# 9 Paired Samples Hypothesis Tests (Cont'd)

MTH 3240 Environmental Statistics

Spring 2020

# Objectives

Objectives:

- Carry out a signed rank test for the difference between two population means.
- Carry out a paired sign test for a population median difference.
- Decide which test (the paired $t$ test, signed rank test test, or paired sign test) is most appropriate for a given set of data.

# Dealing With Non-Normal Data

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  2. Carry out a **nonparametric** test (i.e. one that doesn't require normality).

  We'll look at these two approaches one at a time.

# Transforming Data To Normality

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  (It can be shown that if the $X$ and $Y$ samples are both from **normal** populations, the **differences** will also be **normal**.)

# Carrying Out a Nonparametric Test

- The second approach to testing hypotheses with *paired samples* whose **differences** are **non-normal** is to use a ***nonparametric*** test procedure, i.e. one that **doesn't** rely on a normality assumption.

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  The ***signed rank test*** and ***paired sign test*** described ahead are both **nonparametric** alternatives to the *paired* $t$ *test*.

# The Signed Rank Test

- The ***signed rank test*** is a **paired samples nonparametric** test for the difference between two population **means** $\mu_x$ and $\mu_y$.

- The **null hypothesis** is that there's **no difference** between $\boldsymbol{\mu_x}$ and $\boldsymbol{\mu_y}$.

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(Same hypothesis as for the paired and two-sample $t$ tests.)

- The **alternative hypothesis** is one of the following.

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1. $H_a : \mu_x - \mu_y > 0$ (**upper-tailed test**)
2. $H_a : \mu_x - \mu_y < 0$ (**lower-tailed test**)
3. $H_a : \mu_x - \mu_y \neq 0$ (**two-tailed test**)

depending on what we're trying to verify using the data.

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  **...** are a random sample from a **population** of **differences** whose mean is $\boldsymbol{\mu_d}$.

- The hypotheses can be reformulated in terms of $\mu_d$ as:

| | Hypothesis About $\mu_x - \mu_y$ | Equivalent Hypothesis About $\mu_d$ |
|---|---|---|
| Null | $H_0 : \mu_x - \mu_y = 0$ | $H_0 : \mu_d = 0$ |
| Alternatives | $H_a : \mu_x - \mu_y > 0$ | $H_a : \mu_d > 0$ |
| | $H_a : \mu_x - \mu_y < 0$ | $H_a : \mu_d < 0$ |
| | $H_a : \mu_x - \mu_y \neq 0$ | $H_a : \mu_d \neq 0$ |

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| | $H_a : \mu_x - \mu_y \neq 0$ | $H_a : \mu_d \neq 0$ |

(Same hypotheses as for the paired $t$ test.)

- The ***test statistic***, denoted $W^+$, is obtained by taking the **absolute values** of the **differences**, ***ranking*** them (smallest to largest), and then **summing** the **ranks** of the differences that were **originally positive**.

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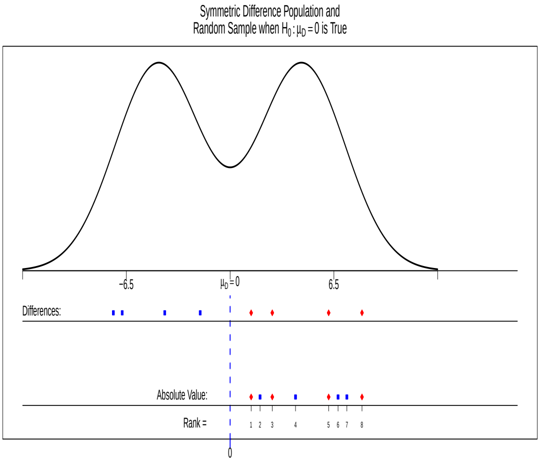
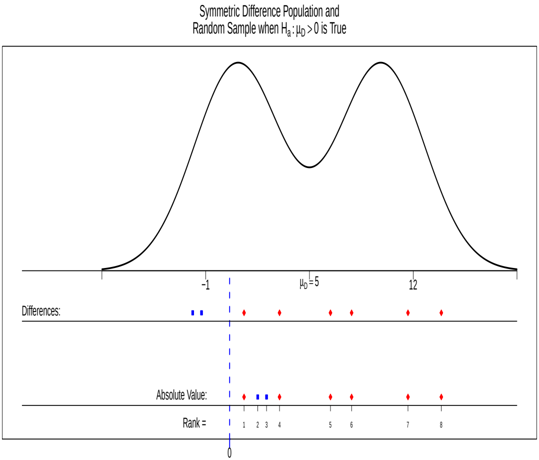
**Signed Rank Test Statistic**:

