Statistical Methods

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Notes

Topics

1 Two-Factor ANOVA with More Than One Observation Per Cell

Notes

Objectives

Objectives:

- State the treatment effects version of the two-factor ANOVA model when $K > 1$.
- Carry out two-factor ANOVA F tests for the interaction effect and main effects of Factors A and B when $K > 1$.
- Interpret the interaction effect and main effects in the two-factor ANOVA model.

Notes

Two-Factor ANOVA with $K \geq 2$

- Two-factor studies usually involve more than one observation per group.

Notes

Example

A software firm was finding that their programmers tended to underestimate the number of programmer-days needed to complete large-scale programming projects.

A study was carried out to determine if either a programmer's **type of experience** or their **years of experience** influence the accuracy of the predictions.

Twenty-four programmers, classified according to **type** and **years** of **experience**, were asked to predict the number of programmer-days required to complete a large project about to be initiated.

Notes

The table below shows their prediction errors (actual minus predicted programmer-days).

		Factor B: Years of Experience			
		Less than 5 ($j = 1$)	5 to 10 ($j = 2$)	More than 10 ($j = 3$)	
Factor A: Type of Experience	Small Systems Only ($i = 1$)	240 206 217 225	110 118 103 95	56 60 68 58	$\bar{X}_{1\cdot\cdot} = 129.7$
	Small and Large Systems ($i = 2$)	71 53 68 57	47 52 31 49	37 33 40 45	$\bar{X}_{2\cdot\cdot} = 48.6$
		$\bar{X}_{\cdot 1 \cdot} = 142.1$	$\bar{X}_{\cdot 2 \cdot} = 75.6$	$\bar{X}_{\cdot 3 \cdot} = 49.6$	$\bar{X}_{\cdot\cdot\cdot} = 89.1$

Notes

Here's a different version of the table showing the so-called **group means**.

		Factor B: Years of Experience			
		Less than 5 ($j = 1$)	5 to 10 ($j = 2$)	More than 10 ($j = 3$)	
Factor A: Type of Experience	Small Systems Only ($i = 1$)	$\bar{X}_{11\cdot} =$ 222.0	$\bar{X}_{12\cdot} =$ 106.5	$\bar{X}_{13\cdot} =$ 60.5	$\bar{X}_{1\cdot\cdot} = 129.7$
	Small and Large Systems ($i = 2$)	$\bar{X}_{21\cdot} =$ 62.3	$\bar{X}_{22\cdot} =$ 44.8	$\bar{X}_{23\cdot} =$ 38.8	$\bar{X}_{2\cdot\cdot} = 48.6$
		$\bar{X}_{\cdot 1 \cdot} = 142.1$	$\bar{X}_{\cdot 2 \cdot} = 75.6$	$\bar{X}_{\cdot 3 \cdot} = 49.6$	$\bar{X}_{\cdot\cdot\cdot} = 89.1$

Notes

The study was designed to find out:

- Does **type of experience** effect prediction accuracy (i.e. is there a ***factor A main effect***)?
- Does **years of experience** effect prediction accuracy (i.e. is there a ***factor B main effect***)?
- Is the effect of **years of experience** ***different*** depending on a programmer's **type of experience** (i.e. is there an ***interaction effect***)?

Notes

- **Notation**:

$\boldsymbol{I}$ = The number of levels of Factor A.
$\boldsymbol{J}$ = The number of levels of Factor B.
$\boldsymbol{K}$ = The number of observations (common sample size) in each of the IJ treatment groups.
$\boldsymbol{X_{ijk}}$ = The kth observation at the ith level of Factor A and jth level of Factor B (i.e. in the i, jth group).

Notes

- The data can be laid out in a table as below:

Factor B

		Level $j=1$	Level $j=2$	$\cdots$	Level $j=J$	
	Level $i=1$	X_{111}, X_{112}, $\vdots$, X_{11K}	X_{121}, X_{122}, $\vdots$, X_{12K}	$\cdots$	X_{1J1}, X_{1J2}, $\vdots$, X_{1JK}	$\bar{X}_{1\cdot\cdot}$
Factor A	Level $i=2$	X_{211}, X_{212}, $\vdots$, X_{21K}	X_{221}, X_{222}, $\vdots$, X_{22K}	$\cdots$	X_{2J1}, X_{2J2}, $\vdots$, X_{2JK}	$\bar{X}_{2\cdot\cdot}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	Level $i=I$	X_{I11}, X_{I12}, $\vdots$, X_{I1K}	X_{I21}, X_{I22}, $\vdots$, X_{I2K}	$\cdots$	X_{IJ1}, X_{IJ2}, $\vdots$, X_{IJK}	$\bar{X}_{I\cdot\cdot}$
		$\bar{X}_{\cdot 1\cdot}$	$\bar{X}_{\cdot 2\cdot}$	$\cdots$	$\bar{X}_{\cdot J\cdot}$	$\bar{X}_{\cdot\cdot\cdot}$

