Notes

Probability and Statistics

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Topics

Notes

1. Measures of Center

2. Measures of Variation

Objectives

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Objectives:

- Compute and interpret the mean and median of a data set
- Know the properties of the mean and median
- Compute and interpret the variance, standard deviation, and interquartile range of a data set
- Know the properties of the variance, standard deviation, and interquartile range
- Construct a boxplot

Measures of Center (1.3)

Notes

- A **sample** of size n will be denoted

$$x_1, x_2, \ldots, x_n$$

- The **sample mean**, denoted $\bar{x}$, is the arithmetic average of the observations in the data set:

> **Sample Mean**:
>
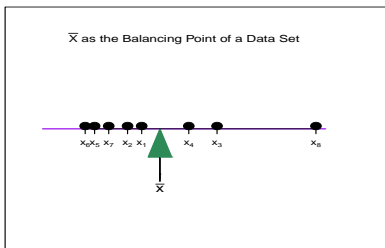> $$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Example

A person's metabolic rate is the rate at which the body consumes energy while at rest. The data below are the metabolic rates (in calories per day) for 7 men who took part in a study of dieting and weight loss.

1792   1666   1362   1614   1460   1867   1439

The sample mean is

$$\bar{x} = \frac{1792 + 1666 + 1362 + 1614 + 1460 + 1867 + 1439}{7}$$
$$= 1600.$$

- The mean is the **"balancing point"** of a data set – weights placed at positions $x_1, x_2, \ldots, x_n$ along a (weightless) axis would balance at $\bar{x}$.

$\overline{X}$ as the Balancing Point of a Data Set

Proposition

Suppose $x_1, x_2, \ldots, x_n$ are a sample and c is any nonzero constant.

1. If
$$y_1 = x_1 + c, \quad y_2 = x_2 + c, \quad \ldots, \quad y_n = x_n + c$$
then
$$\bar{y} = \bar{x} + c.$$

2. If
$$y_1 = cx_1, \quad y_2 = cx_2, \quad \ldots, \quad y_n = cx_n$$
then then
$$\bar{y} = c\bar{x}.$$

where $\bar{x}$ and $\bar{y}$ are the sample means of the x's and y's.

Proposition

Suppose $x_1, x_2, \ldots, x_n$ are a sample. Then

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0.$$

Example (Cont'd)

Here are the metabolic rates data again

1792   1666   1362   1614   1460   1867   1439

The deviations away from the mean (1600) sum to 0:

$$192 + 66 + (-238) + 14 + (-140) + 267 + (-161) = 0.$$

- We say the n deviations

$$x_1 - \bar{x}, \quad x_2 - \bar{x}, \quad \ldots, \quad x_n - \bar{x}$$

have only $n - 1$ **degrees of freedom** because any $n - 1$ of them determines the remaining one.

- For a finite population with N individuals, the **population mean**, denoted μ, is defined as

> **Population Mean**:
>
> $$\mu \;=\; \frac{1}{N} \sum_{i=1}^{N} x_i.$$

- Later we'll see a more general way to compute a population mean.
- A sample mean $\bar{x}$ is used to **estimate** an unknown population mean μ.

- The **sample median** (or **50th percentile**), denoted $\tilde{x}$, is the middle value in the (ordered) sample.

> **Sample Median**:
>
> $$\tilde{x} = \begin{cases} \text{The } (\frac{n+1}{2})\text{th ordered value if } n \text{ is odd.} \\ \text{The average of the } (\frac{n}{2})\text{th and the } (\frac{n+2}{2})\text{th} \\ \text{ordered values if } n \text{ is even.} \end{cases}$$

Example (Cont'd)

Here are the metabolic rates **ordered** from smallest to largest.

$$1362 \quad 1439 \quad 1460 \quad 1614 \quad 1666 \quad 1792 \quad 1867$$

Because $n = 7$ is **odd**, the sample median is the middle value,

$$\tilde{x} \;=\; 1614.$$

Example

Here are data on the lifetimes (in hours) of a certain type of light bulb, **ordered** from smallest to largest.

| 898 | 964 | 970 | 983 | 1003 | 1016 | 1022 | 1029 | 1058 | 1085 |

Because $n = 10$ is **even**, the sample median is the average of the two middle values,

$$\tilde{x} \;=\; \frac{1003 + 1016}{2} \;=\; 1009.5.$$

Notes

- The sample median is the **"middle point"** of a data set – it splits the data set in half.

