MTH 1080, SPRING 2020
HOMEWORK 1 AND WORKSHEET:
EXPONENTIAL GROWTH AND DECAY

The homework problems in this worksheet must be completed **by hand on separate sheets of paper**. This homework assignment is due on the same day as Exam 1, **Wednesday, 2/10/20.** Multiple pages must be stapled together and the original worksheet **should not** be attached to your homework submission. **A good calculator is an essential companion to these notes and homework problems.**

1. Exponential Growth (doubling time formula)

Exponential growth describes quantities that grow faster as the quantity gets larger. More precisely, growth is called **exponential** if the growth rate is proportional to quantity size.

For example, a population of 100 rabbits may grow to a population of about 200 rabbits in six months in an unconfined environment, and if we started with 400 rabbits in the same unconfined environment, it is natural to expect the population to grow to about 800 rabbits over the same six month period (why?). Another way to say this is the *doubling time* of the population is the same for 100 rabbits or 400 rabbits: After six months, the population size doubles. Mathematically, the rabbit population *growth rate* is proportional to the number of rabits.

When a quantity like the rabbit population grows exponentially, there is a fixed time for which any initial quantity will double in size. The **doubling time** of a population that grows exponentially is symbolized as T_{double}, so our rabbit population in the previous paragraph has $T_{double} = 6$ (months). The rabbit population will double in size again over the next six months, so over a year (i.e. two doubling periods), the population doubles itself twice or *quadruples* in size. A population of 100 rabbits will grow to about 400 rabbits over the course of a year if $T_{double} = 6$ months.

There are mathematical equations we can write down for exponentially growing quantities like this one. We use the symbol A for "new value" and the symbol P_0 for "initial value" in our equation. We symbolize time with t, using the same time units that are used to express T_{double} (for rabbits the time units were months). The formulate relates present value, A, of an exponentially growing quantity to the time, t.

Formula for exponential growth with doubling time, T_{double}:

$$A = P_0 \times 2^{(t/T_{double})}$$

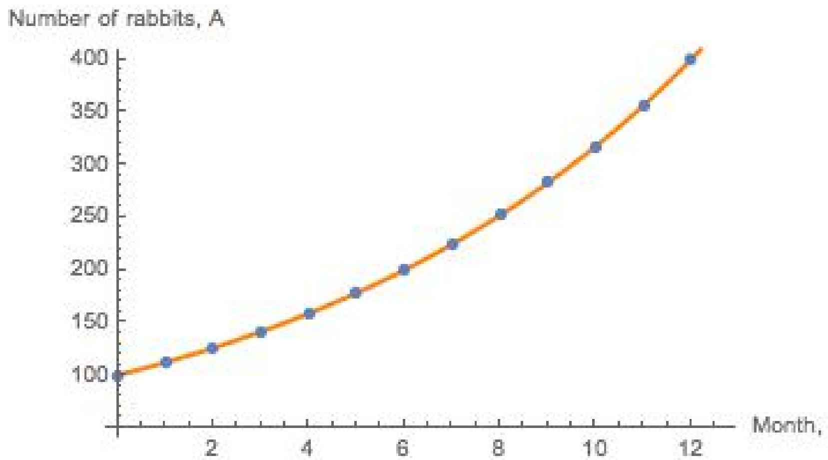
For our rabbit example, the doubling time is six months so $T_{double} = 6$. We can use $P_0 = 100$ if our rabbit population starts with 100 rabbits. In this case, the number of rabbits after t months is given by the equation,

$$A = 100 \times 2^{(t/6)} \qquad \text{where } t \text{ is in months}$$

To see precisely how our rabbit population grows during the first year, we calculate A for each of the times $t = 1$ through $t = 12$.

