition, *J. Algebra* **320** (2008), 2963–2980.

[Mal20] G. Malle. On the number of characters in blocks of quasi-simple groups, *Algebr. Represent. Theor.* **23** (2020), 513–539.

[MM16] G. Malle and A. Maróti, On the number of p'-degree characters in a finite group, *Int. Math. Res. Not.* **20** (2016), 6118–6132.

[MN21] G. Malle and G. Navarro, Brauer's height zero conjecture for principal blocks, *J. Reine Angew. Math.*, to appear. arXiv:2102.08270.

[Mar16] A. Maróti, A lower bound for the number of conjugacy classes of a finite group, *Adv. Math.* **290** (2016), 1062–1078.

[Mar21] J. M. Martínez, Degrees of characters in the principal block, *J. Algebra* **579** (2021), 195–209.

[Mat95] S. Mattarei, On character tables of wreath products, *J. Algebra* **175** (1995), 157–178.

[MO83] G. O. Michler and J. B. Olsson, Character correspondences in finite general linear, unitary and symmetric groups, *Math. Z.* **184**, 203–233.

[Mur94] M. Murai, Block induction, normal subgroups and characters of height zero, *Osaka J. Math* **31** (1994), 9–25.

[Nav98] G. Navarro, *Characters and blocks of finite groups*, *London Mathematical Society Lecture Note Series* **50**, Cambridge University Press, Cambridge, 1998.

[Nav18] G. Navarro, *Character Theory and the McKay Conjecture*, *Cambridge studies in advance mathematics* **175**, Cambridge University Press, Cambridge, 2018.

[Nav04] G. Navarro, The McKay conjecture and Galois automorphisms, *Ann. of Math.* **160** (2004), 1129–1140.

[NST18] G. Navarro, B. Sambale, and P. H. Tiep, Characters and Sylow 2-subgroups of maximal class revisited, *J. Pure Appl. Algebra* **222** (2018), 3721–3732.

[NT19] G. Navarro and P. H. Tiep, Sylow subgroups, exponents, and character values, *Trans. Amer. Math. Soc.* **372** (2019), 4263–4291.

[NT21] G. Navarro and P. H. Tiep, The fields of values of characters of degree not divisible by p, *Forum Math. Pi* **9** (2021), 1–28.

[OW80] T. Okuyama and M. Wajima, Character correspondence and p-blocks of p-solvable groups, *Osaka J. Math.* **17** (1980), 801–806.

[Ols75] J. B. Olsson, On 2-blocks with quaternion and quasidihedral defect groups, *J. Algebra* **36** (1975), 212–241.

[Ols76] J. B. Olsson, McKay numbers and heights of characters, *Math. Scand.* **38** (1976), 25–42.

[Ols84] J. B. Olsson, On the number of characters in blocks of finite general linear, unitary and symmetric groups, *Math. Z.* **186** (1984), 41–47.

[Ols93] J. B. Olsson, *Combinatorics and representations of finite groups*, Universität Essen, Fachbereich Mathematik, Essen, 1993.

[RSV21] N. Rizo, A. A. Schaeffer Fry, and C. Vallejo, Principal blocks with 5 irreducible characters, *J. Algebra*, to appear. arXiv:2010.15422.

[SFT21] A. A. Schaeffer Fry and J. Taylor, Galois automorphisms and classical groups, submitted. arXiv:2005.14088v2.

[Spa13] B. Späth, A reduction theorem for the Alperin-McKay conjecture, *J. Reine Angew. Math.* **680** (2013), 153–189.

[VV85] A. Vera López and J. Vera López, Classification of finite groups according to the number of conjugacy classes, *Israel J. Math.* **51** (1985), 305–338.

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