Notes

- (cont'd)

$\bar{\boldsymbol{X}}_{\boldsymbol{i}\cdot\cdot}$ = The ***Factor A level mean***) of all observations at level i of Factor A.
$\bar{\boldsymbol{X}}_{\cdot\boldsymbol{j}\cdot}$ = The ***Factor B level mean*** of all observations at level j of Factor B.
$\bar{\boldsymbol{X}}_{\boldsymbol{ij}\cdot}$ = The ***group mean*** of the observations in the i, jth group.
$\bar{\boldsymbol{X}}_{\cdot\cdot\cdot}$ = The ***grand mean*** of *all* IJK observations.

Notes

- When the sample sizes per group are all the same, the grand mean $\bar{X}_{\cdot\cdot\cdot}$ can be obtained as:
  - The average of the IJ group means.
  - The average of the I Factor A level means.
  - The average of the J Factor B level means.

Notes

- **Comments**:
  - The sample sizes per group **don't** all have to be the same. But we'll only look at the equal-sample size case.
  - The data can be **samples** from IJ populations (representing combinations of the levels of the factors) **or** responses to treatments in a **randomized experiment**.

Notes

The Two-Factor ANOVA Model

- When $K > 1$, we **no longer** have to assume the effects of the factors are **additive** (which did when $K = 1$ to reduce the number of model parameters that needed to be estimated).

Notes

Two-factor ANOVA Model:

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \qquad (1)$$

where

μ is a constant called the ***true grand mean***.
α_i is the ***effect*** of the ith level of **Factor** A.
β_j is the ***effect*** of the jth level of **Factor** B.
γ_{ij} is the ***interaction effect*** for the ith level of **Factor** A and jth level of **Factor** B.
ϵ_{ijk} are iid $N(0, \sigma)$ ***random errors***.

(More formal definitions on the next slide.)

Notes

- More formally, let

$$\boldsymbol{\mu_{ij}} = \text{The **population mean** for the } i\text{th level of **Factor A** and } j\text{th level of **Factor B**.}$$

Then:

$$\mu = \frac{\sum_i \sum_j \mu_{ij}}{IJ},$$

$$\alpha_i = \mu_{i\cdot} - \mu \qquad \text{and} \qquad \beta_j = \mu_{\cdot j} - \mu,$$

where the **true factor A** and **B levels means**, $\boldsymbol{\mu_{i\cdot}}$ and $\boldsymbol{\mu_{\cdot j}}$, are

$$\mu_{i\cdot} = \frac{\sum_j \mu_{ij}}{J} \qquad \text{and} \qquad \mu_{\cdot j} = \frac{\sum_i \mu_{ij}}{I},$$

and

$$\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j).$$

Notes

- It can be shown that defining the γ_{ij}'s, α_i's, and β_j's as on the previous slide is equivalent to imposing the constraints

$$\sum_i \alpha_i = 0 \quad \text{and} \quad \sum_j \beta_j = 0$$

and

$$\sum_j \gamma_{ij} = 0 \ \text{(for each fixed } i\text{)} \ \text{and} \ \sum_i \gamma_{ij} = 0 \ \text{(for each fixed } j\text{)}.$$

Notes

Sums of Squares and the ANOVA Partition

- We can ***partition*** the **total variation** in the data into four parts:
  - One reflecting variation **between** the levels of **Factor A**.
  - Another reflecting variation **between** the levels of **Factor B**.
  - Another reflecting variation due to the **interaction** between the levels of **Factors A** and **B**.
  - The other reflecting variation **within** the groups.

Notes

- The **partition** will involve the following ***sums of squares*** (shown with their **df**):
  - **SST** is the ***total sum of squares***, defined as

$$\text{SST} = \sum_i \sum_j \sum_k (X_{ijk} - \bar{X}_{...})^2 \qquad \boldsymbol{df = IJK - 1}$$

which measures the **total** variation in the X_{ij}'s.