Please use your own calculator to check that the following table gives the correct values of A when inserting the values $t = 1, 2, 3, \ldots$ 10 into the equation $A = 100 \times 2^{(t/6)}$:

Month, t	**Rabbits, A**
0	100
1	112
2	126
3	141
4	159
5	178
6	200
7	224
8	252
9	283
10	317
11	356
12	400

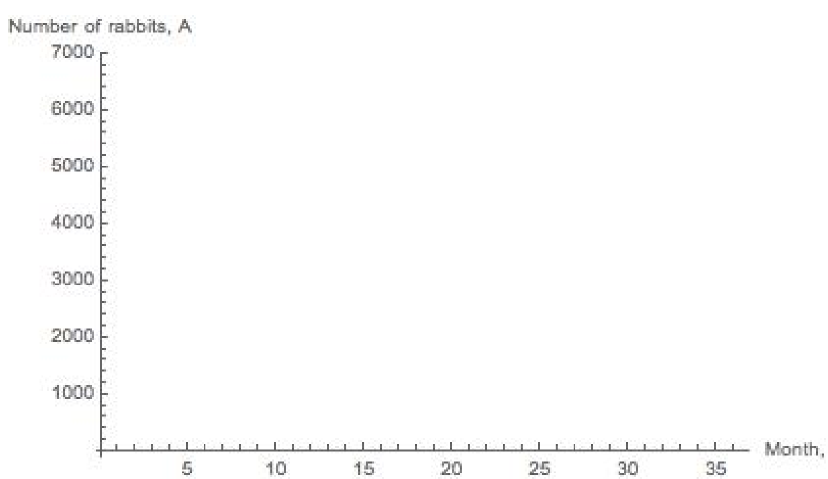


The graph on the right side of the table above gives a plot of the pairs of t (in months) and A (in rabbits) given in each row of the table. This plot is done in *rectangular coordinates*: the horizontal position of each point represents the month, t, and the vertical position of each point represents the number of rabbits, A. The line between the points gives a visualization of the

Exercises:

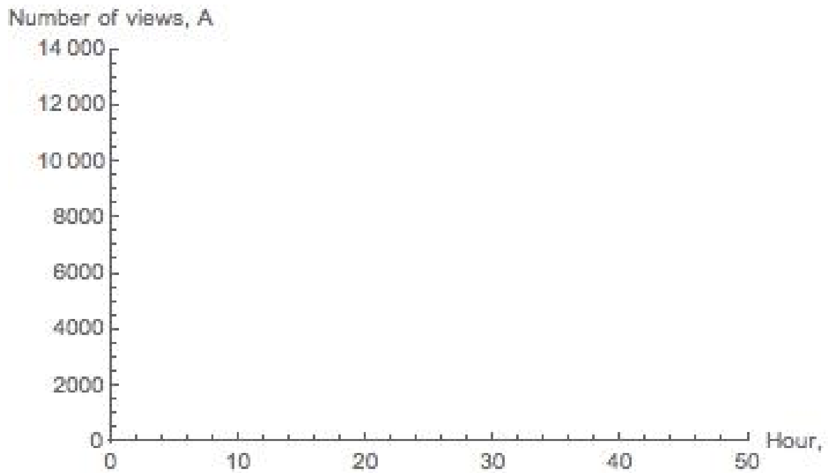
1.1. Consider the example we looked at above where an initial population of 100 rabbits grows with a doubling time of six months. Complete the following table for number of rabbits for each three months over the next three years, then plot the points in rectangular coordinates, at right, connecting the dots to visualize the overall pattern.

Month, t	**Rabbits, A**
0	100
3	
6	
9	
12	
15	
18	
21	
24	
27	
30	
33	
36	



1.2. After being viewed 200 times, a popular video 'goes viral' on the internet. The number of views of the video has a doubling time of eight hours, so $T_{double} = 8$. Calculate the number of views over the next 2 days using the table below, on the left. On the right, plot the table in rectangular coordinates and connect the dots to visualize the overall pattern.

Hour, t	**Views,** A
0	200
4	
8	
12	
16	
20	
24	
28	
32	
36	
40	
44	
48	



1.3. The value of a lucrative stock grows exponentially and doubles in value every 8.5 years. How much is $\$2,500$ of the stock worth after 10 years?

2. Exponential Growth (rate of growth formula)

Exponential growth can also be described in terms of growth rate, r, instead of doubling time, T_{double}. This is the equation used in our textbook, *Math in Society*, in the chapter titled Exponential Growth. Specifically, if r is the exponential growth rate (in decimal), A is the "new value" , P_0 is the "initial value" and time is t, then we have the following formula that relates the quantities.

Formula for exponential growth in terms of growth rate, r:

$$A = P_0 \times (1 + r)^t$$

For example, we can describe the population of 100 rabbits considered in the previous section using a growth rate of 12.25% per month. The decimal version of this that needs to be used for the equation is $r = 0.1225$. One finds that the effect is the same: every six months, the population size doubles.

