MTH 1080, 1/23/2019
HOMEWORK 1: EXPONENTIAL GROWTH AND DECAY

The homework problems in this worksheet must be completed **by hand on separate sheets of paper**. This homework assignment is due on the same day as Exam 1, **Wednesday, 2/6/2019.** Multiple pages must be stapled together.

Important Note: The notes below provide a **supplement** to the exposition of exponential growth and decay provided in our textbook, *Using and Understanding Mathematics*, Chapter 8, Sections A and B. These notes are **not** intended to be a replacement for class notes or an alternative to reading the textbook. **A good calculator is an essential companion to these notes and homework problems.**

1. Exponential Growth

Exponential growth describes quantities that grow faster when the quantity is larger. For example, a population of 100 rabbits may grow to a population of about 200 rabbits in six months in an unconfined environment, so if we started with 400 rabbits in the same unconfined environment, we would expect the population to grow to about 800 rabbits over the same six month period. The *doubling time* of the population is the same for 100 rabbits or 400 rabbits: After six months, the population size doubles. In particular, this means that the population growth of rabbits is faster if you start with more rabbits.

The *doubling time* of a population that grows exponentially is symbolized as T_{double}, so our rabbit population in the previous paragraph has $T_{double} = 6$ (in months). The rabbit population will double again over the next six months, so over a year (i.e. two doubling periods), the population doubles itself twice or *quadruples* in size. A population of 100 rabbits will grow to about 400 rabbits over the course of a year if $T_{double} = 6$ months.

We have a mathematical equation for exponentially growing quantities. Our textbook uses the phrases "new value" and "initial value" in their equations. We will use the symbol A for "new value" and the symbol P_0 for "initial value" in the equations on this worksheet. We symbolize time with t, using the same time units that are used to express T_{double} (for rabbits the time units were months). The present value, A, of an exponentially growing quantity at time t is given as follows.

Formula for exponential growth:

$$A = P_0 \times 2^{(t/T_{double})}$$

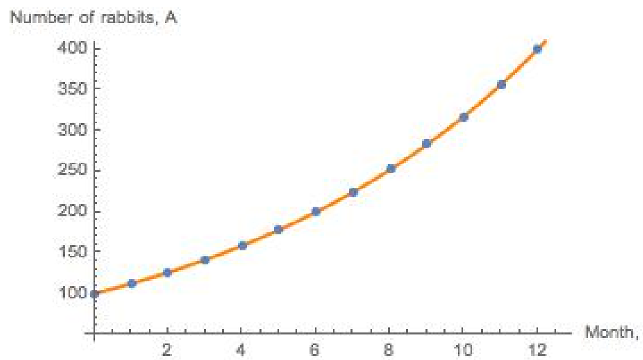
For our rabbit example, the doubling time is six months so $T_{double} = 6$. We can use $P_0 = 100$ if our rabbit population starts with 100 rabbits. In this case, the number of rabbits after t months is given by the equation,

$$A = 100 \times 2^{(t/6)} \qquad \text{where } t \text{ is in months}$$

To see precisely how our rabbit population grows during the first year, we can calculate A for each of the values $T = 1$ through $T = 12$. Turn the page to see the result of these calculations.

Please use your own calculator to check that the following table gives the correct values of A when inserting the values $t = 1, 2, 3, \ldots$ 10 into the equation $A = 100 \times 2^{(t/6)}$:

Month, t	**No. of Rabbits, A**
0	100
1	112
2	126
3	141
4	159
5	178
6	200
7	224
8	252
9	283
10	317
11	356
12	400

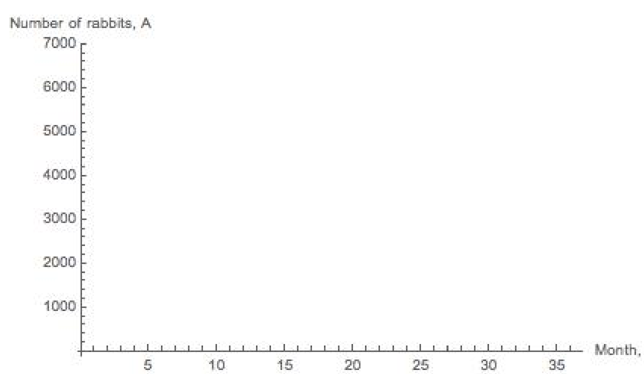


The graph on the right side of the table above gives a plot of the pairs of t (in months) and A (in rabbits) given in each row of the table. This plot is done in *rectangular coordinates*: the horizontal position of each point represents the month, t, and the vertical position of each point represents the number of rabbits, A. The line between the points gives us a visualization of the overall pattern formed by connecting subsequent dots.

