ADDENDUM TO NRS22 PART I

This addendum is in reference to Proposition 3.11 of NRS22 Part I, namely, the proof of the alternating groups case. We thank Annika Bartelt for bringing to our attention two missing cases in the proof and providing many of the ideas for completing these cases. Namely, the issues occur when the partition given gives the sign character of $\mathfrak{S}_n$, and therefore does not restrict to a nontrivial character of $\mathfrak{A}_n$; and when the number S defined in the case $r = 0$ does not actually exist.

The statement should read instead:

Proposition 0.1. *Let p be an odd prime and let $n \geqslant 8$. Then there is a nontrivial element in $\mathrm{Irr}_{2'}(B_2(\mathfrak{S}_n)) \cap \mathrm{Irr}_{p'}(B_p(\mathfrak{S}_n))$ and a nontrivial element in $\mathrm{Irr}_{2'}(B_2(\mathfrak{A}_n)) \cap \mathrm{Irr}_{p'}(B_p(\mathfrak{A}_n))$. In particular, Question 3.2 holds for $q = 2$ and p odd when S is an alternating group $\mathfrak{A}_n$ with $n \geqslant 8$.*

We also remark that Michler–Olsson's height-preserving bijections are not needed, and in fact not actually used, in the proof. In particular, we do not use the assumption that n is even when $r \geqslant 1$. From here, the additional arguments needed are as follows:

Case: $r = 1 = a$**, Case IIa**. In this case, if $r = 1 = a$, the character given in Table 4 is the sign character, which lies in the desired set for $\mathfrak{S}_n$ but does not restrict to a nontrivial character for $\mathfrak{A}_n$. So, we need an alternative argument in this case to complete the statement for $\mathfrak{A}_n$. (In the corresponding situation of the groups of Lie type A_{n-1}, the corresponding situation yielded the Steinberg character, which satisfied our requirements.) Here we may write $n = p^k m_1 + 1 = 2^{a_1} + \cdots + 2^{a_t}$ with $(m_1, p) = 1$, $a_1 < \cdots < a_t$, and note that the condition $r = 1 = a$, in the notation of the proof in *loc. cit*, forces $a_1 \geqslant 1$, so that n is indeed even in this case.

Consider the following alternative partitions:

Case	partition
$p^k > 2^{a_1}$	$(1^{2^{a_1}-2}, 2, n - 2^{a_1})$
$p^k < 2^{a_1}$ and $m_1 \neq 1$	$(1^{n-p^k-1}, p^k + 1)$
$n = p^k + 1 = 2^{a_1}$	$(1^{(n-4)/2}, 2, n/2)$

Note that in particular, the situation of the first line holds when $m_1 = 1$ and $n \neq 2^{a_1}$ and that the second line holds when $n = 2^{a_1}$ unless $m_1 = 1$. The third line holds when $m_1 = 1$ and $n = 2^{a_1}$.

We see that in each case, the 2-core and p-core are the same as (n), so the corresponding characters lie in the principal p- and 2- blocks of $\mathfrak{S}_n$. It suffices to show that in the first two lines, the corresponding character has odd degree prime to p and that in the third line, the character (which restricts to the sum of two characters of the same degree of $\mathfrak{A}_n$ since the partition is self-conjugate) has degree prime to p and 2-part exactly 2.

In the first line, we have degree

$$\frac{n!}{2^{a_1} \cdot m_1 p^k \cdot (n - 2^{a_1}) \cdot \prod_{i=1}^{2^{a_1}-2} i \prod_{i=1}^{n-2^{a_1}-2} i} = \frac{(n - 2^{a_1} - 1)(2^{a_1} - 1) \prod_{i=1}^{2^{a_1}} (n - 2^{a_1} + i)}{m_1 p^k \cdot \prod_{i=1}^{2^{a_1}} i}$$

$$= \frac{(m_1p^k - 2^{a_1})(2^{a_1}-1)n\prod_{i=1}^{2^{a_1}-1}(n-i)}{m_1p^k \cdot \prod_{i=1}^{2^{a_1}} i} = \frac{(m_1p^k - 2^{a_1})n\prod_{i=1}^{2^{a_1}-2}(m_1p^k-i)}{2^{a_1}\prod_{i=1}^{2^{a_1}-2} i}$$

Since $(n-2^{a_1}+i)_2 = (i)_2$ for each $1 \leqslant i \leqslant 2^{a_1}$ and the remaining factors are odd, we see that this expression is odd. Further, since $(m_1p^k - i)_p = (i)_p$ for $1 \leqslant i \leqslant 2^{a_1}-2 < p^k$ and the remaining factors are p', we see this expression is also p'. Hence in the first line, the character lies in $\mathrm{Irr}_{2'}(B_2(\mathfrak{S}_n)) \cap \mathrm{Irr}_{p'}(B_p(\mathfrak{S}_n))$ and restricts irreducibly to $\mathfrak{A}_n$.