Notes

- (cont'd):
  - **SSA** is the ***Factor A sum of squares***, defined as

$$\begin{aligned} \text{SSA} &= \sum_i \sum_j \sum_k (\bar{X}_{i..} - \bar{X}_{...})^2 \\ &= JK \sum_{i=1}^{I} (\bar{X}_{i.} - \bar{X}_{..})^2 \qquad \boldsymbol{df = I - 1} \end{aligned}$$

which measures variation between the **levels** of **Factor A** due to both the **Factor A effect** and **random error**.

Notes

- (cont'd):
  - **SSB** is the ***Factor B sum of squares***, defined as

$$\text{SSB} = \sum_i \sum_j \sum_k (\bar{X}_{\cdot j \cdot} - \bar{X}_{\cdots})^2$$
$$= IK \sum_{i=1}^{I} (\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2 \qquad \boldsymbol{df = J - 1}$$

which measures variation between the **levels** of **Factor B** due to both the **Factor B effect** and **random error**.

Notes

- (cont'd):
  - **SSAB** is the ***interaction sum of squares***, with

$$\text{SSAB} = \sum_i \sum_j \sum_k (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j \cdot} + \bar{X}_{\cdots})^2$$
$$\boldsymbol{df = (I - 1)(J - 1)}$$

which measures variation due to an **interaction** (i.e. **non-additivity**) of the effects of the two factors and **random error**.

Notes

- (cont'd):
  - **SSE** is the ***error sum of squares***, defined as

$$\text{SSE} = \sum_i \sum_j \sum_k (X_{ijk} - \bar{X}_{ij\cdot})^2$$
$$\boldsymbol{df = IJ(K - 1)}$$

which measures variation of the X_{ijk}'s **within** treatment groups due to **random error**.

Notes

Proposition

ANOVA Partition for the Full Model: It can be shown that

$$\text{SST} = \text{SSA} + \text{SSB} + \text{SSAB} + \text{SSE}$$

Notes

Additive Property of Degrees of Freedom:

df for SST $=$ df for SSA $+$ df for SSB $+$ df for SSAB $+$ df for SSE

Notes

Mean Squares

- The ***Factor A mean square***, ***Factor B mean square***, ***interaction mean square***, and ***mean squared error*** are:

$$\text{MSA} = \frac{\text{SSA}}{I-1}$$

$$\text{MSB} = \frac{\text{SSB}}{J-1}$$

$$\text{MSAB} = \frac{\text{SSAB}}{(I-1)(J-1)}$$

$$\text{MSE} = \frac{\text{SSE}}{IJ(K-1)}$$

Notes

The Two-Factor ANOVA F Tests

- We'll want to test **three** sets of hypotheses:
  - Hypotheses about an ***interaction effect*** between the two factors:

$$H_{0AB}: \quad \gamma_{ij} = 0 \text{ for all } i \text{ and } j \qquad (2)$$
$$H_{aAB}: \quad \text{Not all } \gamma_{ij}\text{'s equal zero}$$

  - Hypotheses about a ***Factor A main effect***:

$$H_{0A}: \quad \alpha_i = 0 \text{ for all } i \qquad (3)$$
$$H_{aA}: \quad \text{Not all } \alpha_i\text{'s equal zero}$$

  - Hypotheses about a ***factor B main effect***:

$$H_{0B}: \quad \beta_j = 0 \text{ for all } j \qquad (4)$$
$$H_{aB}: \quad \text{Not all } \beta_j\text{'s equal zero}$$

Notes

- In each case, the **null hypothesis** says there's **no effect** and the **alternative** says there **is an effect**.

Notes

- We always **test for an interaction effect first**, and proceed to the tests for main effects **only if the interaction effect isn't statistically significant**. We'll see why later.

Notes

Two-Factor ANOVA F Test Statistics:

$$F_{AB} = \frac{\text{MSAB}}{\text{MSE}} \qquad F_A = \frac{\text{MSA}}{\text{MSE}} \qquad F_B = \frac{\text{MSB}}{\text{MSE}}$$

Notes

Large values of F_{AB} provide **evidence against H_{0AB} in favor of H_{aAB} : Not all of the γ_{ij}'s are zero**.

Large values of F_A provide **evidence against H_{0A} in favor of H_{aA} : Not all of the α_i's are zero**.

Large values of F_B provide **evidence against H_{0B} in favor of H_{aB} : Not all of the β_j's are zero**.

Notes

- Suppose data in a two-factor study follow the **two-factor ANOVA model**, where the error terms ϵ_{ijk} are iid $N(0, \sigma)$.