Exercises:

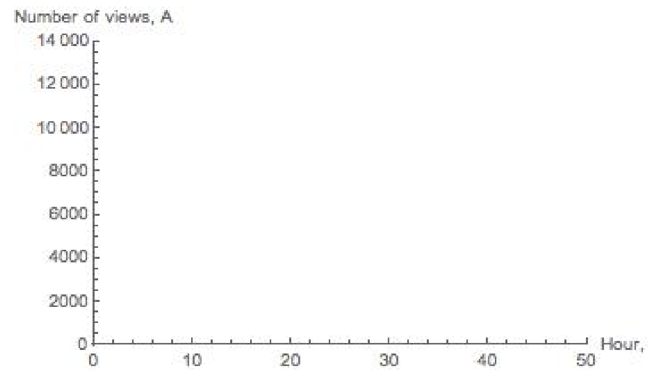
1.1. Consider the example we looked at above where an initial population of 100 rabbits grows with a doubling time of six months. Complete the following table for number of rabbits for each three months over the next three years, then plot the points in rectangular coordinates, at right, connecting the dots to visualize the overall pattern.

Month, t	**No. of Rabbits, A**
0	100
3	
6	
9	
12	
15	
18	
21	
24	
27	
30	
33	
36	



1.2. After being viewed 200 times, a popular video 'goes viral' on the internet. The number of views of the video has a doubling time of eight hours, so $T_{double} = 8$. Calculate the number of views over the next 2 days using the table below, on the left. On the right, plot the table in rectangular coordinates and connect the dots to visualize the overall pattern.

Hour, t	**Views, A**
0	200
4	
8	
12	
16	
20	
24	
28	
32	
36	
40	
44	
48	



1.3. The value of a lucrative stock grows exponentially and doubles in value every 8.5 years. How much is \$2,500 of the stock worth after 10 years?

2. Exponential Decay

The exponentially growing quantities we investigated in the previous section grew quickly because they doubled in value over fixed time periods. The *exponential decaying* quantities we investigate in this section *decrease* more slowly with time because they decay over fixed periods of time. For exponentially decaying quantities, these fixed periods of decay are called half-lives and are denoted T_{half}.

We have a mathematical equation for these exponentially decaying quantities that is very similar to our exponential growth formula above. As stated above, we use the symbol A and P_0 instead of the phrases "new value" and "initial value," respectively, that are used in the textbook. We symbolize time with t and input values of t in the same units as the half life, T_half. The present value, A, of an exponentially decaying quantity at time t is given as follows.

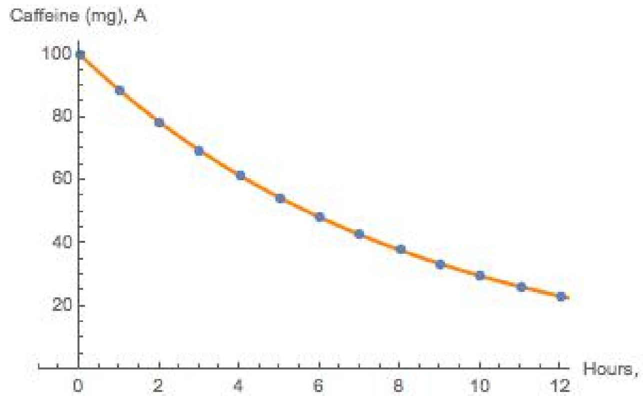
Formula for exponential decay:

$$A = P_0 \times (0.5)^{\left(t/T_{half}\right)}$$

The compound caffeine present in coffee and tea is known to have a half-life of about 5.7 hours (5 hours and 42 minutes) when ingested by humans. A typical cup of coffee has about 100 mg of caffeine. This means that if you drink a cup of coffee at 7 am, then you will still have 50 mg of caffeine in your body at 12:42 am. In fact, we can calculate and graph the amount of caffeine remaining in the human body at any time we choose using our equation above. The table on the following includes clock times, the number of hours since 7 am and the values of the amount of caffeine remaining in the body, A, for each of the elapsed times.

