GALOIS ACTION AND CYCLIC DEFECT GROUPS FOR $\mathrm{Sp}_6(2^a)$

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Abstract. Groups are mathematical objects used to describe the structure of symmetries, with one of the most canonical examples being the set of invertible matrices of a given size. For a given group, a matrix representation leverages this by providing a way to represent each of its elements as an invertible matrix. The information about the (complex) representations of a finite group can be condensed by instead considering the trace of the matrices, yielding a function known as a character. One of the overarching themes in character theory is to determine what properties about a finite group or its subgroups can be obtained by studying its characters. In this paper, we study a conjecture that proposes a correlation between the makeup of a group's irreducible characters and the properties of certain subgroups known as defect groups. In particular, we prove the conjecture for the finite symplectic groups $\mathrm{Sp}_6(2^a)$.

1. Introduction

Given a finite group G and an integer $n \geq 1$, a *complex representation of degree* n of G is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. In other words, ρ is a function such that for each $g \in G$, the image $\rho(g)$ is an $n \times n$ invertible matrix with entries in the complex numbers, and $\rho(gh) = \rho(g)\rho(h)$ for each $g, h \in G$. Here on the left-hand side, multiplication is taken in G, and on the right-hand side, the operation is usual matrix multiplication. We obtain the corresponding *character* for ρ by taking the trace $\mathrm{Tr}(\rho(g))$ of each $\rho(g)$ (that is, by summing the diagonal entries). This gives a function $\chi\colon G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho(g))$ for each $g \in G$. Note here that if $1 \in G$ is the identity element, then $\chi(1) = \mathrm{Tr}(I_n) = n$ is the degree of the original representation.

A character χ is *irreducible* if it cannot be written as $\chi = \chi_1 + \chi_2$, where χ_1 and χ_2 are characters corresponding to representations of G. We refer to the set of irreducible characters of G as $\mathrm{Irr}(G)$. The information about the character theory of G is summarized in the *character table* of G, which is the square table whose columns are indexed by the conjugacy class representatives $\{g_1, \ldots, g_k\}$ of G, rows are indexed by $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$, and whose (i, j)th entry is given by $\chi_i(g_j)$.

One of the main general problems in the representation theory of finite groups is the pursuit of answering the question "what information about G or its subgroups can be obtained from the character table of G?" This general question fits into the framework of so-called "local-global" conjectures in character theory, which seek to find relationships between the character theory of G and properties of certain proper subgroups.

The following standard definitions will be useful. Note that for a finite set X, we use $|X|$ to denote the cardinality of X. Hence, the order of a group G will be given by $|G|$. In analogy to this notation, the order of an element $g \in G$ will be written $|g|$.

We recall that given a subgroup $H \leq G$ of G, the *normalizer of* H *in* G, denoted as $N_G(H)$, is the group

$$N_G(H) := \{x \in G : Hx = xH\}.$$

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Throughout, if ℓ is a prime and n is an integer, we write n_ℓ for the largest power of ℓ dividing n and $n_{\ell'}$ for n/n_ℓ. If ℓ is a prime dividing $|G|$, then any subgroup P of G such that $|P| = |G|_\ell$ is called a *Sylow ℓ-subgroup of G*. We write $P \in \mathrm{Syl}_\ell(G)$.

With this notation established, we may now state one of the earliest and most prominent of these "local-global" conjectures, known as the McKay Conjecture [7]. The McKay conjecture proposes that if G is a finite group, ℓ is a prime that divides $|G|$, and $P \in \mathrm{Syl}_\ell(G)$, then $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))|$, where $\mathrm{Irr}_{\ell'}(G)$ denotes the set of irreducible characters of G with degree prime to ℓ.

Although we only deal with complex representations here, representations over fields of positive characteristic ℓ can also be defined, and these are related to $\mathrm{Irr}(G)$ by so-called ℓ-blocks. For our purposes, we consider ℓ-blocks as a partitioning of the set $\mathrm{Irr}(G)$. Each set in the partition is written $\mathrm{Irr}(B)$, corresponding to an ℓ-block B. (More precisely, the sets $\mathrm{Irr}(B)$ can be obtained as the equivalence classes under the transitive closure of the relation on $\mathrm{Irr}(G)$ such that $\chi, \psi \in \mathrm{Irr}(G)$ are related if $\sum_{\ell \nmid |g|} \chi(g)\psi(g^{-1}) \neq 0$. Here the sum is taken over all elements of G whose order is not divisible by ℓ.)

Each ℓ-block is then associated with a special subgroup of G whose size is a power of ℓ, known as a *defect group* of the block. Although the precise definition of defect groups is technical and not necessary for the results here, we remark that if D is a defect group for B, then every $\chi \in \mathrm{Irr}(B)$ satisfies $\chi(1)$ is divisible by $|G|_\ell/|D|$. The character $\chi \in \mathrm{Irr}(B)$ is called a *height-zero* character if $\chi(1)_\ell = |G|_\ell/|D|$, and hence if $\chi(1)_\ell$ is as small as possible. We write $\mathrm{Irr}_0(B)$ for the set of height-zero characters of B.

The McKay Conjecture, while still unproven, opened the door to a number of stronger conjectures, of which the Alperin-McKay Conjecture [1] (often thought of as the blockwise version of McKay, relating the set $\mathrm{Irr}_0(B)$ to the height-zero characters in a block of $N_G(D)$), McKay–Navarro Conjecture [8] (the Galois version of McKay), and the Alperin–McKay–Navarro Conjecture (a combination of the other two) are most relevant to our work. Although these conjectures are beyond the scope of this article, we deal here with a consequence of the Alperin–McKay–Navarro Conjecture. Namely, in 2019, Rizo, Schaeffer Fry, and Vallejo [9] proved that if the Alperin–McKay–Navarro conjecture holds for $\ell \in \{2, 3\}$, then we can determine from the character table of G whether a defect group is cyclic in the following way:

Conjecture 1.1 (Rizo–Schaeffer Fry–Vallejo [9])**.** *Let $\ell \in \{2, 3\}$. Let G be a finite group and let B be an ℓ-block of G with nontrivial defect group D. Then $|\mathrm{Irr}_0(B)^{\sigma_1}| = \ell$ if and only if D is cyclic.*

Here σ_1 is a specific Galois automorphism, which we define in Section 2.2, and $\mathrm{Irr}_0(B)^{\sigma_1}$ is the set of members of $\mathrm{Irr}_0(B)$ that are fixed under the action of σ_1. In this paper, we prove the following:

Theorem 1.2. *Conjecture 1.1 holds for the group $G = \mathrm{Sp}_6(q)$ and the prime $\ell = 3$, where q is a power of 2.*

Our proof of Theorem 1.2 relies on the known character table for $\mathrm{Sp}_6(q)$ with q even determined by Frank Lübeck [6], as well as the known distribution of characters into blocks and their defect groups by Donald White [13] and the third author [10, 11].

The paper is structured as follows. In Section 2, we introduce some additional notation and definitions and make some preliminary observations. (We remark here that more information on groups and characters can be found in [3, 4].) In Section 3, we provide a series of computational lemmas regarding the irrational values that occur in the character table for $\mathrm{Sp}_6(q)$ and their behavior under that Galois automorphism σ_1. Finally, in Section 4, we complete the proof of Theorem 1.2. We also provide an appendix with examples of character values found in each relevant block.

2. Preliminaries

2.1. **General Linear and Symplectic Groups.** Let q be a power of a prime p, and let $\mathbb{F}_q$ denote a finite field of size q. The *general linear group*, $\mathrm{GL}_n(q)$, is the group of all $n \times n$ invertible matrices with entries in $\mathbb{F}_q$.

With a proper choice of basis, the *symplectic group* $\mathrm{Sp}_{2n}(q)$ can be defined as

$$\mathrm{Sp}_{2n}(q) = \{g \in \mathrm{GL}_{2n}(q) | g^T J g = J\}$$

where

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

I_n is the $n \times n$ identity matrix and g^T is the transpose of g. For the purpose of this paper, we are particularly interested in the case of $\mathrm{Sp}_6(q)$ (i.e. $n = 3$) when q is a power of $p = 2$. In this case, note that $I_n = -I_n$.

2.2. **The Galois Automorphism σ_1.** Let $\mathbb{E}$ be an extension field of $\mathbb{Q}$. Then an *automorphism* of $\mathbb{E}$ is a field isomorphism $\sigma \colon \mathbb{E} \to \mathbb{E}$. That is, σ is a bijective map satisfying $\sigma(a+b) = \sigma(a)+\sigma(b)$, and $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in \mathbb{E}$. Note that any such σ necessarily fixes $\mathbb{Q}$. If $\mathbb{E}$ is *Galois* over $\mathbb{Q}$ (see [2]), then we write $\mathrm{Gal}(\mathbb{E}|\mathbb{Q})$ for the set of automorphisms of $\mathbb{E}$, which in this case are also called *Galois automorphisms* and form a group called a Galois group.

More generally, we can consider the Galois group $\mathrm{Gal}(\mathbb{E}|\mathbb{L})$ of automorphisms of $\mathbb{E}$ fixing all elements of $\mathbb{L}$ when the extensions $\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{E}$ are all Galois. The size of a Galois group $\mathrm{Gal}(\mathbb{E}|\mathbb{L})$ is the same as the index $[\mathbb{E} : \mathbb{L}]$ of $\mathbb{E}$ over $\mathbb{L}$, which is the dimension of $\mathbb{E}$ viewed as a vector space over $\mathbb{L}$. For more information, we refer the reader to an abstract algebra text, such as [2].

Now, given a finite group G, the character values $\chi(g)$ lie in $\mathbb{Q}(e^{2\pi i/|G|})$ for all $g \in G$ and $\chi \in \mathrm{Irr}(G)$. Further, given any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/|\mathrm{G}|})/\mathbb{Q})$ and $\chi \in \mathrm{Irr}(G)$, we obtain another irreducible character χ^σ defined by $\chi^\sigma(g) := \sigma(\chi(g))$ for all $g \in G$. Given a prime ℓ dividing $|G|$, there is a unique $\sigma_1 \in \mathrm{Gal}(\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/|\mathrm{G}|})/\mathbb{Q})$ satisfying that for a root of unity $\xi \in \mathbb{C}^\times$,

$$\sigma_1(\xi) = \begin{cases} \xi^{\ell+1} & \text{if } |\xi| \text{ is a power of } \ell \\ \xi & \text{if } \ell \text{ does not divide } |\xi|. \end{cases} \tag{1}$$

Note that when $|\xi| = \ell$, i.e. ξ is an ℓth root of unity, we have $\xi^{\ell+1} = \xi$. Therefore in this case, ξ is fixed by σ_1. In fact, this is the only case in which a root of unity with order a power of ℓ is fixed by σ_1. Further, note that σ_1 has order a power of ℓ.

