# Statistical Methods

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# Topics

# Objectives

**Objectives**:

- Carry out a Kruskal-Wallis test for $I$ population means.

# Kruskal-Wallis Test for $I$ Population Means $\mu_1, \mu_2, \ldots, \mu_I$

- The ***Kruskal-Wallis test*** is a **nonparametric** alternative to the **one-factor ANOVA $F$ test** for comparing $I$ population means $\mu_1, \mu_2, \ldots, \mu_I$.

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# Kruskal-Wallis Test for $I$ Population Means $\mu_1, \mu_2, \ldots, \mu_I$

- The ***Kruskal-Wallis test*** is a **nonparametric** alternative to the **one-factor ANOVA $F$ test** for comparing $I$ population means $\mu_1, \mu_2, \ldots, \mu_I$.
- We assume that we have *independent* random samples from $I$ continuous populations that all have the **same shape** but possibly different means $\mu_1, \mu_2, \ldots, \mu_I$.

- The **null hypothesis** is that there are no differences among the population means $\mu_1$, $\mu_2$, ..., $\mu_I$:

**Null Hypothesis**:

$$H_0 : \mu_1 \; = \; \mu_2 \; = \; \cdots \; = \; \mu_I$$

- The **alternative hypothesis** is that there's *at least one difference* among the set of means:

**Alternative Hypothesis**: The alternative hypothesis will be

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These are the same hypotheses as the ones tested in a *one-factor ANOVA $F$ test*.

- **Notation**:

$$
\begin{aligned}
I &= \text{The number of populations from which samples are taken.} \\
Y_{ij} &= \text{The } j\text{th observation in the } i\text{th sample.} \\
J_1, J_2, \ldots J_I &= \text{The sample sizes (not necessarily equal).} \\
N &= \text{The total number of observations (in the } I \text{ samples combined), i.e.}
\end{aligned}
$$

$$N = \sum_i J_i \, .$$

- (cont'd)

  Now consider **combining** the $I$ samples and and **ranking** the observations from smallest (rank = 1) to largest (rank = $N$).

$$
\begin{aligned}
\boldsymbol{R_{ij}} &= \text{The rank of } Y_{ij}\,. \\
\boldsymbol{\bar{R}_i} &= \text{The average of the ranks of the observations} \\
&\qquad \text{from the } i\text{th sample.} \\
\boldsymbol{\bar{R}} &= \text{The overall average of all } N \text{ ranks.}
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  For **ties**, use the **average** of the **ranks** that would've been assigned if there weren't any ties.

## Proposition

The overall average of all $N$ ranks is

$$\bar{R} = \frac{N+1}{2}$$

because the sum of the ranks is

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}.$$

**Kruskal-Wallis Test Statistic**:

$$K \;=\; \frac{12}{N(N+1)} \sum_{i=1}^{I} J_i \left(\bar{R}_i - \bar{R}\right)^2 \,.$$

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- $K$ is the **sum of** ($J_i$-*weighted*) **squares** of the **mean ranks** away from the **overall mean rank**, multiplied by the constant $12/N(N+1)$ (explained later).

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  It's analogous to the **treatment sum of squares SSTr** in **one-factor ANOVA**.

- $K$ will be **small** when the **mean ranks** $\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_I$ are approximately **equal**, i.e. if the combined, sorted samples "intermingle", as would be the case if $\mu_1, \mu_2, \ldots, \mu_I$ were equal.

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  $K$ will be **large** when the **mean ranks** $\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_I$ **differ**, i.e. if the combined, sorted samples "segregate", as would be the case if there were differences among $\mu_1, \mu_2, \ldots, \mu_I$

**Large values** of $K$ provide **evidence against $H_0$ in favor of $H_a$ : At least two of the $\mu_i$'s are different.**

- Now suppose the sample sizes $J_1, J_2, \ldots, J_I$ are all *large*.

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  In this case, the sampling distribution of the test statistic is as follows.

**Sampling Distribution of the Test Statistic Under $H_0$:**
If $K$ is the Kruskal-Wallis test statistic, then when

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I$$

is true,

$$K \sim \chi^2(I-1)$$

(approximately), a chi-squared distribution with $(I-1)$ degrees of freedom.

- The $\chi^2(I-1)$ curve gives us:

- The $\chi^2(I - 1)$ curve gives us:
    - The ***rejection region*** as the **extreme largest 100$\alpha$% of $K$ values**.

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  - The ***rejection region*** as the **extreme largest 100$\alpha$% of $K$ values**.
  - The ***p-value*** as the **tail area to the right of the observed $K$ value**.

- **Comment**: The degrees of freedom is $I - 1$ because the $I$ ($J_i$-*weighted*) deviations $J_i(\bar{R}_i - \bar{R})$ sum to zero, so only $I - 1$ of them are "free to vary".

- **Comment**: The degrees of freedom is $I-1$ because the $I$ ($J_i$-*weighted*) deviations $J_i(\bar{R}_i - \bar{R})$ sum to zero, so only $I-1$ of them are "free to vary".

