Probability and Statistics

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March 7, 2019

Topics

Objectives

Objectives:

- Use continuous probability distributions to find probabilities
- For continuous random variables, compute and interpret:
  - The expected value
  - The expected value of a function of the random variable
  - The variance and standard deviation
  - The variance and standard deviation of a linear function of the random variable
- Recognize uniform random variables.
- Use the uniform distribution to find probabilities

Continuous Random Variables (4.1)

- The probability distribution of a **continuous** random variable is represented by a ***probability density function*** (or ***pdf***), denoted $f(x)$ and having the property that for any two numbers a and b, with $a \leq b$,

$$P(a \leq X \leq b) \;=\; \int_a^b f(x)\; dx.$$

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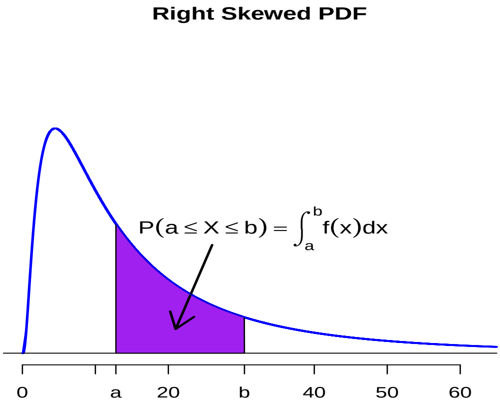
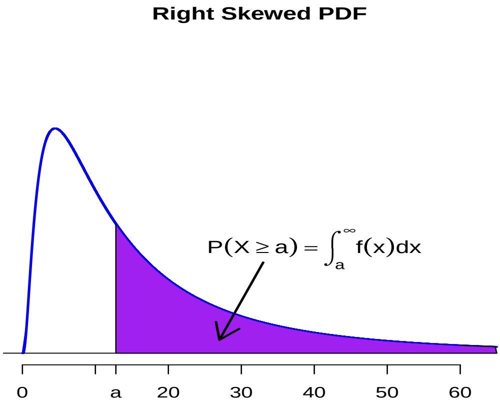
Note that $a = -\infty$ and $b = \infty$ are allowed, and that

