Probability and Statistics

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Topics

Objectives

Objectives:

- Recognize Bernoulli, binomial, and geometric random variables.
- Compute probabilities involving Bernoulli, binomial, and geometric random variables.
- Compute and interpret the expected value, variance, and standard deviation of Bernoulli, binomial, and geometric random variables.

Bernoulli Random Variables (3.1)

- Any random variable X whose only possible values are $\mathbf{0}$ and $\mathbf{1}$ is called a ***Bernoulli random variable***.

Example

You take a pass/fail exam. You either pass or fail. Let

$$X = \begin{cases} 1 & \text{if you pass} \\ 0 & \text{if you fail} \end{cases}$$

Then X is a **Bernoulli random variable.** If you pass with 70% chance, then the **pmf** of X is

$$\begin{aligned} p(1) &= P(\text{you pass}) = \mathbf{0.7} \\ p(0) &= P(\text{you fail}) = \mathbf{0.3} \end{aligned}$$

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- In general, the so-called **success probability** is denoted by $\boldsymbol{p}$ and is called a ***parameter*** of the distribution.

Bernoulli(p) Pmf:

$$\begin{aligned} p(1) &= p \\ p(0) &= 1 - p \end{aligned}$$

which can be written as

$$p(x) = p^x (1-p)^{1-x} \qquad \text{for} \quad x = 0,\ 1.$$

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- The notation

$$\boldsymbol{X} \sim \textbf{Bernoulli}(\boldsymbol{p})$$

  means X follows a Bernoulli(p) distribution.

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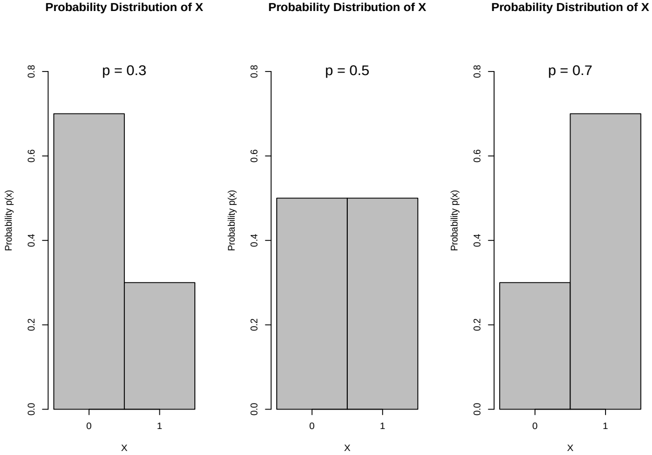
- The notation

  $$X \sim \text{Bernoulli}(p)$$

  means X follows a Bernoulli(p) distribution.
- Each choice of p leads to a different Bernoulli distribution.


Probability Distribution of X
p = 0.3
Probability p(x)
0.0
0.2
0.4
0.6
0.8
0
1
X
Probability Distribution of X
p = 0.5
Probability p(x)
0.0
0.2
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Bernoulli Mean and Variance: If $X \sim$ Bernoulli(p), then

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Proofs:

$$E(X) = \sum x\,p(x) = 0\,(1-p) + 1\,(p) = p.$$

$$\begin{aligned} V(X) &= \sum (x-\mu)^2\,p(x) \\ &= (0-p)^2\,(1-p) + (1-p)^2\,(p) \\ &= p(1-p). \end{aligned}$$

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    0 1 1 0 1 1 1 0 1 1

    gives $\bar{X} = 7/10 = 0.7$.
  - So $E(X)$ is the **long-run proportion** of 1's, which is the **probability** of 1, that is, p.

Example (Cont'd)

You take a pass/fail exam, and $X = 1$ if you pass and $X = 0$ if you fail. Then if you pass with probability $p = 0.7$,

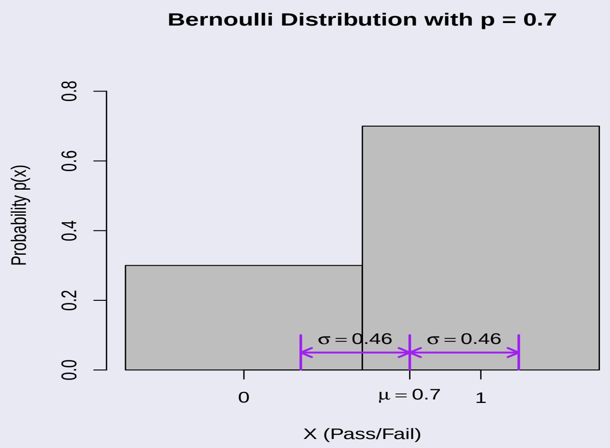
$$E(X) = \mathbf{0.7}$$

and

$$V(X) = 0.7(1 - 0.7) = 0.21$$

so

$$SD(X) = \sqrt{0.21} = \mathbf{0.46}.$$


Bernoulli Distribution with p = 0.7
Probability p(x)
0.0
0.2
0.4
0.6
0.8
σ = 0.46
σ = 0.46
0
μ = 0.7
1
X (Pass/Fail)