In service of Conjecture 1.1, we are concerned with studying when $\chi^{\sigma_1} = \chi$, for certain $\chi \in \mathrm{Irr}(G)$, which means that the value $\chi(g) \in \mathbb{Q}(e^{2\pi i/|G|})$ is fixed by σ_1 for each $g \in G$. In the character table for $\mathrm{Sp}_6(q)$, obtained by F. Lübeck [6] and available in the computer algebra system CHEVIE [5], we often find rational linear combinations of expressions of the form $\xi + \xi^{-1}$, where ξ is some complex root of unity. For this reason, we establish the following observation.

Lemma 2.1. *Let G be a finite group and let ℓ be an odd prime dividing $|G|$. Let ξ be a complex nth root of unity, where $n > 2$ is a divisor of $|G|$. Then σ_1 fixes ξ if and only if σ_1 fixes $\xi + \xi^{-1}$.*

Proof. First, assume that $\sigma_1(\xi) = \xi$. Then note that $\sigma_1(\xi + \xi^{-1}) = \sigma_1(\xi) + \sigma_1(\xi^{-1}) = \sigma_1(\xi) + \sigma_1(\xi)^{-1} = \xi + \xi^{-1}$, and hence σ_1 fixes $\xi + \xi^{-1}$ as well.

Now assume that σ_1 fixes $\xi + \xi^{-1}$. Let $\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{J} \subseteq \mathbb{K}$ be extension fields such that $\mathbb{K} = \mathbb{Q}(e^{2\pi i/|G|})$, $\mathbb{J} = \mathbb{Q}(\xi)$, and $\mathbb{L} = \mathbb{Q}(\xi + \xi^{-1})$. Then $\sigma_1 \in \mathrm{Gal}(\mathbb{K}/\mathbb{L})$. Since $\xi, \xi^{-1} \notin \mathbb{L}$, the polynomial $x^2 - (\xi + \xi^{-1})x + 1 = (x - \xi)(x - \xi^{-1}) \in \mathbb{L}[x]$ has no solutions in $\mathbb{L}$. Therefore $\mathbb{J}$ is a splitting field over $\mathbb{L}$, and the order of the group $\mathrm{Gal}(\mathbb{J}/\mathbb{L})$ is 2. We can then say that $\mathrm{Gal}(\mathbb{J}/\mathbb{L}) = \{\phi_1, \phi_2\}$, where $\phi_1(\xi) = \xi$ and $\phi_2(\xi) = \xi^{-1}$.

Now consider the restriction σ_1' of σ_1 to $\mathrm{Gal}(\mathbb{J}/\mathbb{L})$. That is, σ_1' is the automorphism of $\mathbb{J}$ that is simply the the restriction of σ_1 to the smaller domain $\mathbb{J}$. Then σ_1' must either be ϕ_1 or ϕ_2. For the sake of contradiction assume the latter case. Since we know that the order of σ_1 is a power of ℓ, say ℓ^b, then $\sigma_1^{\ell^b}$ is the trivial automorphism of $\mathrm{Gal}(\mathbb{K}/\mathbb{L})$, so its image in $\mathrm{Gal}(\mathbb{J}/\mathbb{L})$ is also trivial. However, if $\sigma_1' = \phi_2$, then we would have $\phi_2^{\ell^b}(\xi) = \xi^{-1}$, which is a contradiction. Therefore we must have $\sigma_1' = \phi_1$, and so $\sigma_1(\xi) = \xi$. That is, σ_1 also fixes ξ. ☐

Lemma 2.2. *Let G be a finite group and let ℓ be an odd prime dividing $|G|$. Let ξ be a complex nth root of unity, where $n > 2$ is a divisor of $|G|$. Let $\mathcal{I} \subseteq \mathbb{Z}$ be some subset of $\mathbb{Z}$ containing* 1. *Then ξ is fixed by σ_1 if and only if ξ^a is fixed by σ_1 for all $a \in \mathcal{I}$.*

Proof. First, suppose that ξ is fixed by σ_1. Then $\sigma_1(\xi^a) = \sigma_1(\xi)^a = \xi^a$ so ξ^a is still fixed by σ_1 for any $a \in \mathcal{I}$. Now suppose that ξ^a is fixed by σ_1 for all $a \in \mathcal{I}$. Then ξ^a is fixed when $a = 1$, and so ξ is fixed by σ_1. ☐

3. Breaking Down Character Values for $\mathrm{Sp}_6(q)$

3.1. **Notation.** For the remainder of the paper, let q be a power of 2 and let $G = \mathrm{Sp}_6(q)$. Note that $|G| = q^9(q^2-1)(q^4-1)(q^6-1)$. The irrational values in the character table for G, available in the computer algebra system CHEVIE [5] and originally determined in [6], are rational combinations of roots of unity of orders divisible by these polynomials. Namely, the following notation will be used throughout, letting $\epsilon \in \{\pm 1\}$.

$$\zeta_1 := \exp\left(\frac{2\pi\sqrt{-1}}{q-1}\right); \quad \xi_1 := \exp\left(\frac{2\pi\sqrt{-1}}{q+1}\right);$$

$$\omega_1 := \exp\left(\frac{2\pi\sqrt{-1}}{q-\epsilon}\right); \quad \omega_2 := \exp\left(\frac{2\pi\sqrt{-1}}{q+\epsilon}\right);$$

$$\zeta_2 := \exp\left(\frac{2\pi\sqrt{-1}}{q^2-1}\right); \quad \xi_2 := \exp\left(\frac{2\pi\sqrt{-1}}{q^2+1}\right);$$

and

$$\omega_3 := \exp\left(\frac{2\pi\sqrt{-1}}{q^3-\epsilon}\right) = \exp\left(\frac{2\pi\sqrt{-1}}{(q-\epsilon)(q^2+\epsilon q+1)}\right).$$

We note that the roots of unity ζ_i, ξ_i for $i = 1, 2$ are exactly as defined in the character table for G in CHEVIE [5]. The following notation is used in [10, 11], and agrees with that of the CHEVIE character table, to label the blocks and characters of G, where again $\epsilon \in \{\pm 1\}$.

Notation 3.1. Let $I_{q-\epsilon}^0$ be the set $\{i \in \mathbb{Z} : 1 \le i \le q - \epsilon - 1\}$, and let $I_{q-\epsilon}$ be a set of class representatives on $I_{q-\epsilon}^0$ under the equivalence relation $i \sim j \iff i \equiv \pm j \mod (q-\epsilon)$. Let $I_{q^2+1}^0 := \{i \in \mathbb{Z} : 1 \le i \le q^2\}$ and $I_{q^2-1}^0 := \{i \in \mathbb{Z} : 1 \le i \le q^2-1, (q-1) \nmid i, (q+1) \nmid i\}$, and let $I_{q^2-\epsilon}$ be a set of representatives for the equivalence relation on $I_{q^2-\epsilon}^0$ given by $i \sim j \iff i \equiv \pm j$ or $\pm qj \mod (q^2-\epsilon)$. Similarly, let $I_{q^3-\epsilon}^0 := \{i \in \mathbb{Z} : 1 \le i \le q^3 - \epsilon; (q^2+\epsilon q+1) \nmid i\}$ and $I_{q^3-\epsilon}$ a set of representatives for the equivalence relation on $I_{q^3-\epsilon}^0$ given by $i \sim j \iff i \equiv \pm j$, $\pm qj$, or $\pm q^2 j$ mod $(q^3-\epsilon)$.

3.2. **Initial Observations.** We next make some observations about modular relationships that will be useful in what follows. Note that since $3 \nmid q$, we have 3 divides exactly one of $q-1$ or $q+1$. Here and for the remainder of the paper, we let $\epsilon \in \{\pm 1\}$ be such that $3|(q-\epsilon)$ and will write $(q-\epsilon) =: m3^d$ with $m, d \in \mathbb{N}$ and $\gcd(m, 3) = 1$. Note then that 3 divides $(q^2+\epsilon q+1)$ exactly once, and we write $(q^2+\epsilon q+1) =: 3n$, with $\gcd(n, 3) = 1$. (Indeed, we have $q^2+\epsilon q+1 = (q-\epsilon)^2 + 3\epsilon q$, which must be divisible by 3 since both summands are, but cannot be divisible by 9 since then $3q$ is divisible by 9, contradicting that $3 \nmid q$.)

Lemma 3.2. *Let $h, z_1, z_2 \in \mathbb{Z}$, where h is prime to 3. Then $hz_1m3^{d-1} \equiv hz_2m3^{d-1} \mod h(q-\epsilon)$ if and only if $z_1 \equiv z_2 \mod 3$.*

Proof. Since $q-\epsilon = m3^d$, then $hz_1m3^{d-1} \equiv hz_2m3^{d-1} \mod h(q-\epsilon)$ if and only if $hm3^d | h(z_1 - z_2)m3^{d-1}$, which happens if and only if $3|(z_1-z_2)$, and therefore if and only if $z_1 \equiv z_2 \mod 3$. □

Lemma 3.3. *Let $k = x3^d$ for some integer x such that $|x| < m$, let $h \in \mathbb{Z}$, where h is prime to 3, and let $\mu \in \{\pm 1\}$. Then $k + \mu hm3^{d-1} \not\equiv -k + \mu hm3^{d-1} \mod h(q-\epsilon)$, and $k + \mu hm3^{d-1} \not\equiv -k \mod h(q-\epsilon)$.*

Proof. First, it is helpful to notice that m is odd, since $m|(q-\epsilon)$ and q is a power of 2. Suppose then, for the sake of contradiction, that $k + \mu hm3^{d-1} \equiv -k + \mu hm3^{d-1} \mod h(q-\epsilon)$ for some $\mu \in \{\pm 1\}$. Then $hm3^d \mid 2x3^d$, which implies that $hm \mid 2x$ and ultimately $m \mid 2x$. This is a contradiction, since m is odd and $|x| < m$. Now suppose that $k + \mu hm3^{d-1} \equiv -k \mod h(q-\epsilon)$ for some $\mu \in \{\pm 1\}$. Then $hm3^d \mid (2x3^d \pm hm3^{d-1})$. It follows that $m|2x$ and $3|1$, which is again a contradiction, and the proof is complete. □

3.3. **Roots of Unity Fixed by σ_1.** Here we present several lemmas describing when the various roots of unity appearing in the character table for G are fixed by the Galois automorphism σ_1.