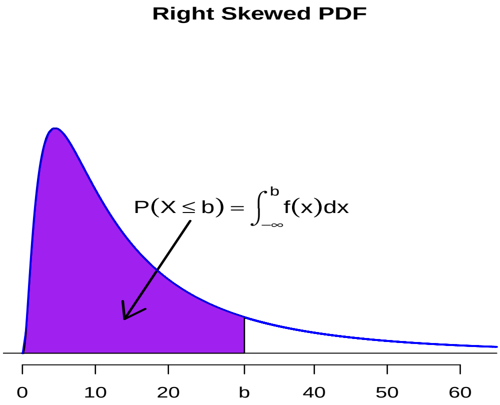
$$P(-\infty \le X \le b) = P(X \le b)$$

and

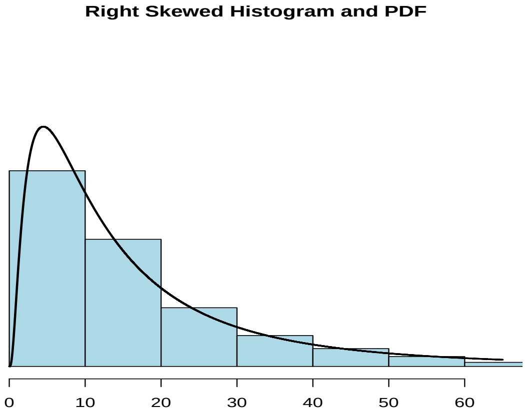
$$P(a \le X \le \infty) = P(X \ge a).$$

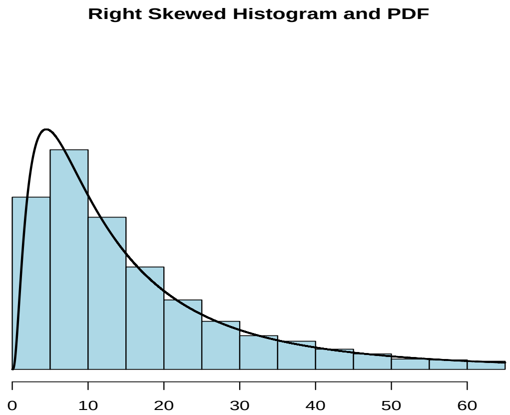

Right Skewed PDF
$P(a \le X \le b) = \int_a^b f(x)dx$
0
a
20
b
40
50
60


Right Skewed PDF
$P(X \geq a) = \int_a^\infty f(x)dx$
0
a
20
30
40
50
60


Right Skewed PDF
$P(X \leq b) = \int_{-\infty}^{b} f(x)dx$
0
10
20
b
40
50
60

- We can think of a **pdf** as mathematical model representing a **population.**


Right Skewed Histogram and PDF
0
10
20
30
40
50
60


Right Skewed Histogram and PDF
0
10
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30
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50
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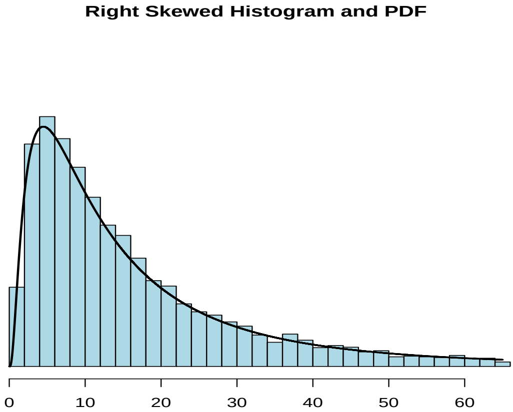
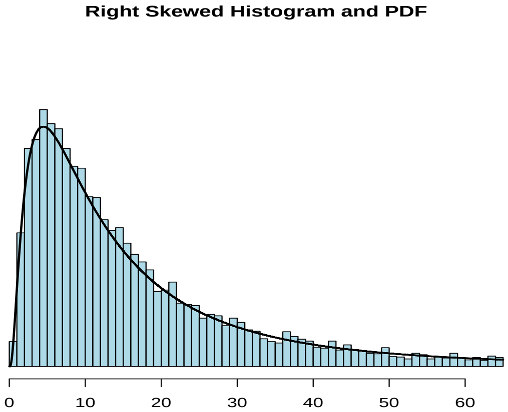

Right Skewed Histogram and PDF
0
10
20
30
40
50
60


Right Skewed Histogram and PDF
0
10
20
30
40
50
60

- In order for a **pdf** to be legitimate, it must satisfy the following conditions:

  1. $f(x) \geq 0$ for all x.
  2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Example

Suppose that the gain in a certain investment, in thousands of dollars, is a **continuous** random variable X that has **pdf** of the form

$$f(x) = \begin{cases} k(3x^2 + 2x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

for some constant $\boldsymbol{k}$.

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for some constant $\boldsymbol{k}$.

To determine the value of k, recall that the pdf has to integrate to 1, i.e.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Thus

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$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= 1 \\ \Rightarrow \quad \int_0^1 k(3x^2+2x)\,dx &= 1 \end{aligned}$$

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Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1 \\
\Rightarrow \quad \int_0^1 k(3x^2+2x)\,dx &= 1 \\
\Rightarrow \quad k(x^3+x^2)\Big|_0^1 &= 1 \\
\Rightarrow \quad \boldsymbol{k} &= \boldsymbol{\frac{1}{2}}
\end{aligned}$$