Binomial Random Variables (3.4)

- A ***binomial experiment*** is when:
  1. There are n **trials**.
  2. Each trial results in one of **two outcomes**, *success* (S) or *failure* (F).
  3. The outcomes of the trials are **independent**.
  4. The **probability of a success**, denoted p, is **constant** from trial to trial.

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- In a binomial experiment, the random variable

$$X = \text{The number of } S\text{'s among the } n \text{ trials}$$

is called a ***binomial random variable***.

Example

According to the *Colorado Springs Gazette*, **90%** of all cars tested for emissions in El Paso, Larimer and Weld counties pass the test. Suppose **four** cars are tested. Let

$$X = \text{The number that pass among the four cars tested}$$

Then X is a **binomial random variable** with 4 **trials** and **success probability 0.9**.

- The binomial distribution has **two parameters**, the **number of trials**, denoted n, and **success probability** on a given trial, denoted p.

Binomial(n, p) **Pmf**:

$$p(x) \;=\; \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for} \quad x = 0, 1, 2, \ldots, n.$$

Example (Cont'd)

Suppose again that **four** cars are tested for emissions, and that each car passes with probability **0.9**. Let

$$X = \text{The number that pass among the four cars tested}$$

Then

$$X \sim \text{binomial}(4, 0.9),$$

so the **probability** that **two** of the four cars will pass the test is

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so the **probability** that **two** of the four cars will pass the test is

$$\begin{aligned} p(2) &= \binom{4}{2}(0.9)^2(1-0.9)^{4-2} \\ &= \frac{4!}{2!(4-2)!}(0.9)^2(0.1)^2 \\ &= \mathbf{0.049}. \end{aligned}$$

- For intuition behind the **binomial pmf**, recall that probability that **two** of the **four** cars will pass the test is

$$p(2) \quad = \quad \frac{4!}{2!(4-2)!} \quad 0.9^2(1-0.9)^{4-2}$$

- For intuition behind the **binomial pmf**, recall that probability that **two** of the **four** cars will pass the test is

$$p(2) \quad = \quad \underbrace{\frac{4!}{2!(4-2)!}}_{\substack{\text{Number of ways}\\ \text{two of the four}\\ \text{cars can pass the}\\ \text{test}}} \quad \underbrace{0.9^2(1-0.9)^{4-2}}_{\substack{\text{Probability of each}\\ \text{of those ways}}}$$

Sequences of Four Cars
with Two Passing the Test

$\frac{4!}{2!(4-2)!}$ Sequences

Sequence Number	Car Number 1	2	3	4
1	S	S	F	F
2	S	F	S	F
3	S	F	F	S
4	F	S	S	F
5	F	S	F	S
6	F	F	S	S

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5	F	S	F	S	
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Sequence Number	Car Number 1	2	3	4	Probability of the Sequence
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2	S	F	S	F	
3	S	F	F	S	
4	F	S	S	F	
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Sequence Number	Car Number 1	2	3	4	Probability of the Sequence
1	P(S	S	F	F)	$(0.9)^2(0.1)^2$
2	+ P(S	F	S	F)	$(0.9)^2(0.1)^2$
3	+ P(S	F	F	S)	$(0.9)^2(0.1)^2$
4	+ P(F	S	S	F)	$(0.9)^2(0.1)^2$
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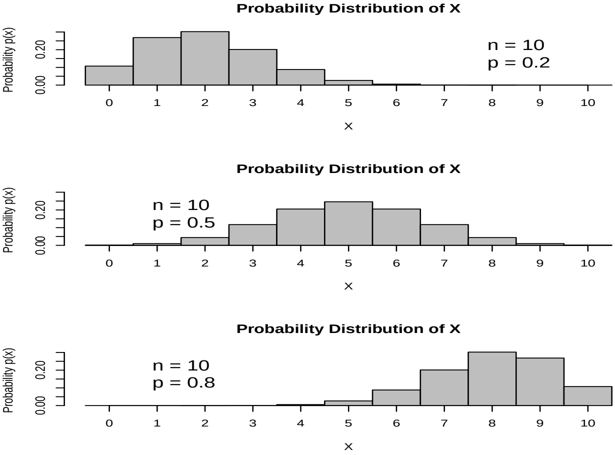
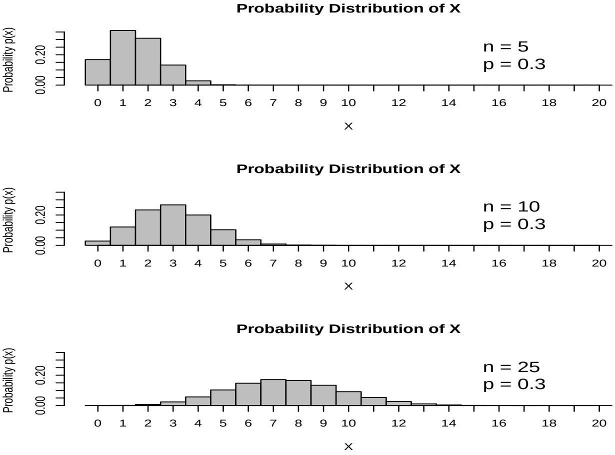
$\frac{4!}{2!(4-2)!}$ Sequences