$\bar{x}$ as the Midpoint of a Data Set

Notes
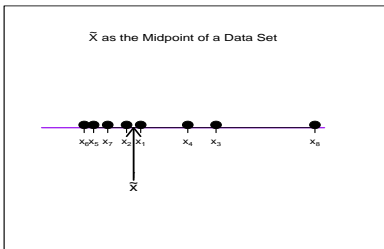
- Both mean and median give some idea of where the data are centered, but they're usually different.

$\bar{x}$ as the Balancing Point of a Data Set
and $\bar{x}$ as the Midpoint

Notes
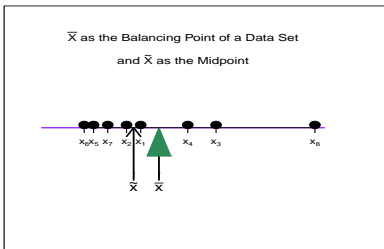
- Why might they be different?
  - The median is not affected by a few outliers (it's *resistant*)
  - The mean can be affected by a few outliers (it's *not resistant*)
  - The median divides the area of the histogram in half
  - The mean is the "balancing point" of the histogram

Symmetric Distribution $\overline{X} \approx \tilde{X}$

$\overline{X} = 51.0$
$\tilde{X} = 50.6$

20   40   60   80
$\overline{X} = \tilde{X}$

Right Skewed Distribution $\overline{X} > \tilde{X}$

$\overline{X} = 80.1$
$\tilde{X} = 51.3$
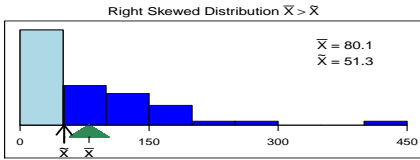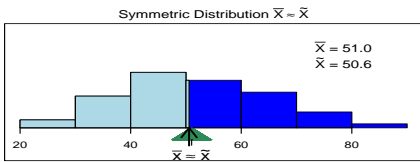
0   150   300   450
$\tilde{X}$ $\overline{X}$

- If the histogram is fairly symmetric, the mean and median will be similar
- If the histogram is right skewed, the mean will be larger
- Comments:
  - If the mean and median are not similar, make sure you report them both
  - The mean is most commonly used in testing and estimation (later in the course)

Measures of Variation (1.4)

- The **sample variance**, denoted s^2, is defined as

Sample Variance:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

- It represents the size of a **typical squared deviation** away from the mean.
- It's measured in the *squares* of the original units.
- We'll see later why we use $n-1$ instead of n.

- The **sample standard deviation**, denoted s, is the square root of the variance.

Sample Standard Deviation:

$$s = \sqrt{s^2} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

- It measures the size of a **typical deviation** away from the mean.
- It's is measured in the original units.

Example

Here are the metabolic rates data again.

| 1792 | 1666 | 1362 | 1614 | 1460 | 1867 | 1439 |

Recall that $\bar{x} = 1600$ and the deviations away from 1600 are

| 192 | 66 | -238 | 14 | -140 | 267 | -161 |

The sample variance is

$$s^2 = \frac{1}{6}(192^2 + 66^2 + (-238)^2 + 14^2 + (-140)^2 + 267^2 + (-161)^2)$$
$$= 35,811.7.$$

The standard deviation is

$$s = \sqrt{35,811.7} = 189.2$$

calories per day.

Metabolic Rates Data Set

| 1300 | 1400 | 1500 | | 1700 | 1800 | 1900 |

$\bar{x} = 1600$
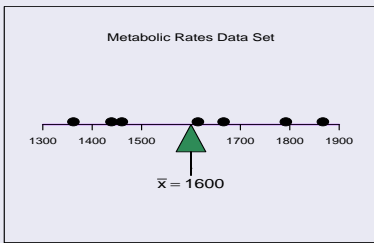
- $s = 0$ if and only if $x_1, x_2, \ldots, x_n$ are all the same value.
- Otherwise $s > 0$, with larger values of s indicating more variation in the data.
- s^2 and s are **not resistant** to outliers.

Proposition

Suppose $x_1, x_2, \ldots, x_n$ are a sample and c is any nonzero constant.