Please use your own calculator to check that the following table gives the correct values of A when inserting the values $t = 1, 2, 3, \ldots$ 10 into the equation $A = 100 \times (.5)^{(t/5.7)}$:

Clock time	**Hour,** t	**Caffeine (mg),** A
7 am	0	100
8 am	1	88.5
9 am	2	78.4
10 am	3	69.4
11 am	4	61.5
12 pm	5	54.4
1 pm	6	48.2
2 pm	7	42.7
3 pm	8	37.8
4 pm	9	33.5
5 pm	10	29.6
6 pm	11	26.2
7 pm	12	23.2

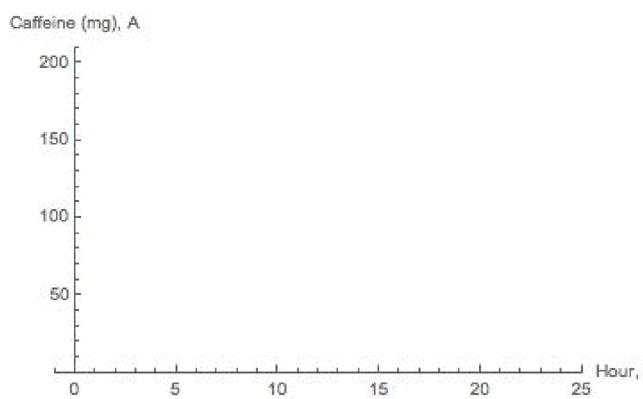


The picture above, on the right, is a graph of our data using rectangular coordinates.

Exercises:

2.1. Consider the example we looked at above concerning the amount of caffeine that remains in the body over the course of a day after drinking one cup of coffee. Suppose instead that we drink 2 cups of coffee at 7 am and ingest a total of 200 mg of caffeine. Recall that the half-life of the amount of caffeine remaining in the body is $T_{half} = 5.7$. Compute the amount of caffeine remaining in our body at each 2 hour interval over the next 24 hours

Clock time	**Hour,** t	**Caffeine (mg),** A
7 am	0	200
9 am	2	
11 am	4	
1 pm	6	
3 pm	8	
5 pm	10	
7 pm	12	
9 pm	14	
11 pm	16	
1 am	18	
3 am	20	
5 am	22	
7 am	24	



2.2. Suppose instead that we drink 1 cup of coffee (100 mg of caffeine) at 7 am and another cup of coffee (100 mg of caffeine) at 1 pm. What amount of caffeine remains in our body at 7 pm? Hint: Add the amount of caffeine left after 12 hours from the first cup of coffee to the amount of caffeine left after 6 hours from the second cup of coffee.

3. Student Project

The last component of this homework assignment is unique to each student. Your assignment is to identify a product meant for human consumption that contains a drug with a known half-life. This product could be a can of soda, cup of coffee, energy drink, pain medication or a THC/CBD product. The product needs to contain one of the drugs listed in the table below and the amount of that drug a serving contains. **It is not necessary for you to consume or to have consumed the product you identify**.

The following (approximate) half-lives of caffeine, acetaminophen (e.g. Tylenol), THC and CBD are given below:

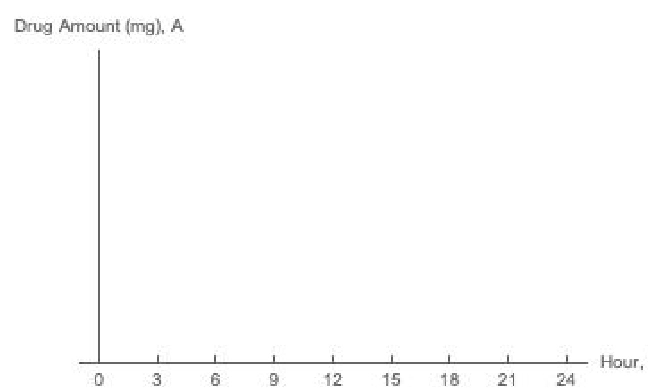
Drug	Half-life in hours
Caffeine	$T_{half} \approx 5.7$
Nicotine	$T_{half} \approx 2$
Acetaminophen	$T_{half} \approx 5$
THC	$T_{half} \approx 32$
CBD	$T_{half} \approx 25$

Exercises:

3.1. Identify the product you will consider for this component of the homework; do not use the same exact product as another student. Identify the amount of the drug (Caffeine, Nicotine, Acetaminophen, THC or CBD) the product contains. Usually these amounts are given in milligrams (mg).

3.2. Assuming that a typical person ingests the product you identified above at 12pm, complete the following table and graph that shows how much of that product remains in their system over the next 24 hour period.

Clock time	**Hour,** t	**Amount (mg),** A
12 pm	0	
3 pm	3	
6 pm	6	
9 pm	9	
12 am	12	
3 am	15	
6 am	18	
9 am	21	
12 pm	24	



3.3. After a typical person ingests the product you identified, estimate the minimum number of hours that need to pass before ...

(1) ...less than 10% of the drug from that product remains in their system.
(2) ...less than 5% of the drug from that product remains in their system.
(3) ...less than 1% of the drug from that product remains in their system.