Lemma 3.4. *For any $k \in \mathbb{Z}$, we have 3 does not divide the order of ω_2^k, $\zeta_2^{k(q-\epsilon)}$, nor ξ_2^k. In particular, these are fixed by σ_1.*

Proof. Since 3 divides $(q-\epsilon)$, then 3 cannot divide $(q+\epsilon) = |\omega_2|$. Further, $|\zeta_2^{k(q-\epsilon)}| = |\omega_2^k| = \frac{q+\epsilon}{\gcd(k,q+\epsilon)}$, which is therefore also prime to 3. Finally, since $q^2 \equiv 1 \mod 3$, it follows that 3 cannot divide q^2+1, so 3 cannot divide $|\xi_2^k| = \frac{q^2+1}{\gcd(k,q^2+1)}$. □

The next two lemmas will be used when the character values contain powers of ω_1, which is the same as $\zeta_2^{q+\epsilon}$. Note that the conditions on $r \in I_{q-\epsilon}$ in these cases are the conditions that appear in the descriptions of the relevant blocks and characters (see Tables 1-14 and the notation preceeding them).

Lemma 3.5. *There is a unique element $r \in I_{q-\epsilon}$ satisfying $m|r$ such that ω_1^r is fixed by σ_1. Namely, this element is $r = m3^{d-1}$.*

Proof. First we will show that the stated value of $r \in I_{q-\epsilon}$ is the only possibility satisfying $m|r$ for which ω_1^r is fixed by σ_1. Assume that $r \in I_{q-\epsilon}$ such that $\sigma_1(\omega_1^r) = \omega_1^r$, and write $r = mf3^x$ with $f, x \in \mathbb{Z}$ and f relatively prime to 3. Notice that $x < d$, as otherwise $r \notin I_{q-\epsilon}$. Suppose, for the sake of contradiction, that $x = d - y$, for some y with $1 < y \leq d$. Then $|\omega_1^r| = \frac{m3^d}{\gcd(m3^d, mf3^{d-y})} = 3^y$, so ω_1^r is not fixed by σ_1. Therefore we must have $r = mf3^{d-1}$.

Now, note that $f \equiv 1$ or $2 \mod 3$. Further, under the equivalence relation defining $I_{q-\epsilon}$, we have i is equivalent to $-i$, but also we see $1 \equiv -2 \mod 3$ and $2 \equiv -1 \mod 3$, so by Lemma 3.2 we have that every r defined as such will be equivalent in the set $I_{q-\epsilon}$. Finally, we see that $\omega_1^{m3^{d-1}}$ has order $\frac{m3^d}{\gcd(m3^d, m3^{d-1})} = 3$, so is fixed by σ_1. □

Lemma 3.6. *Let $k \in I_{q-\epsilon}$, such that $3^d|k$. Then, there are exactly 3 elements $r \in I_{q-\epsilon}$ satisfying $r \equiv \pm k \mod m$ such that ω_1^r is fixed by σ_1.*

Proof. (1) First, we show that there are 6 choices for $r \in I^0_{q-\epsilon}$, under equivalence modulo $q-\epsilon$, satisfying $r \equiv \pm k \mod m$ and such that ω_1^r is fixed by σ_1. Let r be such an element. Since $r \equiv \pm k$ mod m, we can write $r = \pm k + mf$, for some $f \in \mathbb{Z}$. Then, $\omega_1^r = (\omega_1^{\pm k})(\omega_1^{mf})$. Further, since $k \in I_{q-\epsilon}$ and $3^d|k$, we have $k = x3^d$ for some $0 \neq x \in \mathbb{Z}$. Then:

$$|\omega_1^{\pm k}| = |\omega_1^{\pm x3^d}| = \frac{m3^d}{\gcd(x3^d, m3^d)} = \frac{m}{\gcd(x,m)}.$$

Since m is prime to 3, the order of $\omega_1^{\pm k}$ cannot be divisible by 3, so these are fixed by σ_1. Hence, ω_1^r is fixed by σ_1 if and only if ω_1^{mf} is. For $f = 0$ or when f is any multiple of 3^d, we have $\omega_1^{mf} = 1$, so $\omega_1^r = \omega_1^{\pm k}$. Otherwise, we have

$$|\omega_1^{mf}| = \frac{m3^d}{\gcd(mf, m3^d)} = \frac{3^d}{\gcd(f, 3^d)}$$

is some positive power of 3, so ω_1^{mf} is fixed by σ_1 if and only if f is such that $|\omega_1^{mf}| = 3$ exactly.

Note that $\frac{3^d}{\gcd(f,3^d)} = 3$ implies that $\gcd(f, 3^d) = 3^{d-1}$, which implies that $f = z3^{d-1}$, where $z \in \mathbb{Z}$ is prime to 3. So in order for ω_1^r to be fixed by σ_1, r must be of the form $\pm k + zm3^{d-1}$, for some $z \in \mathbb{Z}$ with $z = 0$ or $3 \nmid z$.

Now, by Lemma 3.2, we have that $z_1 m3^{d-1} \equiv z_2 m3^{d-1} \mod (q-\epsilon)$ if and only if $z_1 \equiv z_2 \mod 3$, so we may assume without loss that $z \in \{0, 1, 2\}$. Note that $z = 0$ corresponds to the previous case where $f = 0$ or f is any multiple of 3^d. Therefore, for $r \in I_{q-\epsilon}^0$ with $r \equiv \pm k \mod m$, we have ω_1^r is fixed by σ_1 if and only if r is equivalent modulo $q - \epsilon$ to one of:

$$r = \pm k, \quad r = \pm k + m3^{d-1}, \quad \text{or} \quad r = \pm k + 2m3^{d-1}.$$

(2) Now we will show that these 6 choices of r correspond to at most 3 elements of $I_{q-\epsilon}$. Recall that if $i, j \in I_{q-\epsilon}$, we have $i \sim j$ if and only if $i \equiv \pm j \bmod (q-\epsilon)$. In particular, we have $k \sim (-k)$.

Next, we can see by Lemma 3.2 that $k + 2m3^{d-1} \equiv k - m3^{d-1} \mod (q - \epsilon)$, so $k + 2m3^{d-1} \sim k - m3^{d-1}$. Similarly, we have $-k + 2m3^{d-1} \sim -k - m3^{d-1}$. Then since $k + m3^{d-1} \sim -k - m3^{d-1}$, we also have $-k + 2m3^{d-1} \sim k + m3^{d-1}$. We also have $k + 2m3^{d-1} \sim k - m3^{d-1} \sim -k + m3^{d-1}$ using the same reasoning. For simplicity's sake, we will use the following as our three equivalence class representatives for r:

$$r = k, \quad r = k + m3^{d-1}, \quad \text{or} \quad r = k - m3^{d-1}.$$

(3) Finally, we show that these three choices for r give us distinct class representatives in $I_{q-\epsilon}$. First, suppose that $k+m3^{d-1} \sim k$ in $I_{q-\epsilon}$. Then either $k+m3^{d-1} \equiv k \mod (q-\epsilon)$, or $k+m3^{d-1} \equiv -k \mod (q - \epsilon)$. Then this is a contradiction by Lemmas 3.2 and 3.3, respectively. Second, suppose that $k - m3^{d-1} \sim k$ in $I_{q-\epsilon}$. Then either $k - m3^{d-1} \equiv k \mod (q - \epsilon)$, in which case Lemma 3.2 applies, or $k - m3^{d-1} \equiv -k \mod (q - \epsilon)$, in which case Lemma 3.3 applies, giving us another contradiction. Lastly, suppose that $k + m3^{d-1} \sim k - m3^{d-1}$ in $I_{q-\epsilon}$. Then either $k+m3^{d-1} \equiv k-m3^{d-1} \mod (q-\epsilon)$, in which case Lemma 3.2 applies, or $k+m3^{d-1} \equiv -k+m3^{d-1} \mod q - \epsilon$, in which case Lemma 3.3 applies, giving us our final contradiction. Therefore, the three elements listed indeed yield distinct equivalence class representatives in $I_{q-\epsilon}$, and the proof is complete. □

Due to the nature of the values found in the character table for $\mathrm{Sp}_6(q)$, many of the preceding lemmas will often be used in conjunction with Lemma 2.1. Similarly, Lemmas 3.8 and 3.9 below, which deal with powers of ζ_2, will be used in conjunction with the following:

Lemma 3.7. *Let $r \in I_{q^2-1}$. Then ζ_2^r is fixed by σ_1 if and only if both ω_1^r and $\zeta_2^r + \zeta_2^{rq} + \zeta_2^{-r} + \zeta_2^{-rq}$ are fixed by σ_1.*

Proof. First, if ζ_2^r is fixed by σ_1, then so is any sum of powers of ζ_2^r, so both $\omega_1^r = \zeta_2^{r(q+\epsilon)}$ and $\zeta_2^r + \zeta_2^{rq} + \zeta_2^{-r} + \zeta_2^{-rq}$ are fixed by σ_1.

Conversely, assume that ω_1^r and $\zeta_2^r + \zeta_2^{rq} + \zeta_2^{-r} + \zeta_2^{-rq}$ are fixed by σ_1. Let $\mathbb{F}$ denote the fixed field of $\mathbb{Q}(e^{2\pi i/|G|})$ under the group $\langle\sigma_1\rangle$ generated by σ_1, so that ω_1^r, ω_2^r, and $\zeta_2^r + \zeta_2^{rq} + \zeta_2^{-r} + \zeta_2^{-rq}$ are all elements of $\mathbb{F}$ by assumption and by Lemma 3.4. Assume by way of contradiction that ζ_2^r is not fixed by σ_1, so that $\zeta_2^r + \zeta_2^{-r}$ is also not fixed by σ_1, using Lemma 2.1. Now, since $\mathbb{Q}(\zeta_2^r + \zeta_2^{-r})$ is the (unique) maximal totally real subfield of $\mathbb{Q}(\zeta_2^r)$, we see that, if we let $\alpha_1 := \zeta_2^r + \zeta_2^{-r}$ and

$\alpha_2 := \zeta_2^{qr} + \zeta_2^{-qr}$, then $\mathbb{F}(\alpha_1) = \mathbb{F}(\alpha_2)$. Then since $\alpha_1\alpha_2 = \omega_1^r + \omega_1^{-r} + \omega_2^r + \omega_2^{-r}$, we see $\mathbb{F}(\alpha_1)$ is the splitting field over $\mathbb{F}$ for the polynomial $(x-\alpha_1)(x-\alpha_2) = x^2 + (\alpha_1+\alpha_2)x + \alpha_1\alpha_2$ and $[\mathbb{F}(\alpha_1) : \mathbb{F}] = 2$. From here, we may argue similarly to Lemma 2.1 to obtain a contradiction, unless α_1 (and hence ζ_2^r) is fixed by σ_1. □

Lemma 3.8. *Let $k \in I_{q^2-1}$ such that $3^d|k$. Then there are exactly 3 elements $r \in I_{q^2-1}$ satisfying $r \equiv \pm k$ or $\pm qk \mod m(q+\epsilon)$ such that ω_1^r and ζ_2^r are both fixed by σ_1.*

Proof. First, let r be as in the statement and let $f \in \mathbb{Z}$ such that $r = \pm k + mf(q+\epsilon)$ or $r = \pm qk + mf(q+\epsilon)$. Then we can further write k or qk as $x3^d$ for some $x \in \mathbb{Z}$ with $3 \nmid x$. Therefore, we can write $r = \pm x3^d + mf(q+\epsilon)$.

