9 Paired Samples Hypothesis Tests

MTH 3240 Environmental Statistics

Spring 2020

Objectives

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- Distinguish matched pairs study designs from independent samples designs.
- Carry out a paired t test for the difference between two population means.
- Compute and interpret paired t confidence interval for the difference between two population means.

Matched Pairs Studies

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- In ***matched pairs study designs***, two samples are collected in such a way that each individual in one sample **matches** with one in the other sample.
- In environmental studies, they arise when a variable is measured at **concurrent time points** or **coincident spatial locations** under each of **two conditions.**

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But dredging can alter sediment composition in ways that are detrimental to benthic (bottom dwelling) organisms.

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The **percent clay** in sediment was measured at $n = 8$ sites in the harbor **before** it was dredged and again at the **same eight sites after** dredging.

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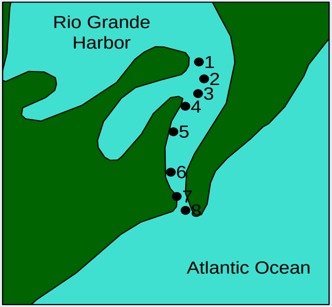
The **percent clay** in sediment was measured at $n = 8$ sites in the harbor **before** it was dredged and again at the **same eight sites after** dredging.

Each of the eight sites forms a **matched pair** of **before** and **after** measurements.

Locations of Sampling Stations in Dredging Impact Assessment Study



The data, including the ***differences*** for the eight sites, are below.

	Clay Percent		
Site	After Dredging	Before Dredging	Difference
1	53.8	61.3	-7.5
2	38.4	60.8	-22.4
3	54.1	49.4	4.7
4	55.7	56.2	-0.5
5	42.0	58.6	-16.6
6	48.1	57.1	-9.0
7	48.7	55.4	-6.7
8	19.3	48.3	-29.0
	$\bar{X} = 45.0$	$\bar{Y} = 55.9$	$\bar{D} = -10.9$
			$S_d = 11.2$.

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The table below shows the data and the ***differences***.

Nitrate Concentration

Date	Upstream	Downstream	Difference
08/15/97	1147.4	995.3	152.1
08/18/97	1412.2	1303.6	108.6
08/31/97	1613.9	1923.3	-309.4
09/18/97	763.3	747.8	15.5
11/04/97	1031.4	1082.9	-51.5
11/07/97	1093.2	1938.7	-845.5
02/27/98	390.8	338.8	52.0
07/14/98	909.8	776.8	133.0
08/25/98	1033.0	676.8	356.2
09/30/98	897.5	1291.0	-393.5
10/29/98	2314.0	1232.9	1081.1
	$\bar{X} = 1146.0$	$\bar{Y} = 1118.9$	$\bar{D} = 27.1$
			$S_d = 480.7$.

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  In the previous example, we **control** for **natural day-to-day variation** in variables that affect the stream's nitrate concentration by pairing **upstream** and **downstream** measurements **by day.**

  In the dredging example, we **control** for **spatial variation** in variables that affect the the harbor's clay percent by pairing **before** and **after** measurements **by location.**

- **Within** a **matched pair**, the two measured values are usually **similar** (compared to values for *unmatched* individuals).

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  .... but in a **matched pairs study**, they're drawn in **pairs** (i.e. **not** independently).

Exercise

For each of the following studies, decide whether a **matched pairs** design or an **independent samples** design was used.

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a) In 1878 Charles Darwin performed an experiment to determine if the height of a *Zea mays* (corn) plant is affected by whether the plant is cross-fertilized or self-fertilized. In each of **15 pots**, two plants were grown, one **self-fertilized** and the other **cross-fertilized**, and their heights later measured.

a) In a study of the impact of agriculture on surface water quality, phosphorus was measured during each of **33 rainstorm events** at the outlets of two adjacent watersheds, one of which contains **farms** and the other **no farms.**

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b) In a study of the health hazards for workers in **two types** of swine confinement **buildings**, dried fecal matter was measured in the air of **12** randomly selected **finishing buildings**, where 50 - 100 kg animals are housed, and also in a random sample of **11 nursery buildings**, which house smaller animals.

c) On April 7, 2000, an **oil pipeline** owned by the Potomac Electric Power Company **ruptured**, spilling 126,000 gallons of oil into marsh areas of Swanson Creek, Maryland. To **assess** the **impact** on benthic (bottom dwelling) communities, benthos were evaluated at **10** randomly selected locations in **Swanson Creek** near the spill and at **10** other randomly selected locations in the nearby, undisturbed **Hunting Creek**.

c) On April 7, 2000, an **oil pipeline** owned by the Potomac Electric Power Company **ruptured**, spilling 126,000 gallons of oil into marsh areas of Swanson Creek, Maryland. To **assess** the **impact** on benthic (bottom dwelling) communities, benthos were evaluated at **10** randomly selected locations in **Swanson Creek** near the spill and at **10** other randomly selected locations in the nearby, undisturbed **Hunting Creek**.

d) In a study of the long-term **effect** on shoreline biology of **effluent** from an oil refinery at Littlewick Bay, Wales, barnacle densities were measured at **10** shoreline **locations near the refinery** in **1974** and again at the **same 10 locations** in **1981**.

e) A study was carried out by the University of Toronto to **assess** the **impact** of **effluent** from a sewage treatment plant near Orangeville, Ontario, Canada into the Credit River.