1. If
$$y_1 = x_1 + c, \quad y_2 = x_2 + c, \quad \ldots, \quad y_n = x_n + c$$

   then
   $$s_y^2 = s_x^2 \qquad \text{and} \qquad s_y = s_x.$$

2. If
$$y_1 = cx_1, \quad y_2 = cx_2, \quad \ldots, \quad y_n = cx_n$$

   then
   $$s_y^2 = c^2 s_x^2 \qquad \text{and} \qquad s_y = |c| s_x.$$

where s_x^2 and s_y^2 are the sample variances of the x's and y's.

- For a finite population with N individuals, the **population variance** and **population standard deviation** , denoted σ^2 and σ, are

**Population Variance and Standard Deviation**:

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$$

and

$$\sigma = \sqrt{\sigma^2}.$$

- Later we'll see a more general way to compute a population variance and standard deviation.

- A sample variance s^2 is used to **estimate** an unknown population variance σ^2 (and s is used to estimate σ).
- We divide by $n - 1$ when computing s^2 but by N when computing σ^2.
- Why? If we divided by n, s^2 would tend to **underestimate** σ^2.
- Dividing by $n - 1$ compensates for this tendency to under estimate σ^2.
- Why would s^2 underestimate σ^2 if we divided by n? The deviations $x_i - \bar{x}$ used to compute s^2 tend to be **smaller** than the deviations $x_i - \mu$ used to compute σ^2.
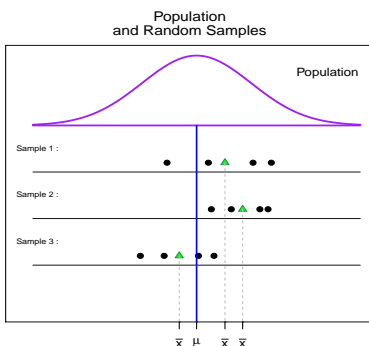
- It turns out that the value of c that makes the quantity

$$\sum_{i=1}^{n}(x_i - c)^2$$

as small as possible is $c = \bar{x}$, so

$$\sum_{i=1}^{n}(x_i - \bar{x})^2$$

will be smaller than

$$\sum_{i=1}^{n}(x_i - \mu)^2$$

- It turns out that the value of c that makes the quantity

$$f(c) \;\; = \;\; \sum_{i=1}^{n}(x_i - c)^2$$

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- It turns out that the value of c that makes the quantity

$$f(c) \;\; = \;\; \sum_{i=1}^{n}(x_i - c)^2 \qquad f'(c) \;=\; 0$$

as small as possible is $c = \bar{x}$, so

$$\sum_{i=1}^{n}(x_i - \bar{x})^2$$

will be smaller than

$$\sum_{i=1}^{n}(x_i - \mu)^2$$

- The *interquartile range* (or *fourth spread*), denoted IQR, is the range of the middle 50% of the data:

> **Interquartile Range**:
>
> $$IQR = Q_3 - Q_1$$
>
> where Q_1 is the *first quartile* (or *25th percentile*) and Q_3 is the *third quartile* (or *75th percentile*), defined by:
>
> $Q_1 \;\; = \;\;$ The median of the observations that are less than or equal to the overall median $\tilde{x}$
>
> $Q_3 \;\; = \;\;$ The median of the observations that are greater than or equal to the overall median $\tilde{x}$

Notes

Example

Here are typical onset times (in days) for locomotion in $n = 11$ different mammal species.

Species	Locomotion Begins (days)
Homo Sapiens	360
Gorilla gorilla	165
Felis catus	21
Canis familiaris	23
Rattus norvegicus	11
Turdus merula	18
Macaca mulatta	18
Pan troglodytes	150
Saimiri sciurens	45
Cercocebus alb.	45
Tamiasciureus hud.	18

Notes

Here are the data again, but **ordered**.

$$11 \quad 18 \quad 18 \quad 18 \quad 21 \quad 23 \quad 45 \quad 45 \quad 150 \quad 165 \quad 360$$

The quartiles are

$$\begin{aligned}
Q_1 &= \text{Median of } 11, 18, 18, 18, 21, 23 \\
&= 18 \\
Q_3 &= \text{Median of } 23, 45, 45, 150, 165, 360 \\
&= 97.5
\end{aligned}$$

so the interquartile range is

$$IQR = 97.5 - 18 = 79.5.$$

Notes

- The interquartile range is **resistant** to outliers.