1. If any of the differences $D_1, D_2, \ldots, D_n$ are **zero**, **discard** them and diminish $n$ by the number of discarded $D_i$'s.
2. Take **absolute values** of the remaining **differences**, keeping track of which ones were originally **positive**.

3. **Sort** the **absolute differences** and ***rank*** them from smallest to largest. If two or more are **tied**, assign to each of them the **average** of the **ranks** they would've been assigned if they hadn't been tied.
4. **Sum** the **ranks** of the absolute differences that were originally **positive.** This gives the **test statistic**:

$$W^+ = \text{Sum of the ranks of the } |D_i| \text{'s for which } D_i \text{ is positive.}$$


Symmetric Difference Population and
Random Sample when $H_0 : \mu_D = 0$ is True
−6.5
$\mu_D = 0$
6.5
Differences:
Absolute Value:
Rank =
1 2 3 4 5 6 7 8
0


Symmetric Difference Population and
Random Sample when $H_a: \mu_D > 0$ is True
-1
$\mu_D = 5$
12
Differences:
Absolute Value:
Rank =
1 2 3 4 5 6 7 8
0

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  - **If $H_0$ was true**, ...

    ... we'd expect the positive and negative differences to be **intermingled**.
  - But **if $H_a$ was true**, ...

    ... we'd expect the positive and negative differences to be **segregated**, and the $\mid D_i \mid$'s for which $D_i$ is positive to mostly lie near the end in the direction specified by $H_a$.

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    1. $\boldsymbol{W}^+$ will be approximately **equal to** $\boldsymbol{n(n+1)/4}$ (most likely), if $H_0$ is true.
    2. $\boldsymbol{W}^+$ will **differ from** $\boldsymbol{n(n+1)/4}$ (most likely) in the direction specified by $H_a$ if $H_a$ is true.

1. *Large* values of $W^+$ (larger than $n(n+1)/4$) provide evidence in favor of $H_a : \mu_x - \mu_y > 0$ (or $H_a : \mu_d > 0$).
2. *Small* values of $W^+$ (smaller than $n(n+1)/4$) provide evidence in favor of $H_a : \mu_x - \mu_y < 0$ (or $H_a : \mu_d < 0$).
3. Both *large and small* values of $W^+$ (larger or smaller than $n(n+1)/4$) provide evidence in favor of $H_a : \mu_x - \mu_y \neq 0$ (or $H_a : \mu_d \neq 0$).

- Now suppose the sample of **differences** is from **any** (continuous) **population** that has a (roughly) **symmetric** shape.

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  In this case, the **null distribution** is as follows.

**Sampling Distribution of $W^+$ Under $H_0$**: If $W^+$ is the signed rank test statistic, then when

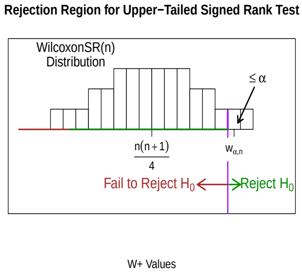
$$H_0 : \mu_x - \mu_y = 0 \qquad \text{(or equivalently } H_0 : \mu_d = 0\text{)}$$