Next, we have $\omega_1^r = (\omega_1^{\pm x3^d})(\omega_1^{mf(q+\epsilon)})$. As in the proof of Lemma 3.6, we then have ω_1^r is fixed by σ_1 if and only if $\omega_1^{mf(q+\epsilon)}$ is. Further, we have $\zeta_2^r = (\zeta_2^{\pm x3^d})(\zeta_2^{mf(q+\epsilon)})$. Notice:

$$|\zeta_2^{\pm x3^d}| = \frac{m3^d(q+\epsilon)}{\gcd(x3^d, m3^d(q+\epsilon))} = \frac{m(q+\epsilon)}{\gcd(x, m(q+\epsilon))}.$$

We know that $m(q+\epsilon)$ is prime to 3, so the order of $\zeta_2^{\pm x3^d}$ is not divisible by 3. Hence we similarly have ζ_2^r is fixed by σ_1 if and only if $\zeta_2^{mf(q+\epsilon)} = \omega_1^{mf}$ is.

Now, since $m(q+\epsilon) = (q^2-1)_{3'}$, arguing exactly as in part (1) of the proof of Lemma 3.6 in this case, we see r is equivalent modulo q^2-1 to one of

$$r = \pm k, \quad r = \pm qk, \quad r = \pm k + m3^{d-1}(q+\epsilon),$$

$$r = \pm k + 2m3^{d-1}(q+\epsilon), \quad r = \pm qk + m3^{d-1}(q+\epsilon), \quad \text{or} \quad r = \pm qk + 2m3^{d-1}(q+\epsilon).$$

(Conversely, we see that these choices of r satisfy the statement.)

Then, in order to partition these choices for r into their respective equivalence classes in $I_{q^2-\epsilon}$, we will use the relation $i \sim j$ if and only if $i \equiv \pm j$ or $\pm qj \mod (q^2-1)$. First, it is again clear that $k \sim -k$, but also that $k \sim qk$ and $k \sim -qk$ under this relation.

For the remaining choices for r, it will be helpful to first notice that $z_1 m3^{d-1}(q+\epsilon) \equiv z_2 m3^{d-1}(q+\epsilon) \mod (q^2-1)$ if and only if $z_1 \equiv z_2 \mod 3$, by Lemma 3.2. We can use this to again substitute $2m$ for $-m$, and then show that these remaining 8 choices for r lie in only two equivalence classes in I_{q^2-1}.

We have $k + \epsilon m3^{d-1}(q+\epsilon) \sim qk + m3^{d-1}(q+\epsilon)$ because (q^2-1) divides $(q^2-1)(-k) - (q-\epsilon)(q+\epsilon)m3^{d-1} = (k+\epsilon m3^{d-1}(q+\epsilon)) - q(qk + m3^{d-1}(q+\epsilon))$. A similar argument shows $-k + \epsilon m3^{d-1}(q+\epsilon) \sim -qk + m3^{d-1}(q+\epsilon)$.

Also note that $k + m3^{d-1}(q+\epsilon) \sim -k - m3^{d-1}(q+\epsilon)$; $-k + m3^{d-1}(q+\epsilon) \sim k - m3^{d-1}(q+\epsilon)$; $qk + m3^{d-1}(q+\epsilon) \sim -qk - m3^{d-1}(q+\epsilon)$; and $qk - m3^{d-1}(q+\epsilon) \sim -qk + m3^{d-1}(q+\epsilon)$. So any $r \in I_{q^2-1}$ such that ω_1^r and ζ_2^r are both fixed by σ_1 is equivalent to one of:

$$r = k, \quad r = k + m3^{d-1}(q+\epsilon), \quad \text{or} \quad r = k - m3^{d-1}(q+\epsilon).$$

It now suffices to show that these elements represent three distinct classes in I_{q^2-1}. First, $k \sim k + m3^{d-1}(q+\epsilon)$ if, and only if, $k \equiv \pm(k + m3^{d-1}(q+\epsilon))$ or $\pm q(k + m3^{d-1}(q+\epsilon)) \mod (q^2-1)$. Applying Lemma 3.2 with $h = (q+\epsilon)$, we see that $k \not\equiv k + m3^{d-1}(q+\epsilon) \mod (q^2-1)$, and we can use Lemma 3.3 with $h = (q+\epsilon)$ to show that $k \not\equiv -(k + m3^{d-1}(q+\epsilon)) \mod (q^2-1)$. Then, $k \equiv qk + qm3^{d-1}(q+\epsilon)$ would imply that $(q^2-1)|(k - qk - qm3^{d-1}(q+\epsilon))$, which gives us $(q^2-1)|(-k(q-1) - qm3^{d-1}(q+\epsilon))$. Similarly, $k \equiv -qk - qm3^{d-1}(q+\epsilon)$ will give us $(q^2-1)|(k(q+1) + qm3^{d-1}(q+\epsilon))$. So, since $3^d|(q^2-1)$ and $3^d|k$, either of these would imply $3|qm(q+\epsilon)$, a contradiction, and therefore, $k \not\sim k + m3^{d-1}(q+\epsilon)$. Using similar calculations, we can also see $k \not\sim k - m3^{d-1}(q+\epsilon)$ and $k + m3^{d-1}(q+\epsilon) \not\sim k - m3^{d-1}(q+\epsilon)$. Therefore, these three elements give distinct $r \in I_{q^2-1}$, and the proof is complete. □

Lemma 3.9. *Let $t \in I_{q+\epsilon}$. Then there is a unique $r \in I_{q^2-1}$ satisfying $r \equiv \pm(q-\epsilon)t \mod m(q+\epsilon)$, such that ω_1^r and ζ_2^r are both fixed by σ_1.*

Proof. Following the strategy from before, we will first show that there are 6 possible choices for r as in the statement such that ω_1^r and ζ_2^r are fixed by σ_1. Then we will show that these actually only give one element of I_{q^2-1}.

We will sometimes write $M := m(q+\epsilon) = (q^2-1)_{3'}$. Since $r \equiv \pm(q-\epsilon)t \mod M$, we can write $r = \pm tm3^d + Mf$, for some $f \in \mathbb{Z}$. Then $\omega_1^r = (\omega_1^{\pm tm3^d})(\omega_1^{Mf})$. We also see that $|\omega_1^{\pm tm3^d}| = \frac{m3^d}{\gcd(tm3^d, m3^d)} = 1$, and $|\omega_1^{Mf}| = \frac{m3^d}{\gcd(mf(q+\epsilon), m3^d)} = \frac{3^d}{\gcd(f(q+\epsilon), 3^d)}$. As in the proof of Lemma 3.6, if f is 0 or any multiple of 3^d, then $\omega_1^{Mf} = 1$ and $\omega_1^r = \omega_1^{\pm tm3^d} = 1$. Otherwise, we must choose f such that $|\omega_1^{Mf}| = 3$ exactly.

Similarly, $\zeta_2^r = (\zeta_2^{\pm tm3^d})(\zeta_2^{Mf}) = (\omega_2^{\pm t})(\zeta_2^{mf(q+\epsilon)}) = (\omega_2^{\pm t})(\omega_1^{mf})$. By Lemma 3.4, we have that $\omega_2^{\pm t}$ is fixed by σ_1, so ζ_2^r is fixed by σ_1 if and only if ω_1^{mf} is. Notice that $|\omega_1^{mf}| = \frac{m3^d}{\gcd(mf, m3^d)} = \frac{3^d}{\gcd(f, 3^d)}$.

Using an argument similar to Lemma 3.6, we see that if ω_1^r and ζ_2^r are both fixed by σ_1, then r is one of:

$$r = \pm(q-\epsilon)t, \quad r = \pm(q-\epsilon)t + m3^{d-1}(q+\epsilon), \quad \text{or} \quad r = \pm(q-\epsilon)t - m3^{d-1}(q+\epsilon).$$

Now, recall that $(q-\epsilon)t \notin I_{q^2-1}$ and $r \sim -r$ in I_{q^2-1}, so in fact we have r represented by one of:

$$r_1 = (q-\epsilon)t + m3^{d-1}(q+\epsilon) \quad \text{or} \quad r_2 = (q-\epsilon)t - m3^{d-1}(q+\epsilon).$$

But notice that $r_1 \equiv -\epsilon q r_2 \mod (q^2-1)$, so these define just one class in I_{q^2-1}. ☐

Lemma 3.10. *Let $k \in I_{q^3-\epsilon}$ such that $3^{d+1}|k$. Then, the following hold:*

(1) There are exactly 3 elements $r \in I_{q^3-\epsilon}$ satisfying $r \equiv \pm k$, $\pm qk$, or $\pm q^2 k$ mod mn, such that ω_3^r is fixed by σ_1.

(2) Let $r \in I_{q^3-\epsilon}$ satisfying $r \equiv \pm k$, $\pm qk$, or $\pm q^2 k$ mod mn and denote by $\chi(r)$ the character $\chi_{63}(r)$ of G if $\epsilon = 1$ and $\chi_{66}(r)$ if $\epsilon = -1$. Then $\chi(r)$ is fixed by σ_1 if and only if ω_3^r is fixed by σ_1.

Proof. First, we notice that $q^3 - \epsilon = (q-\epsilon)(q^2+\epsilon q+1)$, so we will write $q^3-\epsilon$ as $mn3^{d+1}$ when it is useful. Since $3^{d+1}|k$, we write $k = x3^{d+1}$. Note that qk and q^2k are both of the form $x3^{d+1}$ for some (different) $x \in \mathbb{Z}$, so we will write $r = \pm x3^{d+1} + mnf$ for some $f \in \mathbb{Z}$.

