Statistical Methods

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Topics

Objectives

Objectives:

- State the treatment effects version of the three-factor ANOVA model when $L > 1$.
- Carry out three-factor ANOVA F tests for the interaction effects and main effects of Factors A, B, and C when $L > 1$.
- Interpret the two- and three-factor interaction effects and main effects in the three-factor ANOVA model.
- Interpret residuals and fitted values.
- Use residuals to check normality and constant standard deviation assumptions.

Three-Factor ANOVA

- Sometimes we'll want to simultaneously test for the effects of **three** factors.

Example

An **experiment** was carried out to investigate the distance at detection for four different **radar systems**, two different **aircraft**, flying at **day** and at **night.**

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The data are in a three-dimensional table on the next slide.

Factor B: Radar System

Radar System 1 ($j = 1$)

Factor C: Aircraft

Factor A: Time		Aircraft 1 ($k = 1$)	Aircraft 2 ($k = 2$)
	Day ($i = 1$)	49.21	55.11
		49.37	57.44
	Night ($i = 2$)	47.12	54.75
		50.68	55.80

Radar System 2 ($j = 2$)

Factor C: Aircraft

Factor A: Time		Aircraft 1 ($k = 1$)	Aircraft 2 ($k = 2$)
	Day ($i = 1$)	49.22	47.97
		49.57	47.75
	Night ($i = 2$)	49.56	51.56
		49.62	50.52

(Cont'd next slide)

Radar System 3 $(j = 3)$

		Factor C: Aircraft	
		Aircraft 1 $(k = 1)$	Aircraft 2 $(k = 2)$
Factor A: Time	Day $(i = 1)$	51.90	48.27
		50.00	51.93
	Night $(i = 2)$	48.60	53.74
		50.75	50.99

Radar System 4 $(j = 4)$

		Factor C: Aircraft	
		Aircraft 1 $(k = 1)$	Aircraft 2 $(k = 2)$
Factor A: Time	Day $(i = 1)$	56.96	51.11
		52.95	47.87
	Night $(i = 2)$	53.39	48.08
		54.41	49.80

- Each **combination** of levels of the **three** factors is referred to as a ***group*** (e.g. a ***treatment group*** in an experiment).

- **Notation**:

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$$\begin{aligned}
\boldsymbol{I} &= \text{The numbers of levels of Factor A.} \\
\boldsymbol{J} &= \text{The numbers of levels of Factor B.} \\
\boldsymbol{K} &= \text{The numbers of levels of Factor C.} \\
\boldsymbol{L} &= \text{The number of observations (common sample} \\
&\quad \text{size) in each of the } IJK \text{ treatment groups.} \\
\boldsymbol{X_{ijkl}} &= \text{The } l\text{th observation at the } i\text{th level of Factor A,} \\
&\quad j\text{th level of Factor B, and } k\text{th level of Factor C.}
\end{aligned}$$

- (cont'd)

$$
\begin{aligned}
\bar{X}_{....} &= \text{The } \underline{\textbf{\textit{grand mean}}} \text{ of all } IJKL \text{ observations.}\\
\bar{X}_{i...} &= \text{The } \underline{\textbf{\textit{Factor A level mean}}} \text{ of all observations}\\
&\quad \text{at level } i \text{ of Factor A.}\\
\bar{X}_{.j..} &= \text{The } \underline{\textbf{\textit{Factor B level mean}}} \text{ of all observations}\\
&\quad \text{at level } j \text{ of Factor B.}\\
\bar{X}_{..k.} &= \text{The } \underline{\textbf{\textit{Factor C level mean}}} \text{ of all observations}\\
&\quad \text{at level } k \text{ of Factor C.}\\
\bar{X}_{ij..} &= \text{The mean of all observations at level } i \text{ of Factor}\\
&\quad \text{A and } j \text{ of Factor B.}\\
\bar{X}_{i.k.} &= \text{The mean of all observations at level } i \text{ of Factor}\\
&\quad \text{A and } k \text{ of Factor C.}\\
\bar{X}_{.jk.} &= \text{The mean of all observations at level } j \text{ of Factor}\\
&\quad \text{B and } k \text{ of Factor C.}\\
\bar{X}_{ijk.} &= \text{The } \underline{\textbf{\textit{group mean}}} \text{ of the observations in the}\\
&\quad i, j, k\text{th group.}
\end{aligned}
$$

- **Comments**:
  - The sample sizes per group **don't** all have to be the same. But we'll only look at the equal-sample size case.

- **Comments**:
  - The sample sizes per group **don't** all have to be the same. But we'll only look at the equal-sample size case.
  - The data can be **samples** from IJK populations (representing combinations of the levels of the factors) **or** responses to treatments in a **randomized experiment**.