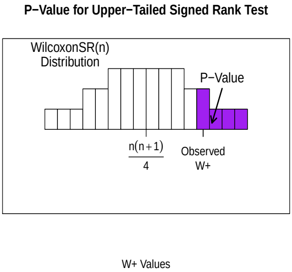
is true, $W^+$ follows a distribution called the ***Wilcoxon signed rank distribution***, which will depend on $n$. We write this as

$$W^+ \sim \text{WilcoxonSR}(n).$$

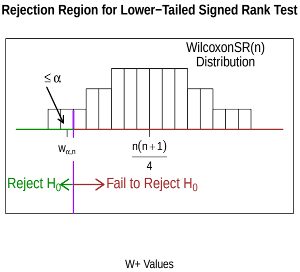
- **P-values** and **rejection regions** are obtained from the appropriate tail(s) of the **WilcoxonSR**$(n)$ **distribution**, as shown on the next slides.

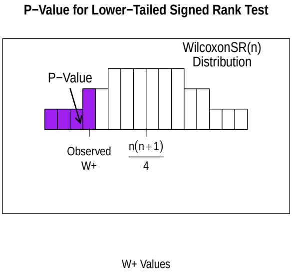
## 1. $H_a : \mu_x - \mu_y > 0$ (Upper-Tailed Test)



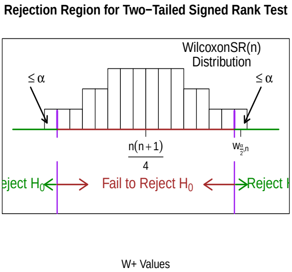



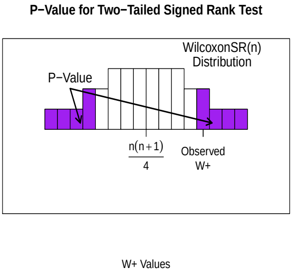
## 2. $H_a : \mu_x - \mu_y < 0$ (Lower-Tailed Test)



## 3. $H_a : \mu_x - \mu_y \neq 0$ (Two-Tailed Test)



## Paired Samples Signed Rank Test for $\mu_d$

**Assumptions**: $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are two random samples that are paired and the differences $d_1, d_2, \ldots, d_n$ form a single sample from a *continuous* population whose distribution is *symmetric*.

**Null hypothesis**: $H_0 : \mu_d = 0$.

**Test statistic value**: $w^+ =$ sum of the ranks of $\mid d_i \mid$'s for which $d_i > 0$.

**Decision rule**: Reject $H_0$ if p-value $< \alpha$ or $w^+$ is in rejection region.

**Null hypothesis**: $H_0 : \mu_d = 0$.

**Test statistic value**: $w^+ =$ sum of the ranks of $| d_i |$'s for which $d_i > 0$.

**Decision rule**: Reject $H_0$ if p-value $< \alpha$ or $w^+$ is in rejection region.

| Alternative hypothesis | P-value = tail probability of the $W^+$ distribution under $H_0$: * | Rejection region = $w^+$ values such that: ** |
|---|---|---|
| $H_a : \mu_d > 0$ | to the right of (and including) $w^+$ | $w^+ \geq w_{\alpha,n}$ |
| $H_a : \mu_d < 0$ | to the left of (and including) $w^+$ | $w^+ \leq w^*_{\alpha,n}$ |
| $H_a : \mu_d \neq 0$ | $2 \cdot$ (the smaller of the tail probabilities to the right of (and including) $w^+$ and to the left of (and including) $w^+$) | $w^+ \leq w^*_{\alpha/2,n}$ or $w^+ \geq w_{\alpha/2,n}$ |

* For a given sample size (after deleting the zero-valued $d_i$'s) $n$, in Table B6, the p-value can be taken to be less than the smallest $\alpha$ for which $H_0$ would be rejected using the rejection region approach.

** For a given level of significance $\alpha$ and sample size (after deleting the zero-valued $d_i$'s) $n$, in Table B6 the upper tail critical value $w_{\alpha,n}$ is the *large* $W$ entry associated with row $n$, column $\alpha$. The lower tail critical value $w^*_{\alpha,n}$ is the *small* $W$ entry.

**Exercise**

A method for measuring ground-level atmospheric mercury (Hg) requires holding air samples for up to 120 h (five days) before analyzing them at a laboratory.