$$\text{Sum} = \frac{4!}{2!(4-2)!}(0.9)^2(1-0.9)^2$$

- Each choice of n and p leads to a different binomial distribution.


Probability Distribution of X
Probability p(x)
0.00
0.20
n = 10
p = 0.2
0 1 2 3 4 5 6 7 8 9 10
x
Probability Distribution of X
Probability p(x)
0.00
0.20
n = 10
p = 0.5
0 1 2 3 4 5 6 7 8 9 10
x
Probability Distribution of X
Probability p(x)
0.00
0.20
n = 10
p = 0.8
0 1 2 3 4 5 6 7 8 9 10
x


Probability Distribution of X
Probability p(x)
0.00
0.20
n = 5
p = 0.3
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20
X
Probability Distribution of X
Probability p(x)
0.00
0.20
n = 10
p = 0.3
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20
X
Probability Distribution of X
Probability p(x)
0.00
0.20
n = 25
p = 0.3
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20
X

Binomial Mean and Variance: If $X \sim$ binomial(n, p), then

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Proofs:

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$$\begin{aligned} V(X) &= \sum (x-\mu)^2\,p(x) \\ &= \sum (x-np)^2 \binom{n}{x} p^x (1-p)^{n-x} = \cdots = np(1-p). \end{aligned}$$

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  - So we'd expect the **number** of success among the n trials, X, to be $\boldsymbol{np}$, on average.

Example (Cont'd)

If **four** cars are tested for emissions, and each car passes with probability **0.9**, then if

$$X = \text{The number that pass among the four cars tested}$$

Then

$$E(X) = n\,p = 4(0.9) = \mathbf{3.6}$$

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Then

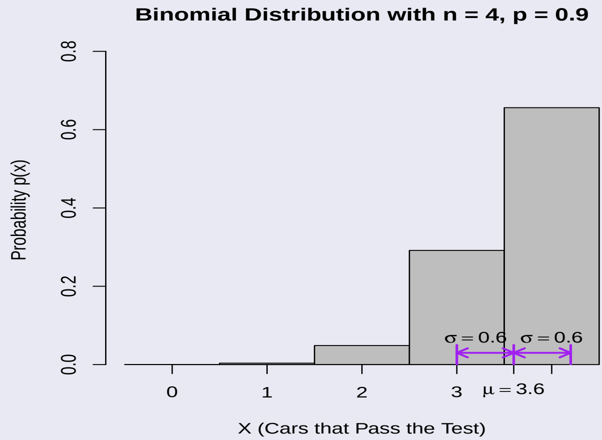
$$E(X) = n\,p = 4(0.9) = \mathbf{3.6}$$

and

$$V(X) = n\,p(1-p) = 4(0.9)(1-0.9) = 0.36$$

so

$$SD(X) = \sqrt{0.36} = \mathbf{0.6}.$$


Binomial Distribution with n = 4, p = 0.9
Probability p(x)
0.0
0.2
0.4
0.6
0.8
σ = 0.6
σ = 0.6
0
1
2
3
μ = 3.6
X (Cars that Pass the Test)

- Note that a **Bernoulli(p)** random variable is a special case of a **binomial(n, p)** random variable for which $n = 1$.

Geometric Random Variables (3.2, 3.5)

- A ***geometric experiment*** is when:
  1. There's a **sequence** of **trials.**
  2. Each trial results in a *success* (S) or *failure* (F).
  3. The trials are **independent**.
  4. The **probability** of a **success**, denoted p, is **constant** from trial to trial.
  5. Trials are performed until the **first success** (S) has been observed.