The Three-Factor ANOVA Model

- When $L > 1$, we use a model that has parameters representing the **effects** of the **three** factors as well as their **interactions**.

Three-Factor ANOVA Model (Full Model):

$$X_{ijkl} = \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC} + \gamma_{ijk}^{ABC} + \epsilon_{ijkl} \,,$$

where

μ is a constant called the ***true grand mean***.
α_i is the ***effect*** of the ith level of **Factor A.**
β_j is the ***effect*** of the jth level of **Factor B.**
δ_k is the ***effect*** of the kth level of **Factor C.**
γ_{ij}^{AB} is the ***two-factor interaction effect*** for the ith level of **Factor A** and jth level of **Factor B.**
γ_{ik}^{AC} is the ***two-factor interaction effect*** for the ith level of **Factor A** and kth level of **Factor C.**
γ_{jk}^{BC} is the ***two-factor interaction effect*** for the jth level of **Factor B** and kth level of **Factor C.**

γ_{ijk}^{ABC} is the ***three-factor interaction effect*** for the ith level of **Factor A**, jth level of **Factor B**, and kth level of **Factor C**.

ϵ_{ijkl} are iid $N(0, \sigma)$ ***random errors***.

(More formal definitions on the next slide.)

- More formally, let

$$\mu_{ijk} = \text{The } \textbf{population mean} \text{ for the } i\text{th level of } \textbf{Factor A}, j\text{th level of } \textbf{Factor B}, \text{ and } k\text{th level of } \textbf{Factor C.}$$

- (cont'd)

  Then the **true grand mean** is:

$$\mu = \frac{\sum_i \sum_j \sum_k \mu_{ijk}}{IJK},$$

and the **Factor A**, **B**, and **C effects** are:

$$\alpha_i = \mu_{i\cdot\cdot} - \mu, \quad \beta_j = \mu_{\cdot j \cdot} - \mu, \quad \text{and} \quad \delta_j = \mu_{\cdot\cdot k} - \mu,$$

where the **true Factor A**, **B**, and **C levels means**, $\boldsymbol{\mu_{i\cdot\cdot}}$, $\boldsymbol{\mu_{\cdot j\cdot}}$ and $\boldsymbol{\mu_{\cdot\cdot k}}$, are defined as:

$$\mu_{i\cdot\cdot} = \frac{\sum_j \sum_k \mu_{ijk}}{JK}, \ \mu_{\cdot j\cdot} = \frac{\sum_i \sum_k \mu_{ijk}}{IK}, \text{ and } \mu_{\cdot\cdot k} = \frac{\sum_i \sum_j \mu_{ijk}}{IJ}.$$

- (cont'd)

  Also, the **three-factor interaction effect** is defined as:

$$\gamma_{ijk}^{ABC} \;=\; \mu_{ijk} \,-\, (\mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC}).$$

- (cont'd)

  Also, the **three-factor interaction effect** is defined as:

$$\gamma_{ijk}^{ABC} \; = \; \mu_{ijk} - (\mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC}).$$

  With this definition,

$$\mu_{ijk} \; = \; \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC} + \gamma_{ijk}^{ABC}.$$

- (cont'd)

  The **two-factor interaction effects** are defined as:

$$
\begin{aligned}
\gamma_{ij}^{AB} &= \mu_{ij\cdot} - (\mu + \alpha_i + \beta_j) \\
&= \mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu\,,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{ik}^{AC} &= \mu_{i\cdot k} - (\mu + \alpha_i + \delta_k) \\
&= \mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot\cdot k} + \mu\,,
\end{aligned}
$$

  and

$$
\begin{aligned}
\gamma_{jk}^{BC} &= \mu_{\cdot jk} - (\mu + \beta_j + \delta_k) \\
&= \mu_{\cdot jk} - \mu_{\cdot j\cdot} - \mu_{\cdot\cdot k} + \mu\,,
\end{aligned}
$$

  where

$$
\mu_{ij\cdot} = \frac{\sum_k \mu_{ijk}}{K}\,, \quad \mu_{i\cdot k} = \frac{\sum_j \mu_{ijk}}{J}\,, \quad \text{and} \quad \mu_{\cdot jk} = \frac{\sum_i \mu_{ijk}}{I}\,.
$$