A quality assurance study was carried out to ensure that the long holding time wouldn't affect the measurement results.

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The table on the next slide shows particulate-bound **Hg measurements** ($\mathrm{pg}/\mathrm{m}^3$) for each of the **10 pairs** along with their **differences**.

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The table on the next slide shows particulate-bound **Hg measurements** ($\mathrm{pg/m^3}$) for each of the **10 pairs** along with their **differences.**

A **normal probability plot** of the **differences** is two slides ahead.

| | Particulate-Bound Hg | | |
|---|---|---|---|
| Air Sample Pair | Long Holding Time | Short Holding Time | Difference |
| 1 | 4.27 | 1.33 | 2.94 |
| 2 | 2.77 | 3.43 | -0.66 |
| 3 | 1.50 | 1.29 | 0.21 |
| 4 | 5.70 | 6.26 | -0.56 |
| 5 | 3.80 | 11.99 | -8.19 |
| 6 | 5.64 | 1.88 | 3.76 |
| 7 | 3.67 | 14.58 | -10.91 |
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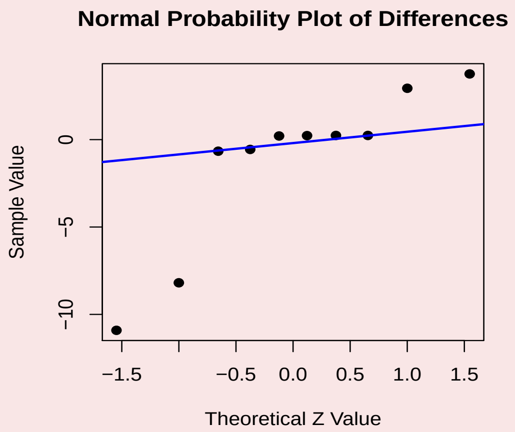
Normal Probability Plot of Differences
Sample Value
0
−5
−10
−1.5
−0.5
0.0
0.5
1.0
1.5
Theoretical Z Value

The ”backward S” shape of the normal probability plot suggests that the normality assumption required for the *paired* $t$ *test* **isn’t** met.

Instead, we’ll carry out a **signed rank test.**

Carry out the **signed rank test** to decide if the long holding period has **any effect** on Hg measurements. Use $\alpha = 0.05$.

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**Hint**: The **combined**, **sorted**, **absolute values** of the **differences** are below.

Differences that were **positive** before taking absolute values are denoted by $+$ and ones that were **negative** by $-$.

| Sign | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Obs. | 0.21 | 0.23 | 0.24 | 0.24 | 0.56 | 0.66 | 2.94 | 3.76 | 8.19 | 10.91 |
| Rank | | | | | | | | | | |

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|---|---|---|---|---|---|---|---|---|---|---|
| Obs. | 0.21 | 0.23 | 0.24 | 0.24 | 0.56 | 0.66 | 2.94 | 3.76 | 8.19 | 10.91 |
| Rank | | | | | | | | | | |

You should get $\boldsymbol{W^+ = 25}$ and **p-value** $= 2(0.423) = \mathbf{0.846}$.

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  We'll act as though the **differences** $D_1, D_2, \ldots, D_n$ are a random sample from a **population** of **differences** whose ***median*** is $\tilde{\mu}_d$.

- The ***paired samples sign test*** is just a **one-sample sign test** for $\tilde{\mu}_d$ based on the **differences**.

- The **null hypothesis** is that $\tilde{\mu}_d$ is **zero**.

**Null Hypothesis**:

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(This says a typical difference is zero.)

- The **alternative hypothesis** is one of the following.

**Alternative Hypothesis**:

1. $\tilde{\mu}_d > 0$ (**upper-tailed test**)
2. $\tilde{\mu}_d < 0$ (**lower-tailed test**)
3. $\tilde{\mu}_d \neq 0$ (**two-tailed test**)

depending on what we're trying to verify using the data.

**Paired Samples Sign Test Statistic**:

$$S^+ \ = \ \text{Number of } D_i\text{'s that are greater than 0.}$$

(If any $D_i$'s equal 0, they're discarded, and $n$ is diminished by the number of discarded $D_i$'s).