Fecal coliform was measured on **each of several days** 1.5 km **upstream** of the plant and on those *same days* 2.5 km **downstream**.

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  2. The **signed rank test**
  3. The **sign test for paired samples**

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  The **paired t test** requires a **normality** assumption (or large sample sizes).

  The **signed rank test** and **sign test** are **nonparametric** tests (i.e. they **don't** rely on a normality assumption).

Paired t Test

- For the ***paired t test***, we suppose we have **paired samples**

$$X_1, X_2, \ldots, X_n \qquad \text{and} \qquad Y_1, Y_2, \ldots, Y_n$$

from **two populations** whose **means** are

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  from **two populations** whose **means** are

  $$\boldsymbol{\mu_x} \qquad \text{and} \qquad \boldsymbol{\mu_y}$$

  Here, X_1 and Y_1 are a **pair**, X_2 and Y_2 are a **pair**, and so on.

- The sample size n is the **same** for the two samples.

- The **null hypothesis** is that there's **no difference** between μ_x and μ_y.

Null Hypothesis:

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(Same hypothesis as for the two-sample t test.)

- The **alternative hypothesis** is one of the following.

Alternative Hypothesis:

1. $H_a : \mu_x - \mu_y > 0$ (**upper-tailed test**)
2. $H_a : \mu_x - \mu_y < 0$ (**lower-tailed test**)
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(Same hypotheses as for the two-sample t test.)

- We'll denote the ***differences*** by $\boldsymbol{D_1, D_2, \ldots, D_n}$, that is,

$$\begin{aligned} D_1 &= X_1 - Y_1 \\ D_2 &= X_2 - Y_2 \\ &\vdots \\ D_n &= X_n - Y_n. \end{aligned}$$

- We'll act as though these **differences** are a **random sample** from a ***population of differences*** whose mean is μ_d.

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- The ***paired*** t ***test*** is just a **one-sample** t **test** for μ_d based on the **differences.**

Fact:

$$\bar{D} = \bar{X} - \bar{Y},$$

where $\bar{X}$ and $\bar{Y}$ are the means of the X and Y samples and $\bar{D}$ is the mean of the sample of differences.

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Thus the hypotheses can be restated in terms of $\boldsymbol{\mu_d}$ as:

	Hypothesis About $\boldsymbol{\mu_x - \mu_y}$	**Equivalent Hypothesis About $\boldsymbol{\mu_d}$**
Null	$H_0 : \mu_x - \mu_y = 0$	$H_0 : \mu_d = 0$
Alternatives	$H_a : \mu_x - \mu_y > 0$	$H_a : \mu_d > 0$
	$H_a : \mu_x - \mu_y < 0$	$H_a : \mu_d < 0$
	$H_a : \mu_x - \mu_y \neq 0$	$H_a : \mu_d \neq 0$

- In a **matched pairs study**, the ***effect size*** is

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It's **estimated** by

$$\bar{X} - \bar{Y} = \bar{D}.$$

Paired t Test Statistic:

$$t \;=\; \frac{\bar{D} \;-\; 0}{S_{\bar{D}}} \;=\; \frac{\bar{D}}{S_{\bar{D}}}$$

where

$$S_{\bar{D}} \;=\; \frac{S_d}{\sqrt{n}}\,,$$

and $\bar{D}$ and S_d are the sample mean and sample standard deviation of the **differences**.

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and $\bar{D}$ and S_d are the sample mean and sample standard deviation of the **differences**.

- Note that t is just the **one-sample t test statistic** based on the **differences.**

1. *Large positive* values of t provide evidence in favor of $H_a : \mu_x - \mu_y > 0$ (or $H_a : \mu_d > 0$).
2. *Large negative* values of t provide evidence in favor of $H_a : \mu_x - \mu_y < 0$ (or $H_a : \mu_d < 0$).
3. *Both large positive and large negative* values of t provide evidence in favor of $H_a : \mu_x - \mu_y \neq 0$ (or $H_a : \mu_d \neq 0$).

- Now suppose either the sample of **differences** is from **normal population** or n is **large.**

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  In this case, the **null distribution** is as follows.

Sampling Distribution of t Under H_0: If t is the paired t test statistic, then when

$$H_0 : \mu_x - \mu_y \;=\; 0 \qquad \text{(or equivalently } H_0 : \mu_d \;=\; 0\text{)}$$

is true,

$$t \;\sim\; t(n-1).$$

- **P-values** and **rejection regions** are obtained from the appropriate tail(s) of the $t(n-1)$ **distribution.**

Paired t Test for μ_d

Assumptions: $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are two random samples that are paired and either the differences $d_1, d_2, \ldots, d_n$ form a single sample from a *normal* population or n is large.