- It can be shown that defining the α_i's, β_j's, δ_k's, γ_{ij}^{AB}'s, γ_{ik}^{AC}'s, γ_{jk}^{BC}'s, and γ_{ijk}^{ABC}'s as on the previous slides is equivalent to imposing the constraints

$$\sum_i \alpha_i = 0, \qquad \sum_j \beta_j = 0, \qquad \sum_k \delta_j = 0,$$

and

$$\sum_j \gamma_{ij}^{AB} = 0 \text{ (for each fixed } i), \qquad \sum_i \gamma_{ij}^{AB} = 0 \text{ (for each fixed } j).$$

$$\sum_k \gamma_{ik}^{AC} = 0 \text{ (for each fixed } i), \qquad \sum_i \gamma_{ik}^{AC} = 0 \text{ (for each fixed } k).$$

$$\sum_k \gamma_{jk}^{BC} = 0 \text{ (for each fixed } j), \qquad \sum_j \gamma_{jk}^{BC} = 0 \text{ (for each fixed } k),$$

- (cont'd)

  and

  $$\sum_i \gamma_{ijk}^{ABC} = 0\,, \quad \sum_j \gamma_{ijk}^{ABC} = 0\,, \quad \text{and} \quad \sum_k \gamma_{ijk}^{ABC} = 0\,,$$

  where in each summation, the other two subscripts are fixed.

Sums of Squares and the ANOVA Partition

- We can ***partition*** the **total variation** in the data into eight parts reflecting:

Sums of Squares and the ANOVA Partition

- We can ***partition*** the **total variation** in the data into eight parts reflecting:
  - Variation **between** the levels of **Factor A**.
  - Variation **between** the levels of **Factor B**:
  - Variation **between** the levels of **Factor C**:
  - Variation due to the **interaction** between **Factors A** and **B**.
  - Variation due to the **interaction** between **Factors A** and **C**.
  - Variation due to the **interaction** between **Factors B** and **C**.
  - Variation due to the **interaction** between **Factors A**, **B**, and **C**.
  - Variation **within** the groups.

- The **partition** will involve the following ***sums of squares*** (shown with their **df**):

- The **partition** will involve the following ***sums of squares*** (shown with their **df**):
  - **SST** is the ***total sum of squares***, defined as

    $$\text{SST} = \sum_i \sum_j \sum_k \sum_l (X_{ijkl} - \bar{X}_{\dots})^2 \qquad \boldsymbol{df = IJKL - 1}$$

    which measures the **total** variation in the X_{ijkl}'s.

- (cont'd):
  - **SSA**, **SSB**, and **SSC** are the ***Factor A***, **B**, and **C sums of squares**, defined as

$$\begin{aligned} \text{SSA} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{i\ldots} - \bar{X}_{\ldots.})^2 \\ &= JKL \sum_{i=1}^{I} (\bar{X}_{i\ldots} - \bar{X}_{\ldots.})^2 \qquad \boldsymbol{df = I - 1} \end{aligned}$$

$$\begin{aligned} \text{SSB} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{.j..} - \bar{X}_{\ldots.})^2 \\ &= IKL \sum_{j=1}^{J} (\bar{X}_{.j..} - \bar{X}_{\ldots.})^2 \qquad \boldsymbol{df = J - 1} \end{aligned}$$

- (cont'd):
  - (cont'd)

$$\begin{aligned} \text{SSC} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{..k.} - \bar{X}_{....})^2 \\ &= IJL \sum_{k=1}^{K} (\bar{X}_{..k.} - \bar{X}_{....})^2 \qquad \boldsymbol{df = K - 1} \end{aligned}$$

which measure, respectively, variation between the **levels** of **Factor A**, between **levels** of **Factor B**, and between **levels** of **Factor C** due to both the **factor effect** and **random error**.

- (cont'd):
  - **SSAB**, **SSAC**, and **SSBC** are the ***two-factor interaction sums of squares***, given by

$$\begin{aligned}\text{SSAB} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....})^2 \\ &= KL \sum_{i=1}^{I} \sum_{j=1}^{J} (\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....})^2\end{aligned}$$

$$\boldsymbol{df = (I-1)(J-1)}$$

- (cont'd):
  - **SSAB**, **SSAC**, and **SSBC** are the ***two-factor interaction sums of squares***, given by

$$\begin{aligned}
\text{SSAB} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....})^2 \\
&= KL \sum_{i=1}^{I} \sum_{j=1}^{J} (\bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....})^2
\end{aligned}$$

$$\boldsymbol{df = (I-1)(J-1)}$$

$$\begin{aligned}
\text{SSAC} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{i.k.} - \bar{X}_{i...} - \bar{X}_{..k.} + \bar{X}_{....})^2 \\
&= JL \sum_{i=1}^{I} \sum_{k=1}^{K} (\bar{X}_{i.k.} - \bar{X}_{i...} - \bar{X}_{..k.} + \bar{X}_{....})^2
\end{aligned}$$

$$\boldsymbol{df = (I-1)(K-1)}$$

- (cont'd):
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$$\text{SSBC} = \sum_i \sum_j \sum_k \sum_l (\bar{X}_{.jk.} - \bar{X}_{.j..} - \bar{X}_{..k.} + \bar{X}_{....})^2$$
$$= IL \sum_{j=1}^{J} \sum_{k=1}^{K} (\bar{X}_{.jk.} - \bar{X}_{.j..} - \bar{X}_{..k.} + \bar{X}_{....})^2$$
$$\boldsymbol{df = (J-1)(K-1)}$$

which measure, respectively, variation due to the **AB**, **AC**, and **BC two-factor interaction effects** and **random error**.