(1) We first consider the first claim. We have $\omega_3^r = (\omega_3^{\pm x3^{d+1}})(\omega_3^{mnf})$ and

$$|\omega_3^{\pm x3^{d+1}}| = \frac{(m3^d)(3n)}{\gcd(x3^{d+1}, (m3^d)(3n))} = \frac{mn}{\gcd(x, mn)}.$$

Since m and n are both prime to 3, the order of $\omega_3^{\pm x3^{d+1}}$ is prime to 3, so this is fixed by σ_1. Hence ω_3^r is fixed by σ_1 if and only if ω_3^{mnf} is fixed by σ_1. Now, replacing the roles of $(3^d, q+\epsilon)$ in Lemma 3.8 with $(3^{d+1}, n)$ here, the situation is analogous, as $z_1 m3^d n \equiv z_2 m3^d n \mod mn3^{d+1}$ if and only if $z_1 \equiv z_2 \mod 3$ arguing like in Lemma 3.2.

In this case, for ω_3^r to be fixed by σ_1, we therefore have r must be of one of the following forms:

$$r = \pm k, \quad r = \pm qk, \quad r = \pm q^2 k,$$

$$r = \pm k + mn3^d, \quad r = \pm qk + mn3^d, \quad r = \pm q^2 k + mn3^d,$$

$$r = \pm k + 2mn3^d, \quad r = \pm qk + 2mn3^d, \quad \text{or} \quad r = \pm q^2 k + 2mn3^d.$$

(Conversely, note that ω_3^r is fixed by σ_1 if r is of any of these forms.)

Now, recall that $k \sim (-k)$, $k \sim (\pm qk)$, and $k \sim (\pm q^2 k)$. Arguing similarly to Lemma 3.8 with the role of $q+\epsilon$ now replaced with n, we obtain that under the relation $\sim$, each value in the list above is equivalent to one of the following three elements of $r \in I_{q^3-\epsilon}$:

$$r = k, \quad r = k + mn3^d, \quad \text{and} \quad r = k - mn3^d.$$

Further, arguing as in the previous lemmas, we again see that these indeed give distinct elements of $I_{q^3-\epsilon}$, completing the proof of (1).

(2) Now we consider the second claim. The character $\chi(r)$ is what is known as a semisimple character, and is indexed by a conjugacy class of G consisting of all elements in G with eigenvalues $\widetilde{\omega}_3^r, \widetilde{\omega}_3^{rq}, \widetilde{\omega}_3^{rq^2}, \widetilde{\omega}_3^{-r}, \widetilde{\omega}_3^{-rq}$, and $\widetilde{\omega}_3^{-rq^2}$, where here $\widetilde{\omega}_3$ is a primitive $q^3-\epsilon$ root of unity in $\mathbb{F}_{q^6}$. (This is the class $g_{31}(r)$ when $\epsilon = 1$, respectively $g_{34}(r)$ when $\epsilon = -1$, defined in [6, Tabelle 19].) Now, since G comes from an algebraic group over $\overline{\mathbb{F}}_q$ whose center is connected, [12, Lemma 3.4] describes how such characters are permuted by members of $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$. In particular, [12, Lemma 3.4] tells us that $\chi(r)$ is fixed by σ_1 if and only if the set $\{\omega_3^r, \omega_3^{rq}, \omega_3^{rq^2}, \omega_3^{-r}, \omega_3^{-rq}, \omega_3^{-rq^2}\}$ is permuted by σ_1.

Now, note that $n \nmid r$, as otherwise $n \mid x$ and hence $3n = q^2 + \epsilon q + 1$ divides k, contradicting that $k \in I_{q^3-\epsilon}$. Suppose that some $\sigma \in \langle \sigma_1 \rangle$ maps ω_3^r to $\omega_3^{r\bar{q}}$, where $\bar{q} \in \{-1, \pm q, \pm q^2\}$. Recall that n is relatively prime to 2, $3^{d+1}m$, $(\pm q^2 - 1)$, and $(\pm q - 1)$. Writing $\omega_3 = y_1 y_2$ for y_1 a primitive $3^{d+1}m$-root of unity and y_2 a primitive nth root of unity, we then see that $(\sigma(y_1^r))y_2^r = y_1^{r\bar{q}} y_2^{r\bar{q}}$, since y_2 is fixed by σ_1. This forces $y_2^{r(\bar{q}-1)}$ to be a $(3^{d+1}m)$'th root of unity. Then y_2^r is also a $(3^{d+1}m)$'th root of unity, since $|y_2|$ is prime to $\bar{q}-1$. Then since $|y_2|$ is prime to $3^{d+1}m$, we see that this forces $y_2^r = 1$, so that $n \mid r$, a contradiction. Hence we see that $\chi(r)$ is fixed by σ_1 if and only if σ_1 fixes ω_3^r. □

4. Proof of Theorem 1.2

Let $G := \mathrm{Sp}_6(q)$ with q a power of 2. To prove Theorem 1.2, we must show that if B is a 3-block of G with cyclic defect groups, then there are exactly three height-zero characters in $\mathrm{Irr}(B)$ that are fixed by σ_1, and that if B has noncyclic defect groups, then the number of such characters is strictly larger than 3.

The defect groups for G are described in [11, Proposition 3.1]. Namely, for the prime 3, the cyclic defect groups are (in the notation of [11]) denoted Q_1, Q_2, and $Q^{(3)}$, and the remaining defect groups are denoted $Q_{1,1}$, $Q_{2,1}$, $Q_{1,1,1}$, and P. Here P is a Sylow 3-subgroup of G.

The sets $\mathrm{Irr}(B)$ for each block B of G are described in [13] for so-called "unipotent" blocks, and in [10, Section 4.4] otherwise. The sets $\mathrm{Irr}_0(B)$ are described in [11, Sections 4.2-4.10] and also in [10, Section 7.4.1]. In Tables 1-14, we list the names of these blocks (with the notation of [13, 10]) and a subset of characters found in $\mathrm{Irr}_0(B)$ (with the notation of the CHEVIE character table and [10]).

With this information in place, and given our work in Section 3, the proof involves considering the character table for $\mathrm{Sp}_6(q)$ due to Frank Lübeck [6] and available on CHEVIE, and analyzing when the character values of the characters in $\mathrm{Irr}_0(B)$ for each block B corresponding to a given defect group are fixed by σ_1. The families of characters and of conjugacy classes for $\mathrm{Sp}_6(q)$ are indexed by the various sets introduced in Notation 3.1. The character values are either rational or sums of complex numbers of the form $x(\xi^{ir} + \xi^{-ir})$, where $i, r \in \mathbb{Z}$ come from one of the indexing sets defined in Notation 3.1 (depending on the index defining the character and the class within their families), ξ is some root of unity, and $x \in \mathbb{C}$ is either rational or otherwise fixed by σ_1. In the appendix, we include examples of specific values for the relevant characters. We have used our lemmas from Section 3 to find the appropriate choices of r so that a given ξ^r will be fixed by σ_1, where again ξ denotes a relevant root of unity.

We apply Lemma 2.2 to say that ξ^r is fixed by σ_1 if and only if ξ^{ir} is fixed by σ_1, for every relevant i. Note that we also apply Lemma 2.1 in conjunction with Lemmas 3.5 and 3.6; Lemma 3.7 in conjunction with Lemmas 3.8 and 3.9; and the two parts of Lemma 3.10 together, to show that in fact the full character values being considered are also fixed by σ_1. Tables 1-14 list the characters being considered for each block and the lemmas from Section 3 that are used for those characters.

For a concrete example, consider the block $B = B_{29}(s, t_1)$ when $\epsilon = 1$ (see Table 3). Here $t_1 \in I_{q+1}$ and $s \in I_{q^2-1}$ is divisible by 3^d. Then the members of $\mathrm{Irr}_0(B)$ are the characters $\chi_{61}(r, t_1)$, where $r \in I_{q^2-1}$ is equivalent to $\pm s$ or $\pm qs$ modulo $m(q+1)$. By Lemma 3.8, there are exactly three choices of such r such that ζ_2^r and ω_1^r are fixed by σ_1, and hence exactly three such choices of r such that $\zeta_1^r + \zeta_1^{-r}$ and $\zeta_2^r + \zeta_2^{-r} + \zeta_2^{qr} + \zeta_2^{-qr}$ are fixed by σ_1, using Lemmas 2.1 and 3.7. Now, the irrational character values for $\chi_{61}(r, t_1)$ take the following forms, where i, i' range through appropriate indexing sets from Notation 3.1 for the conjugacy classes:

- $(\xi_1^{it_1} + \xi_1^{-it_1})$; $(1-q^4)(\xi_1^{it_1} + \xi_1^{-it_1})$; $(1 \pm q^2)(\xi_1^{it_1} + \xi_1^{-it_1})$; $(\xi_1^{ir} + \xi_1^{-ir})$; $(1-q^2)(\xi_1^{ir} + \xi_1^{-ir})$; $(1 \pm q)(\xi_1^{ir} + \xi_1^{-ir})$; $(q^3+1)(\xi_1^{ir} + \xi_1^{-ir})(\xi_1^{it_1} + \xi_1^{-it_1})$; $(\xi_1^{ir} + \xi_1^{-ir})(\xi_1^{it_1} + \xi_1^{-it_1})$; $(1+q)(\xi_1^{ir} + \xi_1^{-ir})(\xi_1^{i't_1} + \xi_1^{-i't_1})$; $(\xi_1^{ir} + \xi_1^{-ir})(\xi_1^{i't_1} + \xi_1^{-i't_1})$, which are always fixed by σ_1 by Lemma 3.4
- $(\zeta_1^{ir} + \zeta_1^{-ir})$; $(q^2 - 2q + 1)(\zeta_1^{ir} + \zeta_1^{-ir})$; $(1-q)(\zeta_1^{ir} + \zeta_1^{-ir})$;
- $(\zeta_2^{ir} + \zeta_2^{-ir} + \zeta_2^{iqr} + \zeta_2^{-iqr})$; $(1 \pm q)(\zeta_2^{ir} + \zeta_2^{-ir} + \zeta_2^{iqr} + \zeta_2^{-iqr})$
- $(\zeta_2^{ir(q+1)} + \zeta_2^{-ir(q+1)})(\zeta_2^{it_1(q-1)} + \zeta_2^{-it_1(q-1)}) = (\zeta_1^{ir} + \zeta_1^{-ir})(\xi_1^{it_1} + \xi_1^{-it_1})$; $(1-q)(\zeta_1^{ir} + \zeta_1^{-ir})(\xi_1^{it_1} + \xi_1^{-it_1})$
- $(\zeta_2^{ir} + \zeta_2^{-ir} + \zeta_2^{iqr} + \zeta_2^{-iqr})(\zeta_2^{i't(q-1)} + \zeta_2^{-i't(q-1)}) = (\zeta_2^{ir} + \zeta_2^{-ir} + \zeta_2^{iqr} + \zeta_2^{-iqr})(\xi_1^{i't} + \xi_1^{-i't})$

Then we see that $\chi_{61}(r, t_1)$ is fixed by σ_1 exactly when r is one of these three choices, showing that B contains exactly three height-zero characters fixed by σ_1. Since this block has defect group Q_2, which is cyclic, this block satisfies the statement.

