11 Tests for the Effects of Two Factors (Cont'd)

MTH 3240 Environmental Statistics

Spring 2020

Objectives

Objectives:

- Write out the group means and treatment effects versions of the two-factor ANOVA model, including any assumptions about the random error term ϵ. (**Optional for Spring 2020**)
- Interpret sums of squares, degrees of freedom, and mean squares in two-factor ANOVA.
- Carry out two-factor ANOVA F tests for the effects of two factors and their interaction effect.
- Interpret the interaction effect in a two-factor study.
- Obtain and interpret fitted values and residuals in two-factor ANOVA.
- Use plots to check the normality and common population standard deviation assumptions required by the ANOVA F tests.

Two-Factor ANOVA Models (Optional for Spring 2020)

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- For **two-factor ANOVA**, we'll suppose each of the ab groups is a sample from a **normal** population.

  The mean of the population corresponding to the i, jth group is $\boldsymbol{\mu_{ij}}$.

  The population standard deviations are required to all the same, $\boldsymbol{\sigma}$.

(Optional for Spring 2020)

Factor B

		Level $j=1$	Level $j=2$	$\cdots$	Level $j=b$	
	Level $i=1$	$\left.\begin{array}{c}Y_{111}\\ Y_{112}\\ \vdots\\ Y_{11n}\end{array}\right\} \sim N(\mu_{11},\sigma)$	$\left.\begin{array}{c}Y_{121}\\ Y_{122}\\ \vdots\\ Y_{12n}\end{array}\right\} \sim N(\mu_{12},\sigma)$	$\cdots$	$\left.\begin{array}{c}Y_{1b1}\\ Y_{1b2}\\ \vdots\\ Y_{1bn}\end{array}\right\} \sim N(\mu_{1b},\sigma)$	$\mu_{1\cdot}$
Factor A	Level $i=2$	$\left.\begin{array}{c}Y_{211}\\ Y_{212}\\ \vdots\\ Y_{21n}\end{array}\right\} \sim N(\mu_{21},\sigma)$	$\left.\begin{array}{c}Y_{221}\\ Y_{222}\\ \vdots\\