Statistical Methods

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Topics

Objectives

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- Identify potentially confounding variables (in observational studies).
- Carry out a paired t test for two population means.
- Compute and interpret a paired t CI for the difference between two population means.

Identifying Causality: Experiments vs Observational Studies

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  - Does a person's income (response) depend on their gender (explanatory variable)?
  - Does a person's risk of colon cancer (response) depend on their diet (explanatory variable)?

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  - ***Observational study***: The investigator merely **observes** whether the two variables vary together.

    No attempt is made to **induce changes** in the response variable.
  - ***Experiment***: **Treatments** are **imposed** on individuals.

    A **deliberate attempt** is made to **induce changes** in the response variable.

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  Such studies suffer from the possible presence of variables whose effects on the response are ***confounded*** with the effect (if any) of the explanatory variable.

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b) List a few possible **confounding** variables that might explain the lower rates of colon cancer.

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b) List a few possible **confounding** variables that might explain the lower rates of colon cancer.

   Hint: Try to identify **other** ways in which people who eat lots of fruits and vegetables might differ from people who don't.

- To establish **cause and effect**, we need to carry out an **experiment.**

Example

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After four years, researchers were surprised to find no significant difference in colon cancer among these groups.

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  1. **Randomization** produces groups that are **similar** with respect to variables whose effects might **otherwise** be **confounded** with antioxidant intake (e.g. amount of exercise, smoking and drinking status, etc.).

  2. **Before** imposing treatments, any differences across groups in propensity for developing colon cancer would be due to **chance**.

  3. Therefore, **after** imposing treatments, any **statistically significant differences** in colon cancer rates could be attributed the **effects** of the treatments (antioxidants).

Paired t Test for Two Population Means μ_1 and μ_2

Paired Samples Study Designs

- The ***paired*** t ***test*** is used with two samples collected using a ***paired samples study design***.

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Which study design is preferred? Why?

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- We'll see how to use the samples to decide if the two **population means** μ_1 and μ_2 are different.
- We denote the first sample by $X_1, X_2, \ldots, X_n$ and the second by $Y_1, Y_2, \ldots, Y_n$, where each X_i is **paired** with it's corresponding Y_i.

- The **null hypothesis** is that no difference between the population means μ_1 and μ_2:

Null Hypothesis:

$$H_0 : \mu_1 - \mu_2 \; = \; 0$$

- The **alternative hypothesis** will depend on what we're trying to "prove":

Alternative Hypothesis: The alternative hypothesis will be one of

1. $H_a : \mu_1 - \mu_2 > 0$ (**one-sided, upper-tailed**)
2. $H_a : \mu_1 - \mu_2 < 0$ (**one-sided, lower-tailed**)
3. $H_a : \mu_1 - \mu_2 \neq 0$ (**two-sided, two-tailed**)

depending on what we're trying to verify using the data.

- Consider the n **differences**

$$\begin{aligned} D_1 &= X_1 - Y_1 \\ D_2 &= X_2 - Y_2 \\ &\vdots \\ D_n &= X_n - Y_n \end{aligned}$$

Example

Here are the data on amount of wear in soles of boys shoes.

Boy	Material B	Material A	Difference
1	14.0	13.2	0.8
2	8.8	8.2	0.6
3	11.2	10.9	0.3
4	14.2	14.3	-0.1
5	11.8	10.7	1.1
6	6.4	6.6	-0.2
7	9.8	9.5	0.3
8	11.3	10.8	0.5
9	9.3	8.8	0.5
10	13.6	13.3	0.3
	$\bar{X} = 11.04$	$\bar{Y} = 10.63$	$\bar{D} = 0.41$
	$s_x = 2.52$	$s_y = 2.45$	$s_D = 0.39$

- We considered $D_1, D_2, \ldots, D_n$ to be a **single random sample** from a ***population of differences*** whose **mean** is $\boldsymbol{\mu_d}$.

Equivalent Ways of Stating the Hypotheses

Proposition

μ_d is related to μ_1 and μ_2 as follows.

$$\mu_d = \mu_1 - \mu_2.$$

- The above fact holds because

$$\mu_d \;=\; E(D_i) \;=\; E(X_i - Y_i)\,,$$

and $X_i - Y_i$ is a **linear combination** of X_i and Y_i, so

$$E(X_i - Y_i) \;=\; E(X_i) - E(Y_i) \;=\; \mu_1 - \mu_2\,.$$

- Hypotheses about $\mu_1 - \mu_2$ can be written in terms of μ_d:

Hypothesis about μ_1 and μ_2	**Equivalent Hypothesis about μ_d**
$H_0 : \mu_1 - \mu_2 = 0$	$H_0 : \mu_d = 0$
$H_a : \mu_1 - \mu_2 > 0$	$H_a : \mu_d > 0$
$H_a : \mu_1 - \mu_2 < 0$	$H_a : \mu_d < 0$
$H_a : \mu_1 - \mu_2 \neq 0$	$H_a : \mu_d \neq 0$

Paired t Test Statistic for $\mu_1 - \mu_2$ (or μ_d):

$$T = \frac{\bar{D} - \mu_d}{S_D/\sqrt{n}},$$

where $\bar{D}$ and S_D are the **sample mean** and **sample standard deviation** of the **differences** $D_1, D_2, \ldots, D_n$.

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- t is just the **one-sample t test statistic** for a test of μ_d.

- Now suppose either the sample of **differences** is from a $N(\mu_d, \sigma_d)$ population or n is *large*.