Thus

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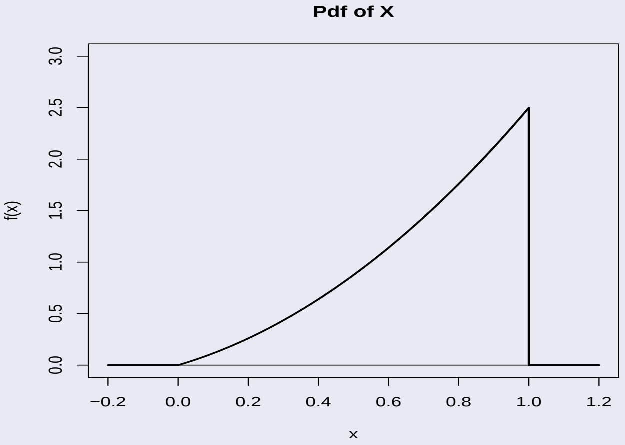
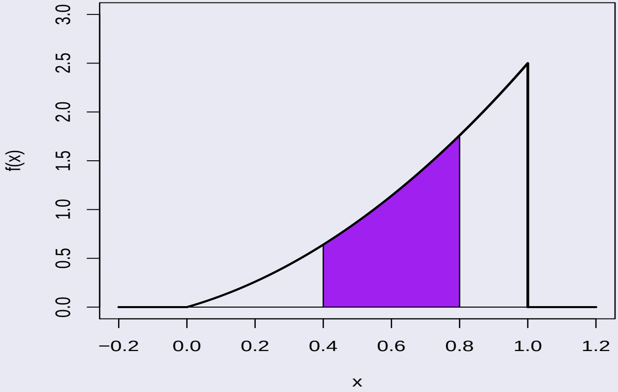
\end{aligned}$$

so the **pdf** is

$$f(x) = \begin{cases} \frac{1}{2}(3x^2+2x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$


Pdf of X
f(x)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x

The probability $P(0.4 \leq X \leq 0.8)$ that the gain is **between $400** and **$800** dollars is the shaded area:



$$P(0.4 \leq X \leq 0.8) \quad = \quad \int_{0.4}^{0.8} f(x)\, dx$$

$$
\begin{aligned}
P(0.4 \leq X \leq 0.8) &= \int_{0.4}^{0.8} f(x)\,dx \\
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\end{aligned}
$$

$$
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P(0.4 \leq X \leq 0.8) &= \int_{0.4}^{0.8} f(x)\,dx \\
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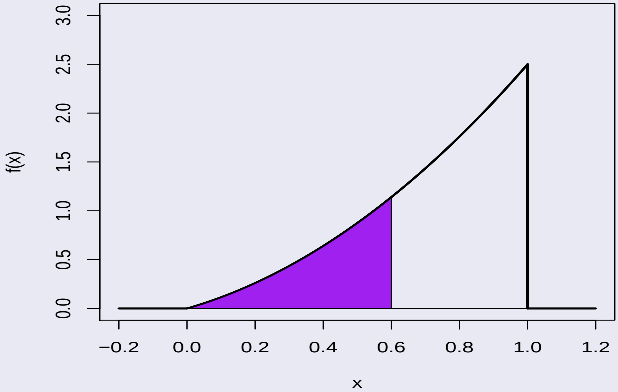
&= \tfrac{1}{2}(x^3 + x^2)\Big|_{0.4}^{0.8}
\end{aligned}
$$

$$\begin{aligned} P(0.4 \leq X \leq 0.8) &= \int_{0.4}^{0.8} f(x)\,dx \\ &= \int_{0.4}^{0.8} \tfrac{1}{2}(3x^2 + 2x)\,dx \\ &= \tfrac{1}{2}(x^3 + x^2)\Big|_{0.4}^{0.8} \\ &= \mathbf{0.464}. \end{aligned}$$

The probability $P(X \leq 0.6)$ that the gain is **less** than **$600** dollars is the shaded area:



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$$\begin{aligned} P(X \leq 0.6) &= \int_{-\infty}^{0.6} f(x)\, dx \\ &= \int_{0}^{0.6} \tfrac{1}{2}(3x^2 + 2x)\, dx \end{aligned}$$