Using $P_0 = 100$ for an initial population of 100 rabbits. In this case, the number of rabbits after t months is given by the equation,

$$A = 100 \times (1 + .1225)^t \qquad \text{where } t \text{ is in months}$$

To see precisely how our rabbit population grows during the first year, we can calculate A for each of the values $t = 1$ through $t = 12$ (see next page).

Please use your own calculator to complete the following table gives the correct values of A when inserting the values $t = 1, 2, 3, \ldots$ 10 into the equation $A = 100 \times (1 + .1225)^t$:

Month, t	**Rabbits,** A
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Exercises:

2.1. The population of a small city grows exponentially with a growth rate of 4.6% per year. Suppose the initial population of this city is 1,345,000 people. Complete the following table that predicts the population size every three years for the next 36 years.

Year, t	**No. of People,** A
0	1,345,000
3	
6	
9	
12	
15	
18	
21	
24	
27	
30	
33	
36	

2.2. The value of a stock grows exponentially with a growth rate of 5.75%. Determine the value of a 5,000 investment in this stock after 16 years.

2.3. The amount of money that a stock market investment is worth grows exponentially with a growth rate of 7.22%. Construct a graph that shows the value of this stock each year for the next 10 years.

3. Exponential Decay

The exponentially growing quantities we investigated in the previous section grew quickly because they doubled in value over fixed time periods. The *exponential decaying* quantities we investigate in this section *decrease* more slowly with time because they decay over fixed periods of time. For exponentially decaying quantities, these fixed periods of decay are called half-lives and are denoted T_{half}.

We have a mathematical equation for these exponentially decaying quantities that is very similar to our exponential growth formula above. As stated above, we use the symbol A and P_0 instead of the phrases "new value" and "initial value," respectively, that are used in the textbook. We symbolize time with t and input values of t in the same units as the half life, T_half. The present value, A, of an exponentially decaying quantity at time t is given as follows.

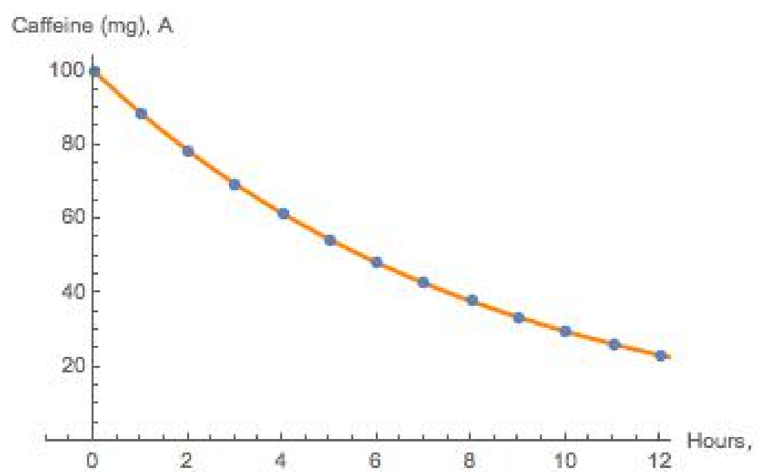
Formula for exponential decay:

$$A = P_0 \times (0.5)^{\left(t/T_{half}\right)}$$

The compound caffeine present in coffee and tea is known to have a half-life of about 5.7 hours (5 hours and 42 minutes) when ingested by humans. A typical cup of coffee has about 100 mg of caffeine. This means that if you drink a cup of coffee at 7 am, then you will still have 50 mg of caffeine in your body at 12:42 am. In fact, we can calculate and graph the amount of caffeine remaining in the human body at any time we choose using our equation above. The table on the following includes clock times, the number of hours since 7 am and the values of the amount of caffeine remaining in the body, A, for each of the elapsed times.