Notes

Sampling Distributions of the Test Statistics Under H_0:

1. If F_{AB} is the F test statistic for the interaction effect, then when

$$H_{0AB} : \gamma_{ij} = 0 \text{ for all } i \text{ and } j$$

is true,

$$F_{AB} \sim F((I-1)(J-1),\, IJ(K-1)).$$

2. If F_A is the F test statistic for Factor A, then when

$$H_{0A} : \alpha_i = 0 \text{ for all } i$$

is true,

$$F \sim F(I-1,\, IJ(K-1)).$$

Notes

3. If F_B is the the F test statistic for Factor B, then when

$$H_{0B} : \beta_j = 0 \text{ for all } j$$

is true,

$$F \sim F(J-1,\, IJ(K-1)).$$

Notes

- The appropriate $\boldsymbol{F}$ curves give us:
  - The ***rejection regions*** as the **extreme largest 100α% of $\boldsymbol{F}$ values**.
  - The ***p-values*** as the **tail areas to the right of the observed $\boldsymbol{F_{AB}}$, $\boldsymbol{F_A}$, and $\boldsymbol{F_B}$ values**.

Notes

- **Comment**: The **ANOVA $\boldsymbol{F}$ tests** can be used even if the samples are from **non-normal** populations as long the per-group sample sizes are large.

Notes

The ANOVA Table

- The results are summarized in an ***ANOVA table***:

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Factor A	$I-1$	SSA	MSA = SSA/$(I-1)$	F_A =MSA/MSE	p
Factor B	$J-1$	SSB	MSB = SSB/$(J-1)$	F_B =MSB/MSE	p
Interaction	$(I-1)(J-1)$	SSAB	MSAB = SSAB/$(I-1)(J-1)$	F_{AB} =MSAB/MSE	p
Error	$IJ(K-1)$	SSE	MSE = SSE/$IJ(K-1)$		
Total	$IJK-1$	SST			

Notes

Example

For the study of factors effecting errors in predicting the completion time for a programming project, the **ANOVA table** is:

Source	df	Sum of Squares	Mean Square	f	P-value
Type	1	39447.0	39447.0	458.0	0.000
Years	2	36412.0	18206.0	211.4	0.000
Interaction	2	20165.3	10082.7	117.1	0.000
Error	18	1550.3	86.1		
Total	23	97574.6			

(We'll interpret the results in a later example.)

Notes

Interpretation of the Interaction Effects γ_{ij}

- **If** the interaction effect $\gamma_{ij} = 0$ for all i and j, then the two-factor ANOVA model reduces to the so-called ***additive effects*** model

$$X_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$$

Notes

- Under the ***additive effects*** model, the **effect** of **each factor**, at any **fixed level** of the **other factor**, is the ***same*** **regardless** of the **level** of that **other factor**.
- By **including** the **interaction** term γ_{ij} in the model (and allowing it to be non-zero), we **allow** the **effect** of **each factor** to be ***different*** depending on the **level** of the **other factor**.

Notes

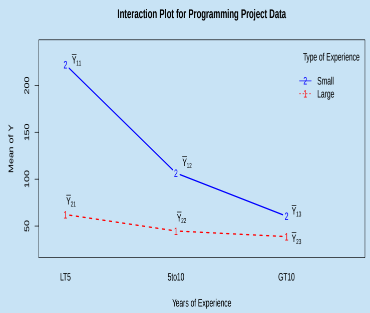
Interaction Plots

- An ***interaction plot*** is a plot of the **group means** $\bar{X}_{ij.}$ on the y-axis, **levels** of **one factor** on the x-axis, and **lines** connecting **group means** at each **fixed level** of the **other factor**.

Notes

Example

For the data from the study of the effects of **type** of experience (Factor A) and **years** of experience (Factor B) on prediction errors, the **interaction plot** is below.



experience and years of experience because **the effect of**

Notes

There appears to be an **interaction effect** between **type** and **years** of experience because the **effect** of **years** is **different** depending on the **type** of experience.

Notes

Interpretation of Interaction Plots

- **Interpretation of Interaction Plots**: If levels of Factor B are marked on the horizontal axis and levels of Factor A are represented by lines, then:
  - The slope of a line indicates the effect of Factor B for a given level of Factor A.
  - The vertical distance between two lines indicates the effect of Factor A for a given level of Factor B.