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$$

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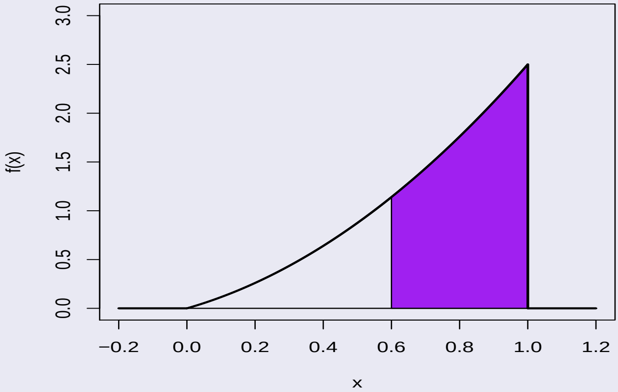
The probability $P(X > 0.6)$ that the gain is **greater** than **$600** dollars is the shaded area:



$$P(X > 0.6) \quad = \quad \int_{0.6}^{\infty} f(x)\, dx$$

$$\begin{aligned} P(X > 0.6) &= \int_{0.6}^{\infty} f(x)\,dx \\ &= \int_{0.6}^{1} \tfrac{1}{2}(3x^2 + 2x)\,dx \end{aligned}$$

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&= \tfrac{1}{2}(x^3 + x^2)\Big|_{0.6}^{1} \\
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\end{aligned}
$$

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Note that

$$P(X > 0.6) \;=\; 1 - P(X \le 0.6).$$

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- It follows that
$$P(X < c) = P(X \leq c)$$
and
$$P(X > c) = P(X \geq c).$$

- What information is given by the **pdf** $f(x)$?

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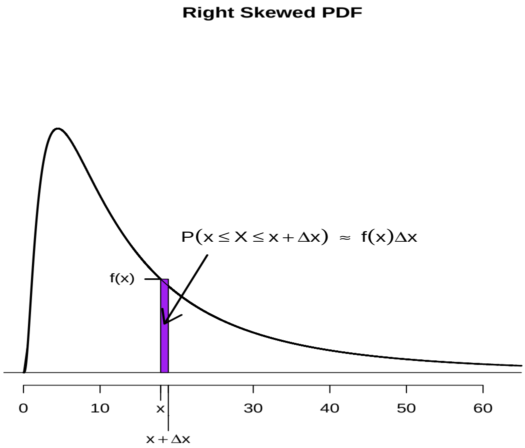
$$f(x) \;\approx\; \frac{P(x \;\leq\; X \;\leq\; x + \Delta x)}{\Delta x},$$

or equivalently

$$P(x \;\leq\; X \;\leq\; x + \Delta x) \;\approx\; f(x)\,\Delta x.$$


Right Skewed PDF
P(x ≤ X ≤ x + Δx) ≈ f(x)Δx
f(x)
0
10
x
30
40
50
60
x + Δx

- More formally, the **pdf** $f(x)$ is

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and so $f(x)$ is measures the **probability per unit of X** at the particular value x.

Expected Values (4.2)

- The ***expected value*** of a **continuous** random variable X, also called the ***mean*** of its distribution, is denoted $\boldsymbol{E(X)}$ or $\boldsymbol{\mu_X}$ and defined as:

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- Compare with the *discrete* case, where

$$E(X) \;=\; \sum x\, p(x)\,.$$

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  - It's the **center** ("balancing point") of the probability distribution.
- When we use probability distributions to represent **populations**, the expected value is the **population mean.**

Example (Cont'd)

Suppose again that the gain in a certain investment, $\boldsymbol{X}$, in thousands of dollars, has **pdf**

$$f(x) = \begin{cases} \frac{1}{2}(3x^2 + 2x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The **expected value** of X is

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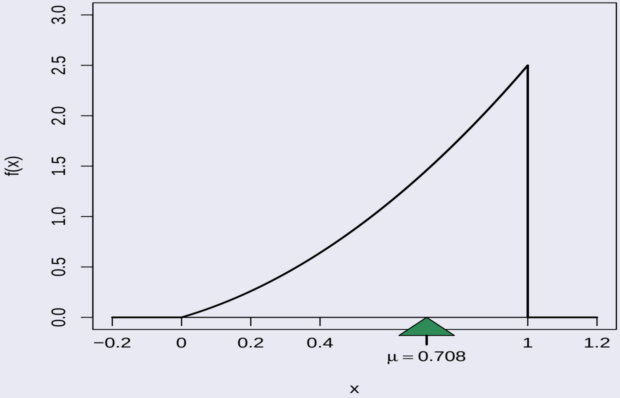
The **expected value** of X is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\ &= \int_0^1 x\, \frac{1}{2}(3x^2+2x)\, dx \\ &= \frac{1}{2}\left(\frac{3}{4}x^4 + \frac{2}{3}x^3\right)\Big|_0^1 \end{aligned}$$

$$= \quad \mathbf{0.708}.$$

This is the **center** ("balancing point") of the distribution.