- **Comment**: The constant $12/N(N+1)$ "rescales" the sum $\sum_i J_i \left(\bar{R}_i - \bar{R}\right)^2$ just enough to force $K$ to follow a $\chi^2(I-1)$ distribution (under $H_0$).

### Example

An agricultural experiment was carried out to examine the effects of **four** soil **treatments** on the soil **phosphorus** levels.

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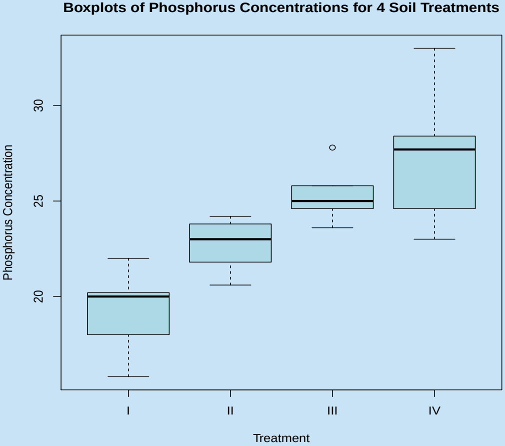
An agricultural experiment was carried out to examine the effects of **four** soil **treatments** on the soil **phosphorus** levels.

**Twenty** plots of land were randomly assigned to receive one of the **four treatments**, with **five** plots per treatment.

The phosphorus concentrations (mg/g) in the topsoils of the plots are shown on the next slide.

| Treatment I | Treatment II | Treatment III | Treatment IV |
|---|---|---|---|
| 20.2 | 23.0 | 23.6 | 23.0 |
| 15.8 | 21.8 | 27.8 | 33.0 |
| 18.0 | 24.2 | 25.8 | 28.4 |
| 20.0 | 20.6 | 24.6 | 24.6 |
| 22.0 | 23.8 | 25.0 | 27.7 |

Side-by-side **boxplots** of the data are on the next slide.


Boxplots of Phosphorus Concentrations for 4 Soil Treatments
Phosphorus Concentration
20
25
30
I
II
III
IV
Treatment

Because the four sample sizes are small, it is difficult to ascertain from plots alone whether the data are normally distributed.

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Suppose we're unwilling to assume normality because previous studies have shown that soil phosphorus concentrations follow **right skewed** distributions.

We'll carry out a **Kruskal-Wallis test** of

$$H_0: \quad \mu_1 = \mu_2 = \cdots = \mu_I$$

$H_a$ : At least two of the $\mu_i$'s are different

to decide if there are *any* significant **differences** in the mean **phosphorus** concentrations for the **four treatments.**

Here are the samples **combined**, **sorted**, and **ranked**.

| Observation | 15.8 | 18.0 | 20.0 | 20.2 | 20.6 | 21.8 | 22.0 | 23.0 | 23.0 | 23.6 |
|---|---|---|---|---|---|---|---|---|---|---|
| Sample | I | I | I | I | II | II | I | II | IV | III |
| Rank | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8.5 | 8.5 | 10 |

| Observation | 23.8 | 24.2 | 24.6 | 24.6 | 25.0 | 25.8 | 27.7 | 27.8 | 28.4 | 33.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| Sample | II | II | III | IV | III | III | IV | III | IV | IV |
| Rank | 11 | 12 | 13.5 | 13.5 | 15 | 16 | 17 | 18 | 19 | 20 |

The $I = 4$ **sample sizes** are all the same:

$$\boldsymbol{J_1 = J_2 = J_3 = J_4 = 5}.$$

The **overall sample size** is $N = 20$.

The **group mean *ranks*** are:

$$\bar{R}_1 = \frac{1+2+3+4+7}{5} = \mathbf{3.4}.$$

$$\bar{R}_2 = \frac{5+6+8.5+11+12}{5} = \mathbf{8.5}.$$

$$\bar{R}_3 = \frac{10+13.5+15+16+18}{5} = \mathbf{14.5}.$$

$$\bar{R}_4 = \frac{8.5+13.5+17+19+20}{5} = \mathbf{15.6}.$$

The **overall mean *rank*** is:

$$\bar{R} = \frac{20+1}{2} = \mathbf{10.5}.$$

The **test statistic** is:

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$$\begin{aligned} K &= \frac{12}{N(N+1)} \sum_{i=1}^{I} J_i \left(\bar{R}_i - \bar{R}\right)^2 \\ &= \frac{12}{20(20+1)} \Big[5\,(3.4 - 10.5)^2 + 5\,(8.5 - 10.5)^2 \\ &\qquad + 5\,(14.5 - 10.5)^2 + 5\,(15.6 - 10.5)^2\Big] \end{aligned}$$

The **test statistic** is:

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From **Table A11** (the **chi-squared distribution** table) with $I - 1 = 3$ **df**, the **p-value** is **between** than **0.001** and **0.005**.

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Using $\alpha = 0.05$, we **reject** $H_0$. There are differences among the phosphorus levels for the four treatments.