**Paired Samples Sign Test Statistic**:

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(If any $D_i$'s equal 0, they're discarded, and $n$ is diminished by the number of discarded $D_i$'s).

This is just the **one-sample sign test statistic** using the sample of **differences** and a null-hypothesized value **zero.**

1. *Large* values of $S^+$ (larger than $n/2$) provide evidence in favor of $H_a : \tilde{\mu}_d > 0$.
2. *Small* values of $S^+$ (smaller than $n/2$) provide evidence in favor of $H_a : \tilde{\mu}_d < 0$.
3. *Both large and small* values of $S^+$ (larger or smaller than $n/2$) provide evidence in favor of $H_a : \tilde{\mu}_d \neq 0$.

## Paired Samples Sign Test for $\tilde{\mu}_d$

**Assumptions**: $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are two random samples that are paired, and the differences $d_1, d_2, \ldots, d_n$ form a single sample from *any continuous* population.

**Null hypothesis**: $H_0 : \tilde{\mu}_d = 0$.

**Test statistic value**: $s^+ =$ number of positive $d_i$'s.

**Decision rule**: Reject $H_0$ if p-value $< \alpha$ or $s^+$ is in rejection region.

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**Null hypothesis**: $H_0 : \tilde{\mu}_d = 0$.

**Test statistic value**: $s^+ =$ number of positive $d_i$'s.

**Decision rule**: Reject $H_0$ if p-value $< \alpha$ or $s^+$ is in rejection region.

| Alternative hypothesis | P-value = tail probability of the binomial$(n, 0.5)$ distribution: * | Rejection region = $s^+$ values such that: ** |
|---|---|---|
| $H_a : \tilde{\mu}_d > 0$ | to the right of (and including) $s^+$ | $s^+ \geq s_{\alpha,n}$ |
| $H_a : \tilde{\mu}_d < 0$ | to the left of (and including) $s^+$ | $s^+ \leq s^*_{\alpha,n}$ |
| $H_a : \tilde{\mu}_d \neq 0$ | $2\cdot$(the smaller of the tail probabilities to the right of (and including) $s^+$ and to the left of (and including) $s^+$) | $s^+ \leq s^*_{\alpha/2,n}$ or $s^+ \geq s_{\alpha/2,n}$ |

## Paired Samples Sign Test for $\tilde{\mu}_d$

* For a given sample size (after deleting the zero-valued $d_i$'s) $n$, the p-value for a one-tailed test is obtained from a binomial$(n,\ 0.5)$ distribution table by locating the upper or lower tail probability (depending on the direction of $H_a$) associated with the observed $S^+$ value. For a two-tailed test, locate both the upper and lower tail probabilities and multiply the smaller of these by two.

** For a given sample size (after deleting zero-valued $d_i$'s) $n$ and level of significance $\alpha$, $s_{\alpha,n}$ is obtained from a binomial$(n,\ 0.5)$ distribution table by locating the smallest $s$ for which the upper tail probability is less than $\alpha$. $s^*_{\alpha,n}$ is obtained by locating the largest $s$ for which the lower tail probability is less than $\alpha$. For the two-tailed test, $s_{\alpha/2,n}$ and $s^*_{\alpha/2,n}$ are defined analogously but with $\alpha/2$ used in place of $\alpha$. In practice, due to the discreteness of the distribution, it's not always possible obtain a rejection region having exact probability $\alpha$.

**Exercise**

Consider again the quality assurance study to ensure that a long holding time wouldn't affect mercury (Hg) measurements in air samples.

The table below shows the data again.

| | Particulate-Bound Hg | | |
|---|---|---|---|
| Air Sample Pair | Long Holding Time | Short Holding Time | Difference |
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Carry out a **sign test for paired samples** to decide if the long holding period has **any effect** on Hg measurements. Use $\alpha = 0.05$.

**Hints**: You should get $S^+ = \mathbf{6}$ and **p-value** $= 2(0.3770) = \mathbf{0.7540}$.