- (cont'd):
  - **SSABC** is the ***three-factor interaction sum of squares***, given by

$$\begin{aligned} \text{SSABC} &= \sum_i \sum_j \sum_k \sum_l (\bar{X}_{ijk\cdot} - \bar{X}_{ij\cdot\cdot} - \bar{X}_{i\cdot k\cdot} - \bar{X}_{\cdot jk\cdot} + \bar{X}_{i\cdots} \\ &\qquad + \bar{X}_{\cdot j\cdot\cdot} + \bar{X}_{\cdot\cdot k\cdot} - \bar{X}_{\cdots\cdot})^2 \\ &= L \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (\bar{X}_{ijk\cdot} - \bar{X}_{ij\cdot\cdot} - \bar{X}_{i\cdot k\cdot} - \bar{X}_{\cdot jk\cdot} + \bar{X}_{i\cdots} \\ &\qquad + \bar{X}_{\cdot j\cdot\cdot} + \bar{X}_{\cdot\cdot k\cdot} - \bar{X}_{\cdots\cdot})^2 \\ &\qquad \boldsymbol{df = (I-1)(J-1)(K-1)} \end{aligned}$$

which measures variation due to the **ABC three-factor interaction effect** and **random error**.

- (cont'd):
  - **SSE** is the ***error sum of squares***, defined as

$$\text{SSE} = \sum_i \sum_j \sum_k \sum_l (X_{ijkl} - \bar{X}_{ijk\cdot})^2$$

$$\boldsymbol{df = IKJ(L-1)}$$

which measures variation of the X_{ijkl}'s **within** treatment groups due to **random error.**

Proposition

ANOVA Partition for the Full Model: It can be shown that

$$\begin{aligned}\text{SST} &= \text{SSA} + \text{SSB} + \text{SSC} + \text{SSAB} + \text{SSAC} \\ &\quad + \text{SSBC} + \text{SSABC} + \text{SSE}\end{aligned}$$

Additive Property of Degrees of Freedom:

$$\begin{aligned} \text{df for SST} &= \text{df for SSA} + \text{df for SSB} + \text{df for SSC} \\ &\quad + \text{df for SSAB} + \text{df for SSAC} \\ &\quad + \text{df for SSBC} + \text{df for SSABC} \\ &\quad + \text{df for SSE} \end{aligned}$$

Mean Squares for the Full Model

- The ***Factor A***, ***Factor B***, ***Factor C***, ***two-factor interaction***, and ***three-factor interaction mean squares***, and the ***mean squared error*** are:

$$\text{MSA} = \frac{\text{SSA}}{I-1} \qquad \text{MSB} = \frac{\text{SSB}}{J-1}$$

$$\text{MSC} = \frac{\text{SSC}}{K-1} \qquad \text{MSAB} = \frac{\text{SSAB}}{(I-1)(J-1)}$$

$$\text{MSAC} = \frac{\text{SSAC}}{(I-1)(K-1)} \qquad \text{MSBC} = \frac{\text{SSAB}}{(J-1)(K-1)}$$

$$\text{MSABC} = \frac{\text{SSAB}}{(I-1)(J-1)(K-1)} \qquad \text{MSE} = \frac{\text{SSE}}{IJK(L-1)}$$

The Three-Factor ANOVA F-Tests

- Suppose data in a three-factor study follow the **three-factor ANOVA model**, where the error terms ϵ_{ijkl} are iid $N(0, \sigma)$.

The Three-Factor ANOVA F-Tests

- Suppose data in a three-factor study follow the **three-factor ANOVA model**, where the error terms ϵ_{ijkl} are iid $N(0, \sigma)$.
- The table below lists the **eight** sets of **hypotheses**, $\boldsymbol{F}$ **test statistics**, and **sampling distributions** of the test statistics under the null hypothesis.