Please use your own calculator to check that the following table gives the correct values of A when inserting the values $t = 1, 2, 3, \ldots$ 10 into the equation $A = 100 \times (.5)^{(t/5.7)}$:

Clock time	**Hour, t**	**Caffeine, A**
7 am	0	100
8 am	1	88.5
9 am	2	78.4
10 am	3	69.4
11 am	4	61.5
12 pm	5	54.4
1 pm	6	48.2
2 pm	7	42.7
3 pm	8	37.8
4 pm	9	33.5
5 pm	10	29.6
6 pm	11	26.2
7 pm	12	23.2

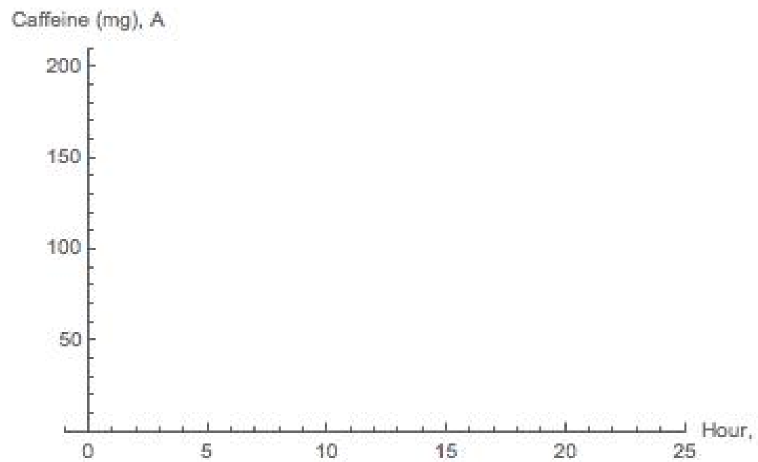


The picture above, on the right, is a graph of our data using rectangular coordinates.

Exercises:

3.1. Consider the example we looked at above concerning the amount of caffeine that remains in the body over the course of a day after drinking one cup of coffee. Suppose instead that we drink 2 cups of coffee at 7 am and ingest a total of 200 mg of caffeine. Recall that the half-life of the amount of caffeine remaining in the body is $T_{half} = 5.7$. Compute the amount of caffeine remaining in our body at each 2 hour interval over the next 24 hours

Clock time	**Hour,** t	**Caffeine,** A
7 am	0	200
9 am	2	
11 am	4	
1 pm	6	
3 pm	8	
5 pm	10	
7 pm	12	
9 pm	14	
11 pm	16	
1 am	18	
3 am	20	
5 am	22	
7 am	24	



3.2. Suppose instead that we drink 1 cup of coffee (100 mg of caffeine) at 7 am and another cup of coffee (100 mg of caffeine) at 1 pm. What amount of caffeine remains in our body at 7 pm? Hint: Add the amount of caffeine left after 12 hours from the first cup of coffee to the amount of caffeine left after 6 hours from the second cup of coffee.

4. Student Project

The last component of this homework assignment is unique to each student. For these problems, we simulate an experiment that mathematical biologists call a "birth process". Birth processes are used by biologists to model rapid cellular reproduction events, and represent a component of many other biological growth models.

Exercises:

4.1. Collect 20 coins and complete the following step-by-step experiment:

(1) Flip one coin.
(2) If the coin shows heads, flip one coin for the next round. Otherwise, if the coin shows tails, flip two coins in the next round.
(3) In each subsequent round of coin flipping, flip all coins and add an extra coin for every coin that shows tails.
(4) Repeat the previous step until you have no coins left to add, keeping track of how many coins are flipped to start each round.

Different students will have experiments that end in different numbers of rounds. While completing your own experiment, construct a table that shows how many coins are flipped in each round for all the rounds required to complete the experiment. Your table should start as follows:

Round	Number of coins
1	1
2	?
3	?
4	?
⋮	⋮

4.2. How many coins were flipped in the last round that involved at most 5 coin flips? How many more rounds did it take for this number of coin flips to double?

4.3. How many coins were flipped in the last round that involved at most 10 coin flips? How many more rounds did it take for this number of coin flips to double?

4.4. This experiment does not demonstrate exact exponential growth, but it does demonstrate something 'close' to exponential growth and is more representative of real world biological growth models. With this in mind, approximate the doubling time of this growth process. Explain your reasoning using complete sentences.

4.5. Based on your doubling time estimate from the previous part, predict how many coins would be in use if you continued this experiment for 100 rounds with no coin limit.

4.6. Based on your doubling time estimate from the previous part, predict how many rounds would be necessary before at least one million coins would be in use.