For each defect group, we include two tables; one for when $\epsilon = 1$ and one for when $\epsilon = -1$. Each table lists all blocks B with the given defect group, additional conditions on indexing, the characters in $\mathrm{Irr}_0(B)$ being considered for that block (in the notation of the CHEVIE character table), and the number of characters in the listed family that are fixed by σ_1, with reference to the lemmas used for those specific characters.

The first six tables are for the cyclic defect groups, Q_1, Q_2, and $Q^{(3)}$. For these groups we list all characters in $\mathrm{Irr}_0(B)$, in order to show that $|\mathrm{Irr}_0(B)^{\sigma_1}| = 3$. The remaining tables correspond to the non-cyclic defect groups, $P, Q_{1,1}, Q_{2,1}$, and $Q_{1,1,1}$. In these cases, we only list enough characters needed to see that $|\mathrm{Irr}_0(B)^{\sigma_1}| > 3$. Therefore in these cases, the column that shows the number of fixed characters refers only to the characters listed, not necessarily the total number fixed in the given block.

4.1. **The Tables.** Throughout, we let $k_1, k_2, k_3 \in I_{q-1}$ with none of k_1, k_2, k_3 the same and let $t_1, t_2, t_3 \in I_{q+1}$ with none of t_1, t_2, t_3 the same. When $\epsilon = 1$, let $3^d | k_i$, and when $\epsilon = -1$, let $3^d | t_i$. Let $u \in I_{q^2+1}$, and $s \in I_{q^2-1}$ with $3^d | s$, where $3^d := (q-\epsilon)_3$. Let $v \in I_{q^3-1}$ and $w \in I_{q^3+1}$. When $\epsilon = 1$, let $(q^3-1)_3 | v$, and when $\epsilon = -1$, let $(q^3+1)_3 | w$. Moreover, let $m := (q-\epsilon)_{3'}$ as before, and let $n := (q^2 + \epsilon q + 1)_{3'}$.

TABLE 1. Blocks with Defect Group Q_1 when $\epsilon = 1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
b_1	N/A	χ_5, χ_{11}	2: rational
	$m\|r$	$\chi_{17}(r)$	1: Lemma 3.5
$B_6(k_1)^{(1)}$	$r \equiv \pm k_1 \mod m$	$\chi_{17}(r)$	3: Lemma 3.6
$B_{23}(t_1, t_2)$	N/A	$\chi_{53}(t_1, t_2), \chi_{54}(t_1, t_2)$	2: Lemma 3.4
	$m\|r$	$\chi_{60}(r, t_1, t_2)$	1: Lemmas 3.4, 3.5
$B_{24}(u)$	N/A	$\chi_{55}(u), \chi_{56}(u)$	2: Lemma 3.4
	$m\|r$	$\chi_{62}(r, u)$	1: Lemmas 3.5, 3.4
$B_{28}(k_1, t_1, t_2)$	$r \equiv \pm k_1 \mod m$	$\chi_{60}(r, t_1, t_2)$	3: Lemmas 3.4, 3.6
$B_{30}(k_1, u)$	$r \equiv \pm k_1 \mod m$	$\chi_{62}(r, u)$	3: Lemmas 3.6, 3.4

TABLE 2. Blocks with Defect Group Q_1 when $\epsilon = -1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
b_1	N/A	χ_4, χ_9	2: rational
	$m\|r$	$\chi_{20}(r)$	1: Lemma 3.5
$B_7(t_1)^{(1)}$	$r \equiv \pm t_1 \mod m$	$\chi_{20}(r)$	3: Lemma 3.6
$B_{17}(k_1, k_2)$	N/A	$\chi_{41}(k_1, k_2), \chi_{42}(k_1, k_2)$	2: Lemma 3.4
	$m\|r$	$\chi_{58}(k_1, k_2, r)$	1: Lemmas 3.4, 3.5
$B_{24}(u)$	N/A	$\chi_{55}(u), \chi_{56}(u)$	2: Lemma 3.4
	$m\|r$	$\chi_{65}(u, r)$	1: Lemmas 3.5, 3.4
$B_{26}(k_1, k_2, t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{58}(k_1, k_2, r)$	3: Lemmas 3.4, 3.6
$B_{33}(u, t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{65}(u, r)$	3: Lemmas 3.6, 3.4

TABLE 3. Blocks with Defect Group Q_2 when $\epsilon = 1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_9(t_1)$	N/A	$\chi_{28}(t_1), \chi_{30}(t_1)$	2: Lemma 3.4
	$r \equiv \pm(q-1)t_1 \mod m(q+1)$	$\chi_{61}(r, t_1)$	1: Lemma 3.9
$B_{22}(t_1, t_2)$	N/A	$\chi_{51}(t_1, t_2), \chi_{52}(t_1, t_2)$	2: Lemma 3.4
	$r \equiv \pm(q-1)t_1 \mod m(q+1)$	$\chi_{61}(r, t_2)$	1: Lemma 3.9
$B_{29}(s, t_1)$	$r \equiv \pm s$ or $\pm qs \mod m(q+1)$	$\chi_{61}(r, t_1)$	3: Lemma 3.8

TABLE 4. Blocks with Defect Group Q_2 when $\epsilon = -1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_8(k_1)$	N/A	$\chi_{25}(k_1), \chi_{27}(k_1)$	2: Lemma 3.4
	$r \equiv \pm(q+1)k_1 \mod m(q-1)$	$\chi_{59}(r, k_1)$	1: Lemma 3.9
$B_{16}(k_1, k_2)$	N/A	$\chi_{39}(k_1, k_2), \chi_{40}(k_1, k_2)$	2: Lemma 3.4
	$r \equiv \pm(q+1)k_1 \mod m(q-1)$	$\chi_{59}(r, k_2)$	1: Lemma 3.9
$B_{27}(s, k_1)$	$r \equiv \pm s$ or $\pm qs \mod m(q-1)$	$\chi_{59}(r, k_1)$	3: Lemma 3.8

TABLE 5. Blocks with Defect Group $Q^{(3)}$ when $\epsilon = 1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_{31}(v)$	$r \equiv \pm v$, $\pm qv$ or $\pm q^2 v \mod mn$	$\chi_{63}(r)$	3: Lemma 3.10

TABLE 6. Blocks with Defect Group $Q^{(3)}$ when $\epsilon = -1$

Block B	Restriction	Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_{34}(w)$	$r \equiv \pm w$, $\pm qw$ or $\pm q^2 w \mod mn$	$\chi_{66}(r)$	3: Lemma 3.10

Table 7. Blocks with Defect Group P when $\epsilon = 1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
b_0	N/A	$\chi_1, \chi_3, \chi_4, \chi_9, \chi_{10}, \chi_{12}$	6: rational
$B_8(k_1)$	$r \equiv \pm k_1 \mod m$	$\chi_{25}(r), \chi_{26}(r), \chi_{27}(r)$	9: Lemma 3.6

Table 8. Blocks with Defect Group P when $\epsilon = -1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
b_0	N/A	$\chi_1, \chi_2, \chi_5, \chi_8, \chi_{11}, \chi_{12}$	6: rational
$B_9(t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{28}(r), \chi_{29}(r), \chi_{30}(r)$	9: Lemma 3.6

Table 9. Blocks with Defect Group $Q_{1,1}$ when $\epsilon = 1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_7(t_1)$	N/A	$\chi_{19}(t_1), \chi_{20}(t_1), \chi_{21}(t_1), \chi_{22}(t_1)$	4: Lemma 3.4
$B_{20}(k_1, t_1)$	$r \equiv \pm k_1 \mod m$	$\chi_{47}(r, t_1), \chi_{48}(r, t_1)$	6: Lemmas 3.4, 3.6
$B_{18}(k_1, t_1)$	$r \equiv \pm k_1 \mod m$	$\chi_{43}(r, t_1), \chi_{44}(r, t_1)$	6: Lemmas 3.4, 3.6
$B_{26}(k_1, k_2, t_1)$	$r_i \equiv \pm k_i \mod m$	$\chi_{58}(r_1, r_2, t_1)$	9: Lemmas 3.4, 3.6

Table 10. Blocks with Defect Group $Q_{1,1}$ when $\epsilon = -1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_6(k_1)$	N/A	$\chi_{13}(k_1), \chi_{15}(k_1), \chi_{16}(k_1), \chi_{17}(k_1)$	4: Lemma 3.4
$B_{20}(k_1, t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{47}(k_1, r), \chi_{48}(k_1, r)$	6: Lemmas 3.4, 3.6
$B_{21}(t_1, k_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{49}(r, k_1), \chi_{50}(r, k_1)$	6: Lemmas 3.4, 3.6
$B_{28}(k_1, t_1, t_2)$	$r_i \equiv \pm t_i \mod m$	$\chi_{60}(k_1, r_1, r_2)$	9: Lemmas 3.4, 3.6

Table 11. Blocks with Defect Group $Q_{2,1}$ when $\epsilon = 1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_{13}(t_1)$	N/A	$\chi_{35}(t_1), \chi_{36}(t_1), \chi_{37}(t_1), \chi_{38}(t_1)$	4: Lemma 3.4
$B_{21}(t_1, k_1)$	$r \equiv \pm k_1 \mod m$	$\chi_{49}(t_1, r), \chi_{50}(t_1, r)$	6: Lemmas 3.4, 3.6
$B_{19}(s)$	$r \equiv \pm s$ or $\pm qs \mod m(q+1)$	$\chi_{45}(r), \chi_{46}(r)$	6: Lemmas 3.4, 3.8
$B_{27}(s, k_1)$	$r \equiv \pm s$ or $\pm qs \mod m(q+1)$ $j \equiv \pm k_1 \mod m$	$\chi_{59}(r, j)$	9: Lemmas 3.4, 3.6, 3.8

Table 12. Blocks with Defect Group $Q_{2,1}$ when $\epsilon = -1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_{11}(k_1)$	N/A	$\chi_{31}(k_1), \chi_{32}(k_1), \chi_{33}(k_1), \chi_{34}(k_1)$	4: Lemma 3.4
$B_{18}(k_1, t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{43}(k_1, r), \chi_{44}(k_1, r)$	6: Lemmas 3.4, 3.6
$B_{19}(s)$	$r \equiv \pm s$ or $\pm qs \mod m(q-1)$	$\chi_{45}(r), \chi_{46}(r)$	6: Lemmas 3.4, 3.8
$B_{29}(s, t_1)$	$r \equiv \pm s$ or $\pm qs \mod m(q-1)$ $j \equiv \pm t_1 \mod m$	$\chi_{61}(r, j)$	9: Lemmas 3.4, 3.6, 3.8

References

[1] J. L. Alperin. The main problem of block theory. In *Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975)*, pages 341–356, New York, 1976. Academic Press.