Effect	Hypotheses	Test Statistic	Distribution of Test Statistic Under H_0
Factor A	H_{0A}: $\alpha_i = 0$ for all i H_{aA}: not all α_i's equal zero	$F = \frac{\text{MSA}}{\text{MSE}}$	F(I-1, IJK(L-1))
Factor B	H_{0B}: $\beta_j = 0$ for all j H_{aB}: not all β_j's equal zero	$F = \frac{\text{MSB}}{\text{MSE}}$	F(J-1, IJK(L-1))
Factor C	H_{0C}: $\delta_k = 0$ for all j H_{aC}: not all δ_k's equal zero	$F = \frac{\text{MSC}}{\text{MSE}}$	F(K-1, IJK(L-1))
AB Interaction	H_{0AB}: $\gamma_{ij} = 0$ for all i and j H_{aAB}: not all γ_{ij}'s equal zero	$F = \frac{\text{MSAB}}{\text{MSE}}$	F((I-1)(J-1), IJK(L-1))
AC Interaction	H_{0AC}: $\gamma_{ik} = 0$ for all i and k H_{aAC}: not all γ_{ik}'s equal zero	$F = \frac{\text{MSAC}}{\text{MSE}}$	F((I-1)(K-1), IJK(L-1))
BC Interaction	H_{0BC}: $\gamma_{jk} = 0$ for all j and k H_{aBC}: not all γ_{jk}'s equal zero	$F = \frac{\text{MSBC}}{\text{MSE}}$	F((J-1)(K-1), IJK(L-1))
ABC Interaction	H_{0ABC}: $\gamma_{ijk}^{ABC} = 0$ for all i, j, and k H_{aABC}: not all γ_{ijk}^{ABC}'s equal zero	$F = \frac{\text{MSABC}}{\text{MSE}}$	F((I-1)(J-1)(K-1), IJK(L-1))

- In each case, the **null hypothesis** says there's **no effect** and the **alternative** says there **is an effect**.

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In each case, a **large value** of the F test statistic provides **evidence against H_0 in favor of H_a.**

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  - The ***rejection regions*** as the **extreme largest 100α% of F values**.

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  - The ***rejection regions*** as the **extreme largest 100α% of F values**.
  - The ***p-values*** as the **tail areas to the right of the observed F values**.

- **Comment**: The **ANOVA F tests** can be used even if the samples are from **non-normal** populations as long the per-group sample sizes are large.

The ANOVA Table

- The results are summarized in an ***ANOVA table***:

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Factor A	$I-1$	SSA	MSA = $\frac{\text{SSA}}{I-1}$	$\frac{\text{MSA}}{\text{MSE}}$	p
Factor B	$J-1$	SSB	MSB = $\frac{\text{SSB}}{J-1}$	$\frac{\text{MSB}}{\text{MSE}}$	p
Factor C	$K-1$	SSC	MSC = $\frac{\text{SSC}}{K-1}$	$\frac{\text{MSC}}{\text{MSE}}$	p
AB Interaction	$(I-1)(J-1)$	SSAB	MSAB = $\frac{\text{SSAB}}{(I-1)(J-1)}$	$\frac{\text{MSAB}}{\text{MSE}}$	p
AC Interaction	$(I-1)(K-1)$	SSAC	MSAC = $\frac{\text{SSAC}}{(I-1)(K-1)}$	$\frac{\text{MSAC}}{\text{MSE}}$	p
BC Interaction	$(J-1)(K-1)$	SSBC	MSBC = $\frac{\text{SSBC}}{(J-1)(K-1)}$	$\frac{\text{MSBC}}{\text{MSE}}$	p
ABC Interaction	$(I-1)(J-1)(K-1)$	SSABC	MSABC = $\frac{\text{SSABC}}{(I-1)(J-1)(K-1)}$	$\frac{\text{MSABC}}{\text{MSE}}$	p
Error	$IJK(L-1)$	SSE	MSE = $\frac{\text{SSE}}{IJK(L-1)}$		
Total	$IJKL-1$	SST			

Interpretation of the Three-Factor Interaction Effect

- ***If*** there's **no three-factor interaction**, then in the **ANOVA model**,

$$\mu_{ijk} \;=\; \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC}$$

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In this case, each **two-factor interaction** effect is the ***same***, **regardless** of the **level** of the **third factor.**

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  In this case, each **two-factor interaction** effect is the ***same***, **regardless** of the **level** of the **third factor.**

- **Including** the **three-factor interaction** term $\boldsymbol{\gamma_{ijk}^{ABC}}$ in the model (and allowing it to be non-zero), allows the **two-factor interaction** effects to be ***different*** depending on the **level** of the **third factor**.

- Consider ***three scenarios*** with $I = 2$ **levels** of **Factor A**, $J = 2$ **levels** of **Factor B**, and $K = 3$ **levels** of **Factor C.**

- **Scenario 1 – Main Effects Only**: There are A, B, and C main effects, but no two- or three-factor interactions.