3.0
2.5
2.0
1.5
1.0
0.5
0.0
f(x)
−0.2
0
0.2
0.4
1
1.2
μ = 0.708
x

- Recall that if X is a random variable, then any **function** $\boldsymbol{h(X)}$ is also a random variable.

Proposition

If X is a continuous random variable with pdf $f(x)$, then the expected value of any function $h(X)$, denoted $\boldsymbol{E\left(h(X)\right)}$ or $\boldsymbol{\mu_{h(X)}}$, is computed by

$$E\left(h(X)\right) \; = \; \mu_{h(X)} \; = \; \int_{-\infty}^{\infty} h(x)\, f(x)\, dx.$$

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$$E\left(h(X)\right) \;=\; \mu_{h(X)} \;=\; \int_{-\infty}^{\infty} h(x)\, f(x)\, dx.$$

- Compare with the *discrete* case, where

$$E\left(h(X)\right) \;=\; \sum h(x)p(x).$$

- The next proposition can be derived from the previous one by setting $h(X) = aX + b$.

Proposition

If X is any random variable, then for any constants a and b,

$$E\left(aX + b\right) \;=\; aE(X) + b$$

(or, using alternative notation, $\mu_{aX\,+\,b} \;=\; a\mu_X + b$).

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- Two special cases (for which $b = 0$ and $a = 1$):
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- Two special cases (for which $b = 0$ and $a = 1$):
  1. $E(aX) \;=\; aE(X)$.
  2. $E(X + b) \;=\; E(X) + b$.

- The ***variance*** and ***standard deviation*** of a **continuous** random variable X, denoted $\boldsymbol{V(X)}$ or $\boldsymbol{\sigma_X^2}$ and $\boldsymbol{SD(X)}$ or $\boldsymbol{\sigma_X}$, are defined as follows.

Variance and Standard Deviation:

$$\begin{aligned} V(X) \quad = \quad \sigma_X^2 \quad &= \quad E\left((X-\mu)^2\right) \\ &= \quad \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x)\, dx, \end{aligned}$$

where $\mu = E(X)$, and

$$SD(X) \; = \; \sigma_X \; = \; \sqrt{V(X)}.$$

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- The **standard deviation** is interpreted as a **typical deviation** of X away from μ.
- Both are measures of the **variation** in X, that is, of the **spread** of the probability distribution of X.
- They're the **population variance** and **population standard deviation** when the probability distribution represents a population.

Example (Cont'd)

Suppose again that X is the gain in a certain investment, in thousands of dollars, with **pdf**

$$f(x) = \begin{cases} \frac{1}{2}(3x^2 + 2x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall that the **mean** of this distribution is

$$\mu = 0.708.$$

The **variance** is

$$V(X) \quad = \quad \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x) \, dx$$

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$$\begin{aligned} V(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x) \, dx \\ &= \int_0^1 (x-0.708)^2 \, \tfrac{1}{2}(3x^2+2x) \, dx \end{aligned}$$

The **variance** is

$$
\begin{aligned}
V(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x) \, dx \\
&= \int_0^1 (x-0.708)^2 \, \tfrac{1}{2}(3x^2+2x) \, dx \\
&= \int_0^1 1.5x^4 - 1.124x^3 - 0.664x^2 + 0.501x \, dx
\end{aligned}
$$

