Important Note: This homework provides a supplement to the exposition of counting provided by the textbook (*Pearson Custom Mathematics Textbook Ch. 1*). These notes are not intended to be a replacement for class notes or an alternative to reading the textbook. A good calculator is an essential companion to these notes and exercises.

1. Three Methods For Counting

When modeling real world systems that have large numbers of possible outcomes, we often need to take account of every possible outcome of an experiment. For example, if one rolls two fair, six-sided dice, there are 36 possible outcomes, listed below:

\[ S = \{ \text{[all possible combinations of dice rolls]} \} \]

In some instances while playing games that involve a pair of six-sided dice, a player is hoping to roll a combination from above that results in a sum of 7 total ‘pips’ on the two dice (a ‘pip’ is one of the dots on a die face, for example \( \langle 5 \rangle \) is one face with 5 pips). By examining the set of outcomes above, we can see that there are exactly six outcomes that result in a sum of 7 total pips on the two dice, namely, the combinations \( \langle 1, 6 \rangle, \langle 6, 1 \rangle, \langle 2, 5 \rangle, \langle 5, 2 \rangle, \langle 3, 4 \rangle \) and \( \langle 4, 3 \rangle \).

Another example of a real world experiment with a large number of possible outcomes is the random selection of a small group in a math class, say a group of three students. For example, if a MTH 1080 class has 35 students enrolled, one might wonder how many distinct groups of three can be formed from students in the class. This turns out to be a large number (6545), but these notes and our textbook explain how to compute this quite easily. In fact, this number can be calculated using a single command on our calculator, namely, the command labeled \( n \text{C}_r \) on most calculators. This command will yield the result \( 35 \text{C}_3 = 6545 \) for the number of groups of 3 that can be chosen from a class with 35 students.

In the previous example, the order in which the students are assigned to the group does not really matter, because the group will be the same even if the same three students are selected in a different order; however, in some applications, the order is important. When order is important, a different counting method is required than the counting method required for problems in which order is unimportant.
Suppose a team of mountaineers is aiming to conquer four famous mountains, Alpamayo, Huascarán Sur, Siula Grande and Ranrapalca, all in Peru. The order in which they climb these mountains is quite important because of weather concerns, travel logistics and varying difficulty of the routes.

The number of ways in which the mountaineers can climb these four mountains turns out to be $4 \times 3 \times 2 \times 1 = 24$. Two common notations calculators use for this calculation are the $n!$ command and the $nPr$ command. For the mountaineers, these commands would proceed in one of the following two ways: we can enter $4! = 24$ or we can enter $4 \text{ nPr } 4 = 24$. Using either approach, we find 24 different orders in which these mountaineers could climb these 4 Peruvian mountains.

The next three sections below will give more detailed presentations of each of the three counting methods described above. Before proceeding to the more technical details in the other sections, work on the introductory exercises below.

**Exercises:**

1.1. How many ways are there of rolling a sum of 9 total pips when rolling two fair six-sided dice? Highlight these possibilities in the set of possible outcomes shown below problem 1.2.

1.2. How many ways are there of rolling both dice with the same number of pips, also known as 'doubles'? Highlight these possibilities in the set of possible outcomes shown below (use a different color to circle these than the one you used for Exercise 1.1, if possible).

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1.3. If there are 25 students in a MTH 1080 class, then how many possible groups of 5 students can we choose from the class? How many groups of 2 students can we choose from the class?

1.4. Explain why a correct way to solve the problem in which the mountaineers are counting the number of orders in which they can climb four Peruvian mountains is to calculate the multiplicative product $4 \times 3 \times 2 \times 1 = 24$. Use complete sentences to explain your answer.

1.5. Suppose the mountaineers climbing in Peru are interested in climbing a total of six mountains while they are there. How many different orders are there in which they can climb all six of the mountains?
2. Counting for dice and coin experiments

There are many real world experiments that require counting techniques to count large numbers of possible outcomes, but dice and coin experiments are a nice place to start because they provide familiar examples.

The key to counting the total number of possible outcomes of these type of experiments is called the multiplication rule. Our textbook provides a thorough description of this rule and how to use it in general.

Suppose we flip three coins, a dime, a nickel and a penny. We can symbolize individual faces of each coin with capitol letters ‘H’ or ‘T’ which represent outcomes of heads or tails, respectively. For each of the three coins (dime, nickel and penny), there are two possible outcomes, so the multiplication rule from our textbook states that there are \(2 \times 2 \times 2 = 8\) possible outcomes.