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  The **true group means** $\boldsymbol{\mu_{ijk}}$ could be written as

$$\mu_{ijk} \;=\; \mu + \alpha_i + \beta_j + \delta_k,$$

  and the **model** as

$$X_{ijkl} \;=\; \mu + \alpha_i + \beta_j + \delta_k + \epsilon_{ijkl}\,.$$

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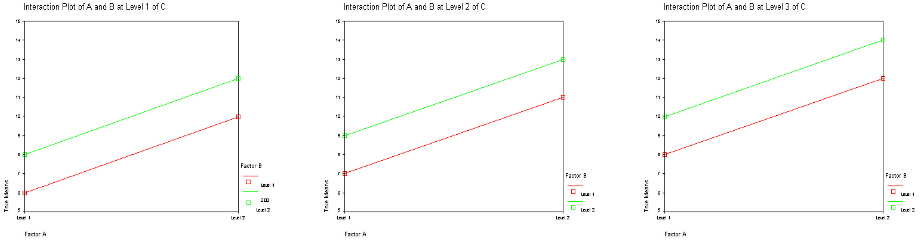
  This so-called ***additive model*** says the **effect** of **each factor** is the ***same*** for ***every combination*** of the **levels** of the **other two factors.**



Above, the **Factor A effect** (represented by the upward slope of the lines) is the ***same*** for **every combination** of **levels** of **Factors B** and **C**.

- **Scenario 2 – Main Effects and AB Interaction**: There are A, B, and C main effects and an AB two-factor interaction, but no other two-factor interactions and no three-factor interaction.

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  The **true group means** $\boldsymbol{\mu_{ijk}}$ could be written as

  $$\mu_{ijk} \;=\; \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB}$$

  and the **model** as

  $$X_{ijkl} \;=\; \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \epsilon_{ijkl} \, .$$

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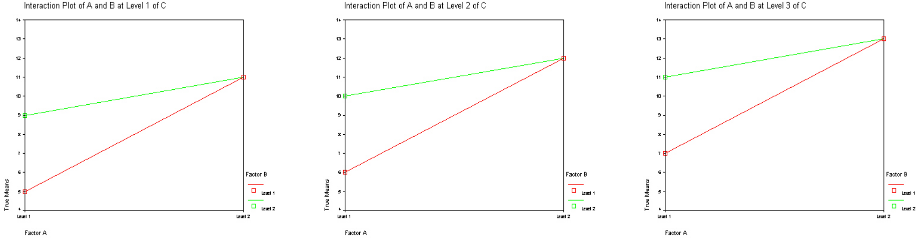
  and the **model** as

$$X_{ijkl} \;=\; \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \epsilon_{ijkl}\,.$$

  This **model** says that there's an **interaction** between **Factors A** and **B**, but the **AB interaction effect** is the ***same*** **regardless** of the **level** of **Factor C**.



Above, the **AB interaction pattern** is the ***same*** for **every level** of **Factor C**.

- **Scenario 3 – Main Effects, AB, BC, and AC Two-Factor Interactions, and ABC Three-Factor Interaction**: There are A, B, and C main effects, AB, AC, and BC two-factor interactions, and an ABC three-factor interaction.

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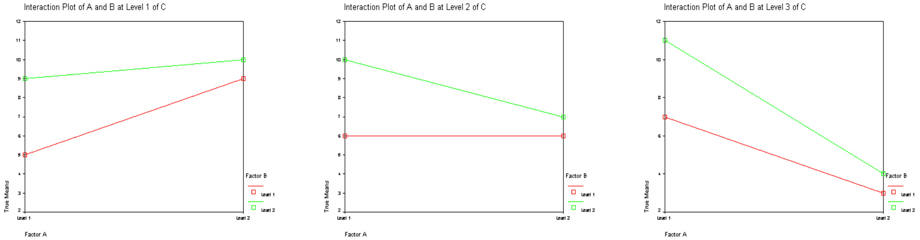
  This so-called **full model** allows **each two-factor interaction effect** to be ***different*** **depending** on the **level** of the **third factor**.



Above, the **three-factor interaction** is apparent because the **AB interaction pattern** is ***different*** **depending** on the **level** of **Factor C.**

Only Test for a Lower-Order Effect If It Isn't Involved in a Significant Higher-Order Interaction Effect

- ***If*** a **higher-order interaction** is significant, all lower-order terms involved in that interaction have effects, ***regardless*** of their **p-values.**

Example

For the study of the effects of four **radar systems**, two different **aircraft**, and two different **time periods** (day and night), the **ANOVA table** is below.

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Time	1	0.235	0.235	0.094	0.764
System	3	40.480	13.493	5.380	0.009
Aircraft	1	2.750	2.750	1.096	0.311
Time:System	3	8.205	2.735	1.091	0.382
Time:Aircraft	1	5.152	5.152	2.054	0.171
System:Aircraft	3	142.532	47.511	18.944	0.000
Time:System:Aircraft	3	5.882	1.961	0.782	0.521
Error	16	40.127	2.508		
Total	31	245.362			

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  The **two-factor interaction** between **System** and **Aircraft** ***is*** **significant** ($F = 18.944$, **p-value = 0.000**).