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&= \int_0^1 (x - 0.708)^2 \, \tfrac{1}{2}(3x^2 + 2x)\, dx \\
&= \int_0^1 1.5x^4 - 1.124x^3 - 0.664x^2 + 0.501x \, dx \\
&= \frac{1.5}{5}x^5 - \frac{1.124}{4}x^4 - \frac{0.664}{3}x^3 + \frac{0.501}{2}x^2 \Big|_0^1
\end{aligned}
$$

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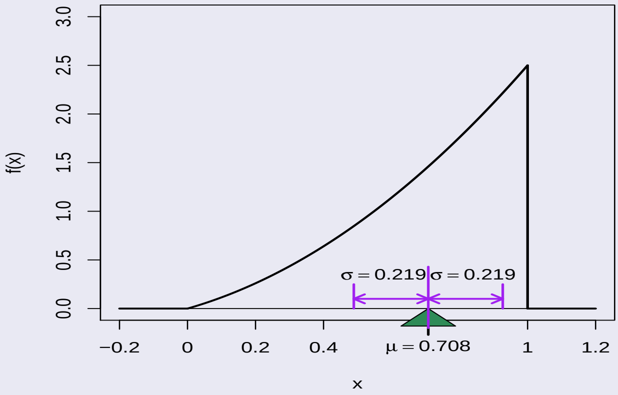
V(X) &= \int_{-\infty}^{\infty}(x-\mu)^2\, f(x)\, dx \\
&= \int_0^1 (x-0.708)^2\, \tfrac{1}{2}(3x^2+2x)\, dx \\
&= \int_0^1 1.5x^4 - 1.124x^3 - 0.664x^2 + 0.501x\, dx \\
&= \frac{1.5}{5}x^5 - \frac{1.124}{4}x^4 - \frac{0.664}{3}x^3 + \frac{0.501}{2}x^2\bigg|_0^1 \\
&= \mathbf{0.048},
\end{aligned}$$

so the **standard deviation** is

$$SD(X) \ = \ \sqrt{0.048} \ = \ \mathbf{0.219}.$$


f(x)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
σ = 0.219
σ = 0.219
−0.2
0
0.2
0.4
μ = 0.708
1
1.2
x

- By expanding the square in the definition

$$V(X) = \int (x-\mu)^2 f(x)\, dx$$

of a variance, we can derive the following.

Proposition

$$V(X) = E(X^2) - \mu^2$$

where $\mu = E(X)$.

- The variance of a **function** $h(X)$ is

$$V\left(h(X)\right) \;=\; E\left(\left(h(X) - \mu_{h(X)}\right)^2\right)$$

Setting $h(X) = aX + b$, we can derive the following.

Proposition

$$V(aX+b) \;=\; \sigma^2_{aX+b} \;=\; a^2\,\sigma^2_X$$

and so

$$SD(aX+b) \;=\; \sigma_{aX+b} \;=\; |a|\,\sigma_X.$$

- Two special cases of the previous proposition (for which $b = 0$ and $a = 1$):

1. $V(aX) \;=\; \sigma^2_{aX} \;=\; a^2\,\sigma^2_X$

   and

   $SD(aX) \;=\; \sigma_{aX} \;=\; |a|\,\sigma_X$.

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2. $V(X+b) \;=\; \sigma^2_{X\,+\,b} \;=\; \sigma^2_X$

   and

   $SD(X+b) \;=\; \sigma_{X\,+\,b} \;=\; \sigma_X$.

The Uniform Distribution (4.1)

- A **uniform** random variable is one that's **equally likely** to fall **anywhere** in an interval from A to B.

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- A **uniform** random variable is one that's **equally likely** to fall **anywhere** in an interval from A to B.
- The ***uniform*** distribution on the interval from A to B has **pdf**:

Uniform(A, B):