- The *five number summary* of a data set is

> **Five Number Summary**:
>
> $$\text{Minimum}, \quad Q_1, \quad \tilde{x}, \quad Q_3, \quad \text{Maximum}$$

- It summarizes the center, spread, *and* (we'll see) skewness of the data.

Notes

- A **boxplot** depicts the five number summary graphically, and consists of:
  - Vertical axis whose units are the those of the data values.
  - A box whose top is level with Q_3 and whose bottom is level with Q_1.
  - A horizontal line through the box level with $\tilde{x}$.
  - "Whiskers" (vertical lines) extending from the box up to the largest data value and from the bottom of the box down to the smallest (unless there are outliers).

Example

On November 28, 2011 a spill of toxic materials from the Suncor Energy oil refinery north of Denver was discovered seeping into Sand Creek.

The Colorado Department of Public Health and Environment monitored the spill by measuring benzene concentrations (ppb) on 13 days at two locations, the confluence of Sand Creek and the South Platte River, and a location on the South Platte farther downstream. The data are below.

Date	Benzene at Confluence with Sand Creek	Benzene Downstream from the Confluence
Dec. 27	640	190
Dec. 28	240	300
Dec. 29	140	130
Dec. 30	190	130
Dec. 31	170	160
Jan. 2	300	240
Jan. 3	730	250
Jan. 4	630	240
Jan. 5	650	240
Jan. 6	190	590
Jan. 7	310	260
Jan. 8	400	260
Jan. 9	720	240

The five number summary of the data from the first location is

Min	Q_1	$\tilde{x}$	Q_3	Max
140	190	310	640	730

and the boxplot (with labels) is shown below.
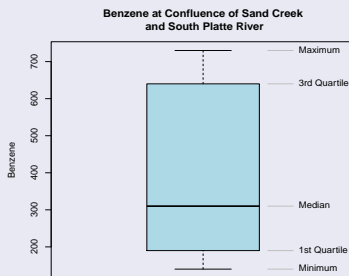
Boxplots are most often used to compare two or more samples.

Benzene at the Confluence and Downstream

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Benzene at the Confluence and Downstream

$> 1.5 \times IQR$

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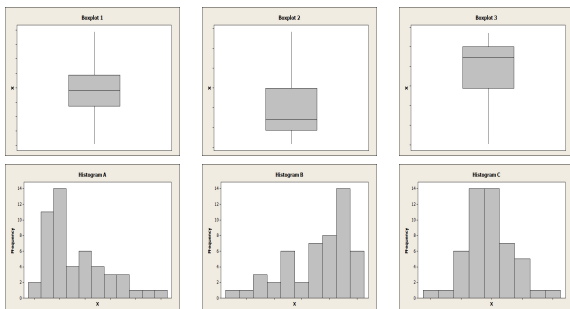Benzene at the Confluence and Downstream

$< 1.5 \times IQR$

$> 1.5 \times IQR$
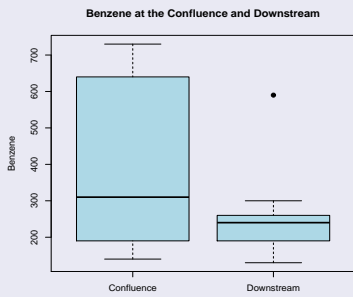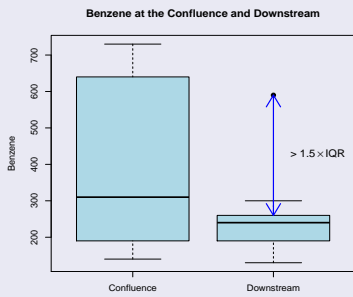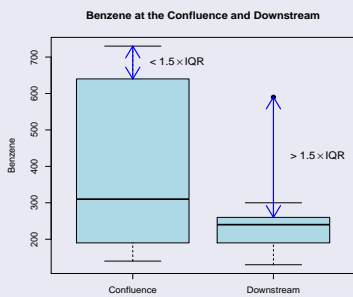
Notes

- Right and left skewed data sets have **asymmetrical** boxplots. The plots below are in jumbled order. Can you match the boxplot with its histogram?