Null hypothesis: $H_0 : \mu_d = 0$.

Test statistic value: $t = \frac{\bar{d}}{s_d/\sqrt{n}}$.

Decision rule: Reject H_0 if p-value $< \alpha$ or t is in rejection region.

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Alternative hypothesis	P-value = area under t-distribution with $n-1$ d.f.:	Rejection region = t values such that:*
$H_a : \mu_d > 0$	to the right of t	$t > t_{\alpha,n-1}$
$H_a : \mu_d < 0$	to the left of t	$t < -t_{\alpha,n-1}$
$H_a : \mu_d \neq 0$	to the left of $-\lvert t\rvert$ and right of $\lvert t\rvert$	$t > t_{\alpha/2,n-1}$ or $t < -t_{\alpha/2,n-1}$

* $t_{\alpha,n-1}$ is the $100(1-\alpha)$th percentile of the t distribution with $n-1$ d.f.

Exercise

For the study of the impact of dredging on the Brazilian harbor, we want to decide if there was **any change** in the sediment's clay percentage.

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For the study of the impact of dredging on the Brazilian harbor, we want to decide if there was **any change** in the sediment's clay percentage.

The summary statistics for the $n = 8$ differences (clay percentage **after** dredging minus **before**) are

$$\bar{D} \ = \ -10.9 \qquad \text{and} \qquad S_d \ = \ 11.2$$

The hypotheses are

$$H_0 : \mu_x - \mu_y \ = \ 0 \qquad \text{(or equivalently } H_0 : \mu_d \ = \ 0)$$

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where $\boldsymbol{\mu_x}$ and $\boldsymbol{\mu_y}$ are the true population mean **clay percents after** and **before** dredging, respectively, and $\boldsymbol{\mu_d}$ is population mean **difference.**

Carry out the **paired t test** using level of significance $\alpha = 0.05$.

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Hints: You should get $\boldsymbol{t = -2.75}$ and **p-value** = 2(0.013) = **0.026**.

Paired t Confidence Interval

- Recall that in a **matched pairs study**, we **estimate** the ***effect size*** $\mu_x - \mu_y$ (or μ_d) by $\bar{X} - \bar{Y}$ (or $\bar{D}$).

Paired t Confidence Interval

- Recall that in a **matched pairs study**, we **estimate** the ***effect size*** $\mu_x - \mu_y$ (or μ_d) by $\bar{X} - \bar{Y}$ (or $\bar{D}$).
- By attaching a **margin of error** to the estimate, we get a **CI**.

Paired t CI: A $100(1-\alpha)\%$ ***paired t CI for $\mu_x - \mu_y$*** (or μ_d) is

$$\bar{D} \pm t_{\alpha/2,n-1} \, S_{\bar{D}} \, ,$$

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$$S_{\bar{D}} \; = \; \frac{S_d}{\sqrt{n}} \, .$$

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- Note that this is just the **one-sample t CI** based on the **differences.**

- The CI is valid if either
    1. The population of **differences** is **normal**, or
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  1. The population of **differences** is **normal**, or
  2. The sample size n is **large**.
- We can be $100(1-\alpha)\%$ confident that the true (unknown) effect size $\mu_x - \mu_y$ (or μ_d) will be contained in the interval.

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Margin of Error: For the paired t CI,

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- A **smaller margin of error** indicates that $\bar{X} - \bar{Y}$ (or $\bar{D}$) is a **more precise estimate** of the (unknown) **effect size** $\mu_x - \mu_y$ (or μ_d).

Example

For the study of dredging in the Brazilian harbor, recall that the summary statistics for the $n = 8$ **differences** are

$$\bar{D} = -10.9 \qquad \text{and} \qquad S_d = 11.2.$$

The **estimated size** of the **effect** of dredging on the sediment's **clay percentage** is

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i.e. a **decrease** of **10.9** percentage points in the clay.

The **standard error** of the estimate $\bar{D}$ is

$$S_{\bar{D}} \;=\; \frac{S_d}{\sqrt{n}} \;=\; \frac{11.2}{\sqrt{8}} \;=\; \mathbf{3.96}.$$

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$$\begin{aligned}\bar{D} \pm t_{\alpha/2,n-1} S_{\bar{D}} &= -10.9 \pm 2.36\,(3.96) \\ &= -10.9 \pm 9.35 \\ &= \mathbf{(-20.25,\ -1.55)}\end{aligned}$$

(where the t critical value $t_{0.025,7} = 2.36$ was obtained from a t distribution table using $n - 1 = 7$ df).

A **95% paired t Ci** for the true (unknown) **effect size $\boldsymbol{\mu_x - \mu_y}$** (or $\boldsymbol{\mu_d}$) is

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(where the t critical value $t_{0.025,7} = 2.36$ was obtained from a t distribution table using $n - 1 = 7$ df).

The **margin of error** in the estimate is **9.35** percentage points.

We can be **95% confident** that the true (unknown) **effect** of dredging is a **decrease** in clay of between **1.55** and **20.25** percentage points.