[2] J. A. Gallian *Contemporary Abstract Algebra, Ninth Edition*, page 531, Boston, 2017. Cengage Learning.

[3] I. M. Isaacs, *Character Theory of Finite Groups.* AMS-Chelsea, Providence, 2006.

[4] G. James, and M. Liebeck, *Representations and Characters of Groups* (2nd ed.). Cambridge: Cambridge University Press, 2001.

Table 13. Blocks with Defect Group $Q_{1,1,1}$ when $\epsilon = 1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_6(k_1)^{(0)}$	$r \equiv \pm k_1 \mod m$	$\chi_{13}(r), \chi_{14}(r), \chi_{15}(r), \chi_{16}(r)$	12: Lemma 3.6
$B_{11}(k_1)$	$r \equiv \pm k_1 \mod m$	$\chi_{31}(r), \chi_{32}(r), \chi_{33}(r), \chi_{34}(r)$	12: Lemma 3.6
$B_{17}(k_1, k_2)$	$r_i \equiv \pm k_i \mod m$	$\chi_{41}(r_1, r_2), \chi_{42}(r_1, r_2)$	18: Lemma 3.6
$B_{16}(k_1, k_2)$	$r_i \equiv \pm k_i \mod m$	$\chi_{39}(r_1, r_2), \chi_{40}(r_1, r_2)$	18: Lemma 3.6
$B_{25}(k_1, k_2, k_3)$	$r_i \equiv \pm k_i \mod m$	$\chi_{57}(r_1, r_2, r_3)$	27: Lemma 3.6

Table 14. Blocks with Defect Group $Q_{1,1,1}$ when $\epsilon = -1$

Block B	Restriction	Selection of Characters in $\mathrm{Irr}_0(B)$	# Fixed by σ_1
$B_7(t_1)^{(0)}$	$r \equiv \pm t_1 \mod m$	$\chi_{19}(r), \chi_{21}(r), \chi_{22}(r), \chi_{23}(r)$	12: Lemma 3.6
$B_{13}(t_1)$	$r \equiv \pm t_1 \mod m$	$\chi_{35}(r), \chi_{36}(r), \chi_{37}(r), \chi_{38}(r)$	12: Lemma 3.6
$B_{23}(t_1, t_2)$	$r_i \equiv \pm t_i \mod m$	$\chi_{53}(r_1, r_2), \chi_{54}(r_1, r_2)$	18: Lemma 3.6
$B_{22}(t_1, t_2)$	$r_i \equiv \pm t_i \mod m$	$\chi_{51}(r_1, r_2), \chi_{52}(r_1, r_2)$	18: Lemma 3.6
$B_{32}(t_1, t_2, t_3)$	$r_i \equiv \pm t_i \mod m$	$\chi_{64}(r_1, r_2, r_3)$	27: Lemma 3.6

[5] M. Geck, G. Hiss, G. Lübeck, G. Malle, and G. Pfeiffer. **CHEVIE** – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. *Appl. Algebra Engrg. Comm. Comput.*, 7:175–210, 1996.

[6] F. Lübeck. Charaktertafeln für die Gruppen $CSp_6(q)$ mit ungeradem q und $Sp_6(q)$ mit geradem q, Dissertation, Universität Heidelberg, 1993.

[7] J. McKay. Irreducible representations of odd degree. *J. Algebra*, 20:416–418, 1972.

[8] G. Navarro. The McKay conjecture and Galois automorphisms. *Ann. of Math.* (2) **160** (2004), 1129–1140.

[9] N. Rizo, A. A. Schaeffer Fry and C. Vallejo. Galois action on the principal block and cyclic Sylow subgroups. *Algebra Number Theory*, v. 14, n.7, 2020.

[10] A. A. Schaeffer Fry. *Irreducible representations of finite groups of Lie type: on the irreducible restriction problem and some local-clobal conjectures.* Ph.D. Thesis, The University of Arizona (2013). `http://hdl.handle.net/10150/293407`

[11] A. A. Schaeffer Fry. $Sp_6(2^a)$ is “good” for the McKay, Alperin weight, and related local–global conjectures. *J. Algebra* **401** (2014), 13–47.

[12] A. A. Schaeffer Fry and J. Taylor. On self-normalising Sylow 2-subgroups in type *A*. *J. Lie Theory* **28** (2018), no. 1, 139–168.

[13] D. White. Decomposition numbers of unipotent blocks of $Sp_6(2^a)$ in odd characteristics. *J. Algebra* **227** (2000), 172–194.

Appendix A. Some Character Values

Although it would be unreasonable to include the entire character table, here we list a character value on a single family of conjugacy classes for some relevant characters, to help illustrate the use of the lemmas listed in Tables 1-14. We follow the order they are listed in those Tables. In many cases, only one character family from a line in Tables 1-14 is listed, as the character values for the other characters on the line take similar forms. All notation is taken from the CHEVIE character table for $\mathrm{Sp}_6(q)$.

Character	Class	Value
χ_5, χ_{11}		all rational values
$\chi_{17}(k_1)$	$C_{17}(i_1)$	$\frac{1}{2}q(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})$
$\chi_{60}(k_1, k_2, k_3)$	$C_{44}(i_1, i_2)$	$(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})[\xi_1^{i_2k_2} + \xi_1^{-i_2k_2} + \xi_1^{i_2k_3} + \xi_1^{-i_2k_3}]$
$\chi_{55}(u)$	$C_{53}(i_1)$	$\xi_2^{i_1k1} + \xi_2^{-i_1k1} + \xi_2^{qi_1k1} + \xi_2^{-qi_1k1}$
$\chi_{62}(k_1, k_2)$	$C_{62}(i_1, i_2)$	$(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\xi_2^{i_2k_2} + \xi_2^{-i_2k_2} + \xi_2^{qi_2k_2} + \xi_2^{-qi_2k_2})$
χ_4, χ_9		all rational values
$\chi_{20}(k_1)$	$C_{20}(i_1)$	$-\frac{1}{2}(q^2 + q)(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1})$
$\chi_{58}(k_1, k_2, k_3)$	$C_{44}(i_1, i_2)$	$-(\xi_1^{i_2k_3} + \xi_1^{-i_2k_3})[\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1} + \zeta_1^{i_1k_2} + \zeta_1^{-i_1k_2}]$
$\chi_{56}(u)$	$C_{53}(i_1)$	$q(\xi_2^{i_1k1} + \xi_2^{-i_1k1} + \xi_2^{qi_1k1} + \xi_2^{-qi_1k1})$
$\chi_{65}(k_1, k_2)$	$C_{65}(i_1, i_2)$	$-(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})(\xi_2^{i_1k_1} + \xi_2^{-i_1k_1} + \xi_2^{qi_1k_1} + \xi_2^{-qi_1k_1})$
$\chi_{28}(t_1)$	$C_{19}(i_1)$	$(-q^3 + q^2 - q + 1)(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1})$
$\chi_{61}(r, t_1)$	$C_{45}(i_1)$	$(-q + 1)(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{52}(t_1, t_2)$	$C_{20}(i_1)$	$(-q^2 + 2q - 1)(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1}) + q(\xi_1^{i_1k_2} + \xi_1^{-i_1k_2})$
$\chi_{25}(t_1)$	$C_{17}(i_1)$	$\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1}$
$\chi_{59}(r, t_1)$	$C_{45}(i_1)$	$(-q - 1)(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{40}(k_1, k_2)$	$C_{16}(i_1)$	$(2q + 1)(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1}) + q(\zeta_1^{i_1k_2} + \zeta_1^{-i_1k_2})$
$\chi_{63}(k_1)$	$C_{63}(i_1)$	$\zeta_3^{q^2i_1k_1} + \zeta_3^{-q^2i_1k_1} + \zeta_3^{q^{i}_{1}k_1} + \zeta_3^{-qi_1k_1} + \zeta_3^{i_1k_1} + \zeta_3^{-i_1k_1}$
$\chi_{66}(k_1)$	$C_{66}(i_1)$	$-\xi_3^{q^2i_1k_1} - \xi_3^{-q^2i_1k_1} - \xi_3^{q^{i}_{1}k_1} - \xi_3^{-qi_1k_1} + \xi_3^{i_1k_1} - \xi_3^{-i_1k_1}$