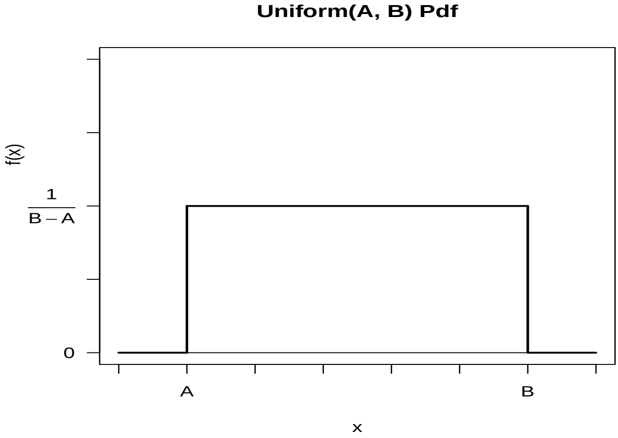
$$f(x) = \begin{cases} \frac{1}{B-A} & \text{for } A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

- The notation

$$\boldsymbol{X} \sim \text{uniform}(\boldsymbol{A},\, \boldsymbol{B})$$

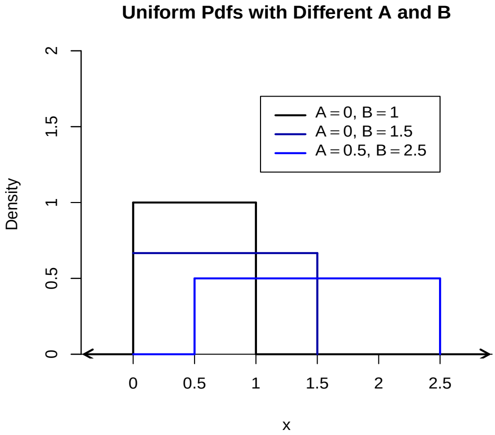
means X follows a uniform(A, B) distribution.

- The graph of one **uniform pdf** is shown on the next slide.



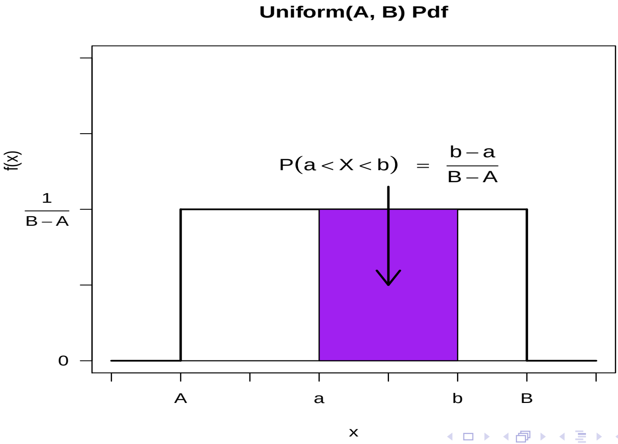
- The interval endpoints A and B are called ***parameters*** of the **uniform distribution.**

- The interval endpoints A and B are called ***parameters*** of the **uniform distribution.**
- Each choice of A and B leads to a different **uniform distribution.**


Uniform Pdfs with Different A and B
A = 0, B = 1
A = 0, B = 1.5
A = 0.5, B = 2.5
Density
0
0.5
1
1.5
2
0
0.5
1
1.5
2
2.5
x

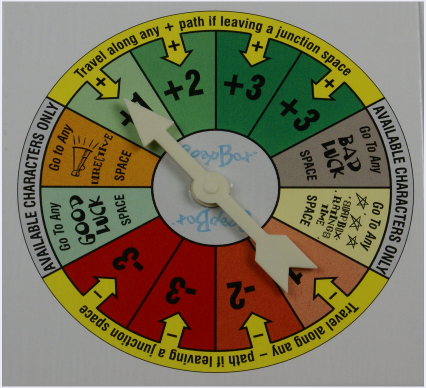
- For a ***uniform*** random variable X,

$$P(a \leq X \leq b) = \frac{b-a}{B-A}.$$



Example

When a board game spinner is spun, the pointer is equally likely to point in any direction (radians) over the range 0 to 2π.