All eight individual outcomes, in the order dime, nickel then penny, can be listed as follows:

\[ S = \{ HHH, HHT, HTH, THH, TTT, TTH, THT, HTT \} \]

We should make a quick note about the notation used above for mathematical sets; the same notation is used in our set of outcomes for two fair dice. The brackets ‘{’ and ‘}’ are used in mathematics to express sets of objects that are not ordered in any particular way. If you rearranged any of the eight outcomes in our sample space, it does not change the set of objects. The name we are giving that set above is \(S\) (in words, ‘script S’).

The multiplication rule that gave us \(2 \times 2 \times 2 = 8\) possible outcomes may not seem so impressive in this particular example, but it is a powerful rule because it can be generalized to predict the number of possible outcomes for as many coins as may want to flip. This is explored in Exercise 2.1, below.

With coin and dice examples, we also want to be able to count subsets of possible outcomes of these experiments. For example, we may want to count how many three-coin flips result in exactly two tails. By scanning all eight of the possible outcomes listed above, we can see that there are exactly three outcomes with exactly two T’s, namely, TTH, THT and HTT.

There are tricks for counting this type of subset of a coin flip experiment that we will learn in Section 3 of these notes.

We have discussed the possible outcomes of two, fair six-sided dice in Section 1. As with the coin experiment, the multiplication rule can be used to predict the total number of possible outcomes. Since there are six faces on the first die and six faces on the second die, the total number of possible outcomes is \(6 \times 6 = 36\). Exercise 2.3, in the exercises below, considers the case of rolling five six-sided dice instead of only two.

While playing games associated with dice, we are often interested in the likelihood of certain subsets of the set of all possible outcomes. In order to determine the likelihood, we must count the possible outcomes in the subsets we are interested in. If the total number of possible outcomes can be listed, then this is simply a matter of identifying the possible outcomes in the subset of interest.

For example, while playing a dice game, there may be a situation where we need to roll at least one \(\square\). How many ways can this occur? Below, I have highlighted all of the outcomes...
that correspond to “at least one □” and can see from the picture that there are exactly 11 outcomes in this subset.

\[ S = \{ \square \square, \square \circ, \square \diamond, \square \blacklozenge, \square \bigstar, \square \Box, \square \\text{ } \} \]

We may also be interested in the number of outcomes that result in a sum of at least nine pips on the two regular, six-sided dice. Again, by scanning the whole sample space for outcomes that include at least nine pips, we see that there are exactly 10 such outcomes, as seen below.

\[ S = \{ \square \square, \square \circ, \square \circ, \square \bigstar, \square \bigstar, \square \Box, \square \\text{ } \} \]

Exercises:

2.1. Suppose we flip six coins, a penny, nickel, dime, quarter, half-dollar and silver dollar, in that order. In terms of sequences of heads and tails, how many unique outcomes can occur?

2.2. If we flip six coins, as in Exercise 2.1, how many possible outcomes began with a tails and end with a heads (i.e. the penny shows tails and silver dollar shows heads)?

2.3. Suppose we roll four regular, six-sided dice. Use the multiplication rule to predict how many possible outcomes of this experiment are possible.

2.4. If we roll two regular, six-sided dice, then how many ways are there to roll exactly one □? How many ways are there to roll at most one □?

2.5. In the sample space for rolling two regular, six-sided dice, shown below, identify all of the possible outcomes that include least one □ or at least one □.

\[ S = \{ \square \square, \square \circ, \square \circ, \square \bigstar, \square \bigstar, \square \Box, \square \\text{ } \} \]
3. Counting when order is not important (combinations)

Fourteen people are barely surviving on a life raft that is over capacity and is surrounded by sharks in the South Pacific (I apologize if this brings back any bad memories for readers). It is decided that the fourteen people will ‘draw straws’ to pick three passengers to be thrown overboard for the sake of the remaining survivors. How many possible ways can we pick these three passengers and, more importantly, how many of these outcomes include you among the three that are fed to the sharks!?

There is a good mathematical explanation of how to compute this number by hand using just multiplication and division in our textbook, but these notes are focused on the practical way of computing this number with your calculator. The commands on your calculator that allow you to count these types of outcomes practically are $\binom{n}{r}$ and $\text{nPr}$. Both of these were already exemplified in Section 1, but we need only one of these commands in this section. The trick to many of these problems is in identifying which of these two commands to use.