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    5. Trials are performed until the **first success** (S) has been observed.

- The random variable

X = The number trials *up to and including* the first success

is called a ***geometric random variable***.

Example

Suppose we roll a die repeatedly until a 6 occurs. Let

$$X \;=\; \text{The number rolls } \textit{up to and including} \text{ the first 6}$$

Then X is a **geometric random variable.** We can derive the **pmf** of X:

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p(1) &= P(S) = \frac{1}{6} \\
p(2) &= P(F)P(S) = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) \\
p(3) &= P(F)P(F)P(S) = \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) \\
&\vdots
\end{aligned}$$

$$\vdots$$

$$p(x) \quad = \quad \underbrace{P(F)P(F)\cdots P(F)}_{x-1 \quad \text{F's}} P(S) \quad = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$$

- The geometric distribution has **one parameter**, the **success probability** on a given trial, denoted p.

Geometric(p) **Pmf**:

$$p(x) \;=\; (1-p)^{x-1}p \qquad \text{for} \quad x = 1, 2, 3, \ldots.$$

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Geometric(p) Pmf:

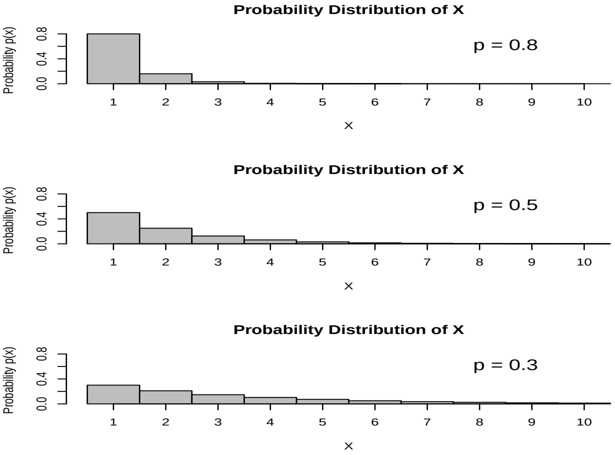
$$p(x) \;=\; (1-p)^{x-1}p \qquad\qquad \text{for} \quad x = 1, 2, 3, \ldots.$$

- Note that some textbooks (including ours) define a geometric random variable to be

$$Y \;=\; \text{The number trials } \textit{up to } \textbf{\textit{but not}} \textit{ including} \text{ the first success}$$

i.e. $Y = X - 1$.

- Each choice of p leads to a different geometric distribution.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x

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Proofs:

$$E(X) = \sum x\, p(x) = \sum x\,(1-p)^{x-1}p = \cdots = \frac{1}{p}\,.$$

Geometric Mean and Variance: If $X \sim$ geometric(p), then

$$E(X) = \frac{1}{p}$$
$$V(X) = \frac{1-p}{p^2}$$

Proofs:

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Example (Cont'd)

Suppose again that we roll a die repeatedly until a 6 occurs, and we let

$$X = \text{The number rolls } \textit{up to and including} \text{ the first 6}$$

Then

$$E(X) = \frac{1}{1/6} = \mathbf{6}$$

Example (Cont'd)

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Then

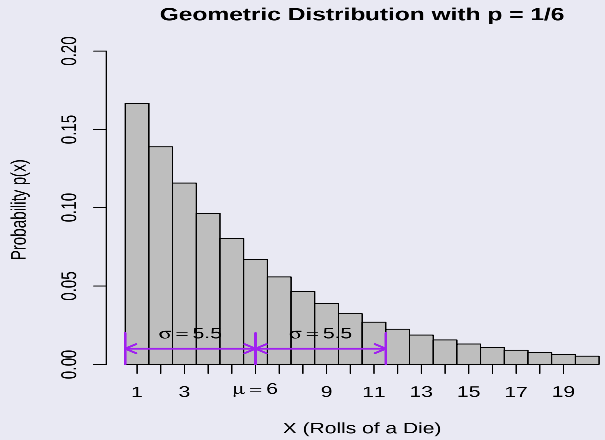
$$E(X) = \frac{1}{1/6} = \mathbf{6}$$

and

$$V(X) = \frac{1 - 1/6}{(1/6)^2} = 30$$

so

$$SD(X) = \sqrt{30} = \mathbf{5.5}.$$


Geometric Distribution with p = 1/6
Probability p(x)
0.00
0.05
0.10
0.15
0.20
σ = 5.5
σ = 5.5
1
3
μ = 6
9
11
13
15
17
19
X (Rolls of a Die)