If we let

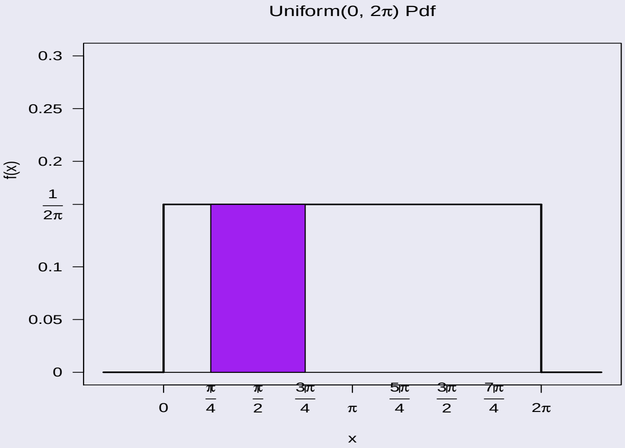
$$X = \text{the direction of the pointer in radians}$$

then

$$\boldsymbol{X} \sim \textbf{uniform}(\mathbf{0},\ \mathbf{2\pi}).$$

Thus,

$$P\left(\frac{\pi}{4} \le X \le \frac{3\pi}{4}\right) = \frac{3\pi/4 - \pi/4}{2\pi - 0} = \frac{1}{4}.$$


Uniform(0, 2π) Pdf
f(x)
0.3
0.25
0.2
1/2π
0.1
0.05
0
0
π/4
π/2
3π/4
π
5π/4
3π/2
7π/4
2π
x

Uniform Mean and Variance: If $X \sim$ uniform(A, B), then

$$\begin{aligned} E(X) &= \frac{A+B}{2} \\ V(X) &= \frac{(B-A)^2}{12} \end{aligned}$$

Uniform Mean and Variance: If $X \sim$ uniform(A, B), then

$$
\begin{aligned}
E(X) &= \frac{A+B}{2} \\
V(X) &= \frac{(B-A)^2}{12}
\end{aligned}
$$

Proofs:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_A^B x\, \frac{1}{B-A}\, dx = \cdots = \frac{A+B}{2}.$$

Uniform Mean and Variance: If $X \sim$ uniform(A, B), then

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Proofs:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_A^B x \frac{1}{B-A}\, dx = \cdots = \frac{A+B}{2}.$$

$$\begin{aligned} V(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \\ &= \int_{-\infty}^{\infty} \left(x - \frac{A+B}{2}\right)^2 \frac{1}{B-A}\, dx \cdots = \frac{(B-A)^2}{12}. \end{aligned}$$

Example

Let

$$X = \text{The wait time for a bus at a certain stop (in minutes).}$$

and suppose

$$X \sim \text{uniform}(0, 15).$$

Then the **expected value** of the wait time is

$$E(X) = \frac{0+15}{2} = \mathbf{7.5}$$

minutes.

The **variance** and **standard deviation** are

$$V(X) \;=\; \frac{(15-0)^2}{12} \;=\; \mathbf{18.75}$$

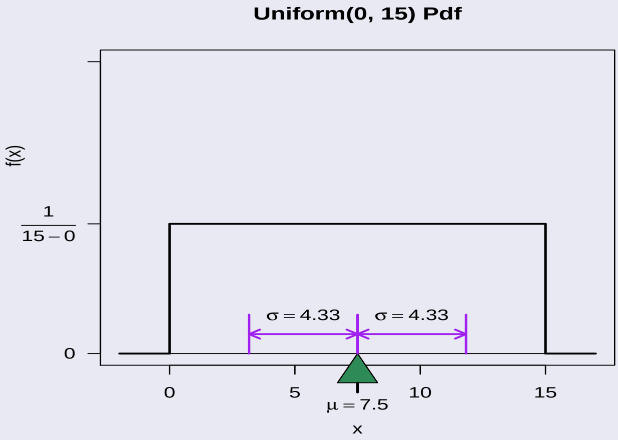
and

The **variance** and **standard deviation** are

$$V(X) \;=\; \frac{(15-0)^2}{12} \;=\; \mathbf{18.75}$$

and

$$SD(X) \;=\; \sqrt{18.75} \;=\; \mathbf{4.33}.$$


Uniform(0, 15) Pdf
f(x)
$\frac{1}{15-0}$
0
$\sigma = 4.33$
$\sigma = 4.33$
0
5
10
15
$\mu = 7.5$
x