In the second line, we obtain a character of degree $\binom{m_1p^k}{p^k}$, which is equivalent to $m_1 \mod p$ by Lucas's theorem, and hence is prime to p. Rewriting the expression as $\frac{(n-1)\cdots(n-p^k)}{\prod_{i=1}^{p^k} i}$, we see that this is also odd, since $(n-i)_2 = (i)_2$ for each $1 \leqslant i \leqslant p^k < 2^{a_1}$.

Finally, we consider the third line. Here the degree of the character is

$$\frac{n!}{p^k \cdot (n/2)^2 \left(\prod_{i=1}^{n/2-2} i\right)^2} = \frac{(2^{a_1-1}-1)^2\prod_{i=2^{a_1-1}+1}^{2^{a_1}} i}{p^k\prod_{i=1}^{2^{a_1-1}} i}.$$

Since $(2^{a_1-1}+i)_2 = (i)_2$ for $1 \leqslant i < 2^{a_1-1}$, $2^{a_1}/2^{a_1-1} = 2$, and the remaining factors are odd, we see this degree is divisible by 2 exactly once, so that the two characters lying in $\mathfrak{A}_n$ below the character corresponding to this self-conjugate partition each lie in $\mathrm{Irr}_{2'}(B_2(\mathfrak{A}_n))$ as well. Further, we may rewrite the expression in the numerator as $(2^{a_1-1}-1)^2(p^k+1)\prod_{i=1}^{2^{a_1-1}-1}(p^k+1-i) = ((p^k-1)/2)^2(p^k+1)\prod_{i=1}^{2^{a_1-1}-1}(p^k+1-i)$, so we obtain the degree:

$$\frac{(2^{a_1-1}-1)^2(p^k+1)\prod_{i=1}^{2^{a_1-1}-2}(p^k-i)}{\prod_{i=1}^{2^{a_1-1}} i} = \frac{(p^k-1)/2 \cdot (p^k+1)\prod_{i=1}^{2^{a_1-1}-2}(p^k-i)}{2^{a_1-1}\prod_{i=1}^{2^{a_1-1}-2} i},$$

which is p' since $(p^k-i)_p = (i)_p$ for $1 \leqslant i \leqslant 2^{a_1-1}-2 < p^k$ and the remaining factors are prime to p.

Case: $r = 0$ and S is undefined In this case, we have $n = mp$ and $p > 2^{a_t}$, so $mp > 2^{a_t+1} > n$ if $m \geqslant 2$. Hence we must have $n = p = 2w+1$ for some positive integer w. Here, again the sign character lies in both principal blocks, so we wish to exhibit a partition whose corresponding character restricts to the sum of one or two nontrivial members of $\mathrm{Irr}_{2'}(B_2(\mathfrak{A}_n)) \cap \mathrm{Irr}_{p'}(B_p(\mathfrak{A}_n))$. Consider the partition $(1^\ell, p-\ell)$ for ℓ some integer $1 \leqslant \ell < p-1$. This partition certainly corresponds to a character in $\mathrm{Irr}_{p'}(B_p(\mathfrak{S}_p))$, so it suffices to show that there exists some ℓ such that the character also lies in $\mathrm{Irr}_{2'}(\mathfrak{S}_p)$. Note that the character lies in $\mathrm{Irr}(B_2(\mathfrak{S}_p))$ if and only if ℓ is even. The character has degree $\binom{p-1}{\ell}$, so using Lucas's theorem again, we see that such an even ℓ exists yielding an odd-degree character, unless $p-1 = 2^k$. That is, $n = 2^k+1 = p$. In this case, consider the partition $(1^{2^{k-1}}, 2^{k-1}+1)$ - that is, $\ell = 2^{k-1}$. Our assumption $n \geqslant 8$ forces $k \geqslant 2$, so ℓ is even and this character lies in $\mathrm{Irr}(B_2(\mathfrak{S}_n)) \cap \mathrm{Irr}_{p'}(B_p(\mathfrak{S}_n))$. We claim that it has degree divisible by 2 exactly once, so that again the two characters lying in $\mathfrak{A}_n$ below the character corresponding to this self-conjugate partition each lie in $\mathrm{Irr}_{2'}(B_2(\mathfrak{A}_n))$. Indeed, the degree is

$$\frac{\prod_{i=1}^{2^{k-1}}(2^{k-1}+i)}{\prod_{i=1}^{2^{k-1}} i},$$

which satisfies the claim since $2^k/2^{k-1} = 2$ and $(2^{k-1}+i)_2 = (i)_2$ for $1 \leqslant i < 2^{k-1}$.