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  In this case, the sampling distribution of the test statistic is as follows.

Sampling Distribution of the Test Statistic Under H_0:
If t is the paired t test statistic, then when

$$H_0 : \mu_1 - \mu_2 = 0 \qquad \text{(or } H_0 : \mu_d = 0)$$

is true,

$$t \sim t(n-1).$$

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  - The ***rejection region*** as the **extreme 100α% of t values** (in the direction(s) specified by H_a).
  - The ***p-value*** as the **tail area(s) beyond the observed t value** (in the direction(s) specified by H_a).

- **Comment**: Sometimes we want to test

$$H_0 : \mu_d = \Delta_0$$

where Δ_0 is some non-zero value. In this case the test statistic is

$$T = \frac{\bar{D} - \Delta_0}{S_D/\sqrt{n}},$$

which follows a $t(n-1)$ distribution when H_0 is true.

Exercise

Here (again) are the data on amount of wear in soles of boys shoes.

Boy	Material B	Material A	Difference
1	14.0	13.2	0.8
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	$s_x = 2.52$	$s_y = 2.45$	$s_D = 0.39$

A histogram of the $n = 10$ **differences** suggests the **normality assumption** is tenable.

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Hint: You should get t **= 3.324** and **p-value = 0.0089**.

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   Hint: You should get t **= 3.324** and **p-value = 0.0089**.

b) If you found a difference in Part *a*, which material is preferred?

- The **differences** $D_1, D_2, \ldots, D_n$ will be **normally distributed** if the X_i's and Y_i's are drawn from **normal** populations.

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Proposition

Suppose $X_i \sim \mathrm{N}(\mu_1, \sigma_1)$ and $Y_i \sim \mathrm{N}(\mu_2, \sigma_2)$. Let

$$D_i = X_i - Y_i\,.$$

Then

$$D_i \sim \mathrm{N}(\mu_d,\ \sigma_d)$$

where $\mu_d = \mu_1 - \mu_2$ (and σ_d is discussed later).

- The above fact holds because **linear combinations** of **normally distributed** random variables are **normally distributed** (and $X_i - Y_i$ is a linear combination of X_i and Y_i).

(Optional Section) Advantage of Matched Pairs Study Designs

Proposition

It can be shown that

$$\bar{D} = \bar{X} - \bar{Y},$$

where $\bar{D}$ is the sample mean of $D_1, D_2, \ldots, D_n$ and $\bar{X}$ and $\bar{Y}$ are the sample means of $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$.

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Paired t Test Statistic:

$$T = \frac{\bar{X} - \bar{Y} - 0}{S_D/\sqrt{n}}$$

Two-Sample t Test Statistic:

$$T = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{S_1^2/n + S_2^2/n}}$$

It can be shown that usually $S_D^2 < S_1^2 + S_2^2$

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Paired t Test Statistic:	**Two-Sample t Test Statistic**:
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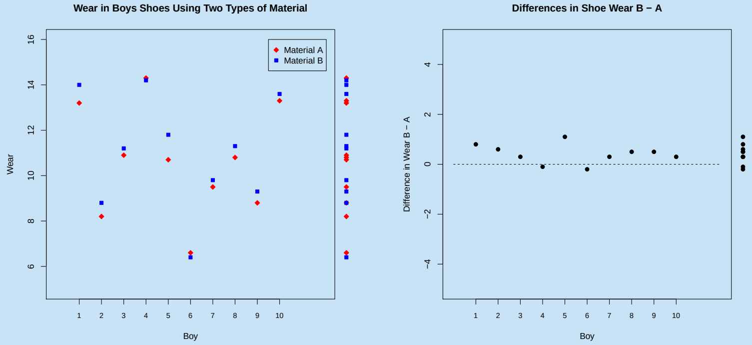
(because $\sigma_d^2 = \sigma_1^2 + \sigma_2^2 - \rho\sigma_1\sigma_2$, where $0 < \rho < 1$).

Example

The figure below shows two representations of the data from the study comparing two materials for soles of boys shoes.



Compare:

Material A:	**Material B:**	**Differences**
$S_1^2 = 6.35$	$S_2^2 = 6.00$	$S_D^2 = 0.15$

Compare:

Two-Sample t:	**Paired t:**
$t = 0.37$	$t = 3.32$
df $= 17$	df $= 9$
p-value $= 0.7165$	p-value $= 0.0089$

Paired t Confidence Interval for $\mu_1 - \mu_2$

Paired t CI: When the differences $D_1, D_2, \ldots, D_n$ in **paired samples** can be treated as a sample from a population whose mean is μ_d $(= \mu_1 - \mu_2)$ a $100(1-\alpha)\%$ ***paired t confidence interval for $\mu_1 - \mu_2$*** is:

$$\bar{D} \pm t_{\alpha/2,n-1} \cdot \frac{S_D}{\sqrt{n}}.$$

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- In either case, we can be $100(1-\alpha)\%$ confident that $\mu_1 - \mu_2$ will be contained in the CI.
- The CI is just the **one-sample t CI** based on the **differences**.

Exercise

For the boys' shoe wear data set, the **sample mean** and **standard deviation** of the **differences** was

$$\bar{D} = 0.41 \qquad \text{and} \qquad S_D = 0.39.$$

a) Give the (point) **estimate** of the true difference in means $\mu_1 - \mu_2$ (or μ_d).

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c) Does the CI contain the value **zero**? What does this say about the shoe wear for the two materials?