  Neither of the other **two-factor interactions** is significant ($F = 2.054$, **p-value = 0.171**, and $F = 1.091$, **p-value = 0.382**).

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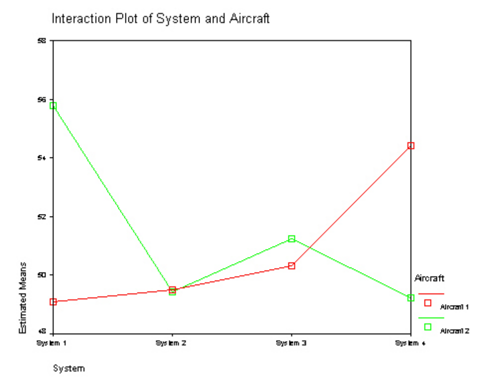
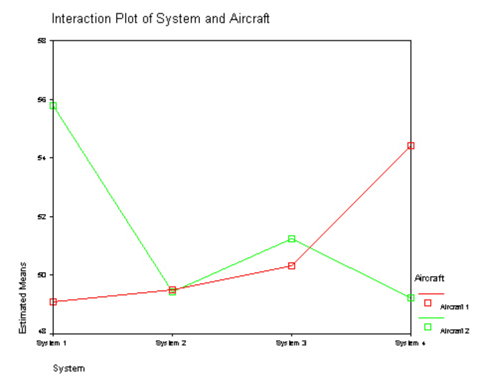
The next step is to examine the nature of the **significant** effects using plots.

An **interaction plot** of **radar system** and **aircraft** is on the next slide.


Interaction Plot of System and Aircraft
Estimated Means
58
56
54
52
50
48
System 1
System 2
System 3
System 4
System
Aircraft
Aircraft 1
Aircraft 2



Based on the plot, for **Aircraft 1**, the best **radar system** is **System 4.** But for **Aircraft 2**, the best system is **System 1.**

Estimating Parameters in the Full Model

- Here are the estimators for the model parameters.

Model Parameter Estimators: We estimate the unknown model parameters μ, α_i, β_j, δ_k, γ_{ij}^{AB}, γ_{ik}^{AC}, γ_{jk}^{BC}, γ_{ijk}^{ABC} and σ using the **estimators** $\boldsymbol{\hat{\mu}}$, $\boldsymbol{\hat{\alpha}_i}$, $\boldsymbol{\hat{\beta}_j}$, $\boldsymbol{\hat{\delta}_k}$, $\boldsymbol{\hat{\gamma}_{ij}^{AB}}$, $\boldsymbol{\hat{\gamma}_{ik}^{AC}}$, $\boldsymbol{\hat{\gamma}_{jk}^{BC}}$, $\boldsymbol{\hat{\gamma}_{ijk}^{ABC}}$, and $\boldsymbol{\hat{\sigma}}$, defined as:

Model Parameter	Estimator
μ	$\hat{\mu} = \bar{X}_{....}$
$\alpha_i = \mu_{i..} - \mu$	$\hat{\alpha}_i = \bar{X}_{i...} - \bar{X}_{....}$
$\beta_j = \mu_{.j.} - \mu$	$\hat{\beta}_j = \bar{X}_{.j..} - \bar{X}_{....}$
$\delta_k = \mu_{..k} - \mu$	$\hat{\delta}_k = \bar{X}_{..k.} - \bar{X}_{....}$
$\gamma_{ij}^{AB} = \mu_{ij.} - \mu_{i..} - \mu_{.j.} + \mu$	$\hat{\gamma}_{ij}^{AB} = \bar{X}_{ij..} - \bar{X}_{i...} - \bar{X}_{.j..} + \bar{X}_{....}$
$\gamma_{ik}^{AC} = \mu_{i.k} - \mu_{i..} - \mu_{..k} + \mu$	$\hat{\gamma}_{ik}^{AC} = \bar{X}_{i.k.} - \bar{X}_{i...} - \bar{X}_{..k.} + \bar{X}_{....}$
$\gamma_{jk}^{BC} = \mu_{.jk} - \mu_{.j.} - \mu_{..k} + \mu$	$\hat{\gamma}_{jk}^{BC} = \bar{X}_{.jk.} - \bar{X}_{.j..} - \bar{X}_{..k.} + \bar{X}_{....}$
$\gamma_{ijk}^{ABC} = \mu_{ijk} - (\mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC})$	$\hat{\gamma}_{ijk}^{ABC} = \bar{X}_{ijk.} - \left(\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_k + \hat{\gamma}_{ij}^{AB} + \hat{\gamma}_{ik}^{AC} + \hat{\gamma}_{jk}^{BC}\right)$
σ	$\hat{\sigma} = \sqrt{\text{MSE}}$