The command $\binom{n}{r}$ counts combinations and the command $\text{nPr}$ counts permutations. Combinations are used when the order in which objects are picked is not important; permutations are used if the order is important. In this section, we look at examples in which order is not important, so we will use the $\binom{n}{r}$ command.

When choosing which three passengers on our life raft will be thrown to the sharks, order is not really too important because all three are going to be eaten by sharks. We have 14 total passengers and we must choose exactly 3 of these 14 passengers, so we calculate the number of combinations of 3 passengers by inputting $14 \binom{n}{r} 3$. Inputting this into our calculator gives us $14 \binom{n}{r} 3 = 364$.

Now, if we are unfortunate enough to be in this situation, how many ways can you be chosen? Since there are 13 other passengers, there are $13 \binom{n}{r} 3 = 286$ groups of 3 passengers that do not include yourself, so it must be that $364 - 286 = 78$ of the groups include yourself. If each group is equally likely, the probability that you are fed to the sharks is $\frac{78}{364} = 21.4\%$

Exercises:

3.1. A sock drawer has 6 remaining socks and none of them match. How many different pairs of unmatched socks can be chosen?

3.2. If we roll four dice, how many ways are there to get exactly two faces showing $\spadesuit$ and the other two faces showing $\clubsuit$? Hint: There are four ‘slots,’ and we need to pick two of the four ‘slots’ to place the $\spadesuit$’s in.

3.3. An art student goes to Paris and wants to visit 6 of the 12 major art museums in Paris. How many ways can she choose 6 of the 12 museums to visit (the order of visits is not important)?
4. COUNTING WHEN ORDER IS IMPORTANT (REARRANGEMENTS AND PERMUTATIONS)

An art student intends to see five different museums in a city they are visiting and wants to visit the museums in an order that is efficient and practical. How many different possible orders of visiting the five museums are possible?

In turns out that we can use the multiplication rule to count the number of possibilities, some of which are likely to be much more practical than others. The number of choices the student will have as to which museum she visits first is 5. After the first museum has been seen, she has 4 choices of remaining museums to visit next. After the fourth museum has been seen, she has only 3 choices remaining and then 2 and then only 1; using the multiplication rule, she has \(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\) possible orders in which she can visit all five of the museums; it would be in her interest to investigate which of these orders would be most practical and efficient.

Our calculator has a command that also computes this quickly; namely, the \(n\text{Pr}\) command that counts permutations. Since she has 4 choices of museums to visit and she will visit all 4 museums, the calculator gives us that she has \(4\text{Pr} 4 = 24\) possible orders in which she can visit the museums.

In more general applications, the number of museums a student visits does not have to include every museum in the city. For example, there are 12 major art museums in Paris. A student visiting Paris may only have time to visit 4 art museums in the time that he is there. To count the number of orders in which he can visit the 4 of the 12 art museums, we input \(12\text{Pr} 4\) into our calculator and find that there are \(12\text{Pr} 4 = 11880\) orders in which he can visit 4 of Paris’ twelve major art museums.

For the previous example, it is interesting to note that there are \(12\text{Cr} 4 = 495\) ways the student can choose 4 of the 12 major art museums in Paris to visit. For each of those choices of 4 museums, there are \(4\text{Pr} 4 = 24\) orders in which he can visit those particular 4 museums; therefore, there are \((12\text{Cr} 4) \times (4\text{Pr} 4) = 495 \times 24 = 11880\) total orders in which he can visit 4 of Paris’ 12 major art museums, just like we got above. Usually we write \(4\text{Pr} 4\) as 4!, so we have just used a special case of the formula below:

\[
(n\text{Cr} r) \times (r!) = n\text{Pr} r
\]

for all values of \(r\) less than or equal to \(n\)

Exercises:

4.1. A delivery person needs to drop off eight packages addressed to eight different locations. In how many different orders can the delivery person drive to the eight locations to deliver the packages?

4.2. In ice cream stand has 17 flavors available. If order of scoops is important, how many different two-scoop ice cream cones with two different flavors are possible? How many three-scoop ice cream cones with three different flavors are possible?

4.3. A human resources manager has 14 candidates they need to interview, but only has time to interview 8 candidates each day. In how many ways can the manager schedule the first 8 interviews if the order in which she schedules the interviews is important?