  Thus:
  - **Lines that are close to parallel suggest** that the effect of Factors A is the same regardless of the level of Factor B, i.e. that **the effects of the two factors are *additive***.
  - **Lines that aren't parallel suggest that there's an *interaction* effect between the two factors**.

Notes

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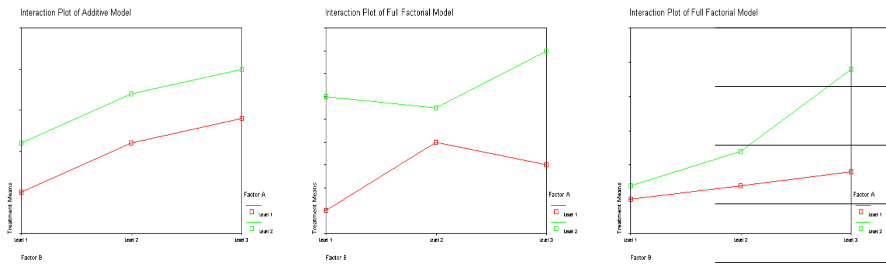


Figure: No interaction between two factors (i.e. their effects are *additive*) (left). Interactions between the two factors (i.e. the effects are *not additive*) (center and right).

Notes

Only Test for Main Effects if the Interaction Isn't Significant

- If we reject H_{0AB}, this **automatically** tells us that **both factors have effects**, **regardless** of whether or not we reject H_{0A} and H_{0B}.

  But the **effect** of **each factor** is **different** depending on the **level** of the **other factor**.

Notes

Example

Here's the **ANOVA table** (again) for our running example:

Source	df	Sum of Squares	Mean Square	f	P-value
Type	1	39447.0	39447.0	458.0	0.000
Years	2	36412.0	18206.0	211.4	0.000
Interaction	2	20165.3	10082.7	117.1	0.000
Error	18	1550.3	86.1		
Total	23	97574.6			

Notes

We can conclude:

- There is an **interaction effect** between **type** of and **years** of experience (**p-value = 0.000**).
- There's **no need to proceed** to the tests for **main effects** because we know, since there's an **interaction**, that **regardless** of what their **p-values** are, both **type** and **years** of experience have effects.

  Their effects each differ depending on the level of the other factor.

The **interaction plot** depicts the **nature** of the **interaction effect**.

Main Effects *Masked* by an Interaction Effect

- We **only proceed** to tests for **main effects** if the **interaction *isn't* significant**.

  When the **interaction *is* significant**, the tests for main effects have **little practical meaning**.

Notes

Example

Consider the following data from a two-factor experiment.

		Factor B: Level 1	Factor B: Level 2	
Factor A	Level 1	20 25	10 14	$\bar{X}_{1\cdot\cdot} = 17.25$
	Level 2	11 14	21 24	$\bar{X}_{2\cdot\cdot} = 17.50$
		$\bar{X}_{\cdot 1\cdot} = 17.50$	$\bar{X}_{\cdot 2\cdot} = 17.25$	

Notes

The resulting **ANOVA table** is below.

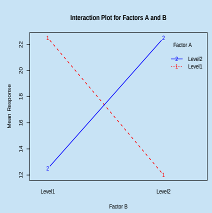
Source	df	Sum of Squares	Mean Square	f	P-value
Factor A	1	0.12	0.12	0.017	0.9027
Factor B	1	0.13	0.13	0.017	0.9027
Factor A:Factor B	1	210.13	210.13	28.492	0.0059
Error	4	29.50	7.37		
Total	7	239.88			

The **interaction** is **significant**, but **neither** of the **main effects is**.

Notes

The table and interaction plot below show that the Factor A and B main effects are **masked** by the interaction effect.

		Factor B: Level 1	Factor B: Level 2	
Factor A	Level 1	$\bar{X}_{11\cdot} =$ **22.5**	$\bar{X}_{12\cdot} =$ **12.0**	$\bar{X}_{1\cdot\cdot} = 17.25$
	Level 2	$\bar{X}_{21\cdot} =$ **12.5**	$\bar{X}_{22\cdot} =$ **22.5**	$\bar{X}_{2\cdot\cdot} = 17.50$
		$\bar{X}_{\cdot 1\cdot} = 17.50$	$\bar{X}_{\cdot 2\cdot} = 17.25$	



Notes

- When the **interaction *is* significant**, we can investigate the effect of each factor **separately** for each fixed **level** of the **other factor**.

  One way to do this is to carry out **Tukey's multiple comparison procedure** on the IJ group means.

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