- **Comment**: From parameter **estimates** above, the **sums of squares** can be written as:

$$\text{SSA} = \sum_i \sum_j \sum_k \sum_l \ \hat{\alpha}_i^2 \qquad \text{SSB} = \sum_i \sum_j \sum_k \sum_l \ \hat{\beta}_j^2$$

$$\text{SSC} = \sum_i \sum_j \sum_k \sum_l \ \hat{\delta}_k^2 \qquad \text{SSAB} = \sum_i \sum_j \sum_k \sum_l \ (\hat{\gamma}_{ij}^{AB})^2$$

$$\text{SSAC} = \sum_i \sum_j \sum_k \sum_l \ (\hat{\gamma}_{ik}^{AC})^2 \qquad \text{SSBC} = \sum_i \sum_j \sum_k \sum_l \ (\hat{\gamma}_{jk}^{BC})^2$$

$$\text{SSABC} = \sum_i \sum_j \sum_k \sum_l \ (\hat{\gamma}_{ijk}^{ABC})^2$$

Fitted Values and Residuals

- The ***fitted value*** (or ***predicted value***) for the lth individual in the i, j, kth group, $\boldsymbol{\hat{X}_{ijkl}}$, is

$$\begin{aligned}\hat{X}_{ijkl} &= \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_k + \hat{\gamma}_{ij}^{AB} + \hat{\gamma}_{ik}^{AC} + \hat{\gamma}_{jk}^{BC} + \hat{\gamma}_{ijk}^{ABC} \\ &\vdots \quad \text{(using definitions of the estimators)} \\ &= \bar{X}_{ijk\cdot}\,.\end{aligned}$$

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  It's just the $\boldsymbol{i, j, k}$**th group mean** $\boldsymbol{\bar{X}_{ijk\cdot}}$ (which is also the estimate of the true group mean μ_{ijk}).

- The ***residual*** for the lth observation in the i, j, kth group, $\boldsymbol{e_{ijkl}}$, is defined as

$$\begin{aligned} e_{ijkl} &= X_{ijkl} - \hat{X}_{ijkl} \\ &= X_{ijkl} - (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_k + \hat{\gamma}_{ij}^{AB} + \hat{\gamma}_{ik}^{AC} + \hat{\gamma}_{jk}^{BC} + \hat{\gamma}_{ijk}^{ABC}) \\ &= X_{ijkl} - \bar{X}_{ijk\cdot}\,. \end{aligned}$$

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  The **residual** $\boldsymbol{e_{ijkl}}$ corresponds to the **random error** term $\boldsymbol{\epsilon_{ijkl}}$ in the model.

  Note that a **residual** is just the **deviation** of an observed response $\boldsymbol{X_{ijkl}}$ **away from** the **group mean** $\boldsymbol{\bar{X}_{ijk\cdot}}$.

- **Comment**: The **error sum of squares** is the **sum of squared residuals**, i.e.

$$\text{SSE} = \sum_i \sum_j \sum_k \sum_l e_{ijkl}^2 \,.$$

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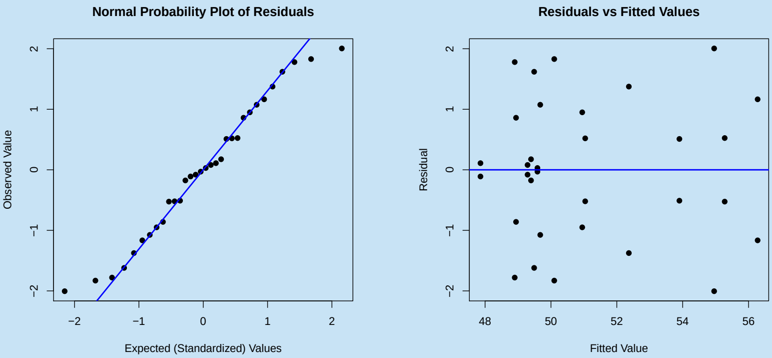
- **Checking the Normality Assumption**: Use a **histogram** or **normal probability plot** of the **residuals.**

- **Checking the Constant σ Assumption**: Plot the **residuals** versus the **fitted values.**

  Usually, when σ *isn't* constant, it increases with the group mean.

Example

For the study of aircraft radar systems, a **normal probability plot** of the **residuals** and a plot of **residuals** versus **fitted values** are shown below.



The first plot indicates that the assumption of **normality** of the error term ϵ_{ijkl} is valid, and the second indicates that the assumption of a **constant standard deviation** σ of ϵ_{ijkl} is valid.