Character	Class	Value
$\chi_1, \chi_3, \chi_4, \chi_9, \chi_{10}, \chi_{12}$		all rational values
$\chi_{26}(k_1)$	$C_{25}(i_1)$	$(q^3 + 2q^2 + 2q + 1)(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1}) + (q^2 + q)(\zeta_1^{3i_1k_1} + \zeta_1^{-3i_1k_1})$
$\chi_1, \chi_2, \chi_5, \chi_8, \chi_{11}, \chi_{12}$		all rational values
$\chi_{28}(k_1)$	$C_{28}(i_1)$	$(q^2 - q + 1)(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1}) + \xi_1^{3i_1k_1} + \xi_1^{-3i_1k_1}$
$\chi_{21}(k_1)$	$C_{21}(i_1)$	$-q - \frac{1}{2}(q^2 + q)(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1})$
$\chi_{47}(k_1, k_2)$	$C_{47}(i_1, i_2)$	$(-q - 1)(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})$
$\chi_{44}(k_1, k_2)$	$C_{44}(i_1, i_2)$	$-(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})$
$\chi_{58}(k_1, k_2, k_3)$	$C_{58}(i_1, i_2, i_3)$	$(\xi_1^{i_3k_3} + \xi_1^{-i_3k_3})[(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\zeta_1^{i_2k_2} + \zeta_1^{-i_2k_2}) + (\zeta_1^{i_1k_2} + \zeta_1^{-i_1k_2})(\zeta_1^{i_2k_1} + \zeta_1^{-i_2k_1})]$
$\chi_{17}(k_1)$	$C_{13}(i_1)$	$(\frac{1}{2}q^3 - q^2 + \frac{1}{2}q)(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})$
$\chi_{48}(k_1, k_2)$	$C_{48}(i_1, i_2)$	$-(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})$
$\chi_{49}(k_1, k_2)$	$C_{49}(i_1, i_2)$	$(q - 1)(\zeta_1^{i_2k_2} + \zeta_1^{-i_2k_2}) - (\xi_1^{2i_1k_1} + \xi_1^{-2i_1k_1})(\zeta_1^{i_2k_2} + \zeta_1^{-i_2k_2})$
$\chi_{60}(k_1, k_2, k_3)$	$C_{60}(i_1, i_2, i_3)$	$(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})[(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})(\xi_1^{i_3k_3} + \xi_1^{-i_3k_3}) + (\xi_1^{i_2k_3} + \xi_1^{-i_2k_3})(\xi_1^{i_3k_2} + \xi_1^{-i_3k_2})]$

Character	Class	Value
$\chi_{37}(k_1)$	$C_{56}(i_1, i_2)$	$\xi_1^{2i_1k_1} + \xi_1^{-2i_1k_1} + \xi_1^{(i_1+i_2)k_1} + \xi_1^{-(i_1+i_2)k_1} + \xi_1^{(i_1-i_2)k_1} + \xi_1^{-(i_1-i_2)k_1} + 1$
$\chi_{49}(k_1, k_2)$	$C_{50}(i_1, i_2)$	$-(\xi_1^{2i_1k_1} + \xi_1^{-2i_1k_1} + 1)(\zeta_1^{i_2k_2} + \zeta_1^{-i_2k_2})$
$\chi_{46}(k_1)$	$C_{45}(i_1)$	$-q(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{59}(k_1, k_2)$	$C_{59}(i_1, i_2)$	$(\zeta_1^{i_2k_2} + \zeta_1^{-i_2k_2})(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{33}(k_1)$	$C_{41}(i_1, i_2)$	$q(\zeta_1^{2i_1k_1} + \zeta_1^{2i_1k_1}) + (q + 1)(\zeta_1^{(i_1+i_2)k_1} + \zeta_1^{-(i_1+i_2)k_1} + \zeta_1^{(i_1-i_2)k_1} + \zeta_1^{-(i_1-i_2)k_1}) + 1 + q$
$\chi_{43}(k_1, k_2)$	$C_{44}(i_1, i_2)$	$-(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1})(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})$
$\chi_{45}(k_1)$	$C_{46}(i_1)$	$-(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{61}(k_1, k_2)$	$C_{61}(i_1, 1_2)$	$(\xi_1^{i_2k_2} + \xi_1^{-i_2k_2})(\zeta_2^{qi_1k_1} + \zeta_2^{-qi_1k_1} + \zeta_2^{i_1k_1} + \zeta_2^{-i_1k_1})$
$\chi_{14}(k_1)$	$C_{57}(i_1, i_2, i_3)$	$2(\zeta_1^{i_1k_1} + \zeta_1^{-i_1k_1} + \zeta_1^{i_2k_1} + \zeta_1^{-i_2k_1} + \zeta_1^{i_3k_1} + \zeta_1^{-i_3k_1})$
$\chi_{33}(k_1)$	$C_{58}(i_1, i_2, i_3)$	$\zeta_1^{(i_1+i_2)k_1} + \zeta_1^{-(i_1+i_2)k_1} + \zeta_1^{(i_1-i_2)k_1} + \zeta_1^{-(i_1-i_2)k_1}$
$\chi_{41}(r_1, r_2)$	$C_{41}(i_1, i_2)$	see (2) below
$\chi_{39}(r_1, r_2)$	$C_{39}(i_1, i_2)$	see (3) below
$\chi_{57}(r_1, r_2, r_3)$	$C_{57}(i_1, i_2, i_3)$	see (4) below
$\chi_{19}(k_1)$	$C_{64}(i_1, i_2, i_3)$	$-(\xi_1^{i_1k_1} + \xi_1^{-i_1k_1} + \xi_1^{i_2k_1} + \xi_1^{-i_2k_1} + \xi_1^{i_3k_1} + \xi_1^{-i_3k_1})$
$\chi_{36}(k_1)$	$C_{64}(i_1, i_2, i_3)$	see (5) below
$\chi_{53}(r_1, r_2)$	$C_{55}(i_1, i_2)$	see (6) below
$\chi_{51}(r_1, r_2)$	$C_{51}(i_1, i_2)$	see (7) below
$\chi_{64}(r_1, r_2, r_3)$	$C_{64}(i_1, i_2, i_3)$	see (8) below

$$(q+1)[(\zeta_1^{i_1k_1}+\zeta_1^{-i_1k_1})(\zeta_1^{i_2k_2}+\zeta_1^{-i_2k_2})+(\zeta_1^{i_2k_1}+\zeta_1^{-i_2k_1})(\zeta_1^{i_1k_2}+\zeta_1^{-i_1k_2})+(\zeta_1^{i_1k_1}+\zeta_1^{-i_1k_1})(\zeta_1^{i_1k_2}+\zeta_1^{-i_1k_2})] \tag{2}$$

$$\begin{aligned}(q+1)[\zeta_1^{(i_1+i_2)k_1}+\zeta_1^{-(i_1+i_2)k_1}+\zeta_1^{(i_1-i_2)k_2}+\zeta_1^{-(i_1-i_2)k_2}+(\zeta_1^{i_1k_1}+\zeta_1^{-i_1k_1})(\zeta_1^{i_2k_2}+\zeta_1^{-i_2k_2})+\\(\zeta_1^{i_2k_1}+\zeta_1^{-i_2k_1})(\zeta_1^{i_1k_2}+\zeta_1^{-i_1k_2})]\end{aligned} \tag{3}$$

$$\begin{aligned}(\zeta_1^{i_1k_1}+\zeta_1^{-i_1k_1})[(\zeta_1^{i_2k_2}+\zeta_1^{-i_2k_2})(\zeta_1^{i_3k_3}+\zeta_1^{-i_3k_3})+(\zeta_1^{i_2k_3}+\zeta_1^{-i_2k_3})(\zeta_1^{i_3k_2}+\zeta_1^{-i_3k_2})]+\\(\zeta_1^{i_1k_2}+\zeta_1^{-i_1k_2})[(\zeta_1^{i_2k_1}+\zeta_1^{-i_2k_1})(\zeta_1^{i_3k_3}+\zeta_1^{-i_3k_3})+(\zeta_1^{i_2k_3}+\zeta_1^{-i_2k_3})(\zeta_1^{i_3k_1}+\zeta_1^{-i_3k_1})]+\\(\zeta_1^{i_1k_3}+\zeta_1^{-i_1k_3})[(\zeta_1^{i_2k_2}+\zeta_1^{-i_2k_2})(\zeta_1^{i_3k_1}+\zeta_1^{-i_3k_1})+(\zeta_1^{i_2k_1}+\zeta_1^{-i_2k_1})(\zeta_1^{i_3k_2}+\zeta_1^{-i_3k_2})]\end{aligned} \tag{4}$$

$$\begin{aligned}\xi_1^{(i_1+i_2)k_1}+\xi_1^{-(i_1+i_2)k_1}+\xi_1^{(i_1-i_2)k_1}+\xi_1^{-(i_1-i_2)k_1}+\xi_1^{(i_1+i_3)k_1}+\xi_1^{-(i_1+i_3)k_1}+\\\xi_1^{(i_1-i_3)k_1}+\xi_1^{-(i_1-i_3)k_1}+\xi_1^{(i_2+i_3)k_1}+\xi_1^{-(i_2+i_3)k_1}+\xi_1^{(i_2-i_3)k_1}+\xi_1^{-(i_2-i_3)k_1}\end{aligned} \tag{5}$$

$$-(q-1)[(\xi_1^{i_1k_1}+\xi_1^{-i_1k_1})(\xi_1^{i_2k_2}+\xi_1^{-i_2k_2}+\xi_1^{i_2k_1}+\xi_1^{i_1k_2}+\xi_1^{-i_1k_2})+(\xi_1^{i_1k_2}+\xi_1^{-i_1k_2})(\xi_1^{i_2k_1}+\xi_1^{-i_2k_1})] \tag{6}$$

$$\begin{aligned}-(q-1)[(\xi_1^{i_1k_1}+\xi_1^{-i_1k_1})(\xi_1^{i_2k_2}+\xi_1^{-i_2k_2})+(\xi_1^{i_2k_1}+\xi_1^{-i_2k_1})(\xi_1^{i_1k_2}+\xi_1^{-i_1k_2})+\\(\xi_1^{(i_1+i_2)k_1}+\xi_1^{-(i_1+i_2)k_1})(\xi_1^{(i_1-i_2)k_2}+\xi_1^{-(i_1-i_2)k_2})]\end{aligned} \tag{7}$$

$$\begin{aligned}(\xi_1^{i_1k_1}+\xi_1^{-i_1k_1})[(\xi_1^{i_2k_2}+\xi_1^{-i_2k_2})(\xi_1^{i_3k_3}+\xi_1^{-i_3k_3})+(\xi_1^{i_2k_3}+\xi_1^{-i_2k_3})(\xi_1^{i_3k_2}+\xi_1^{-i_3k_2})]+\\(\xi_1^{i_1k_2}+\xi_1^{-i_1k_2})[(\xi_1^{i_2k_1}+\xi_1^{-i_2k_1})(\xi_1^{i_3k_3}+\xi_1^{-i_3k_3})+(\xi_1^{i_2k_3}+\xi_1^{-i_2k_3})(\xi_1^{i_3k_1}+\xi_1^{-i_3k_1})]+\\(\xi_1^{i_1k_3}+\xi_1^{-i_1k_3})[(\xi_1^{i_2k_2}+\xi_1^{-i_2k_2})(\xi_1^{i_3k_1}+\xi_1^{-i_3k_1})+(\xi_1^{i_2k_1}+\xi_1^{-i_2k_1})(\xi_1^{i_3k_2}+\xi_1^{-i_3k_2})]\end{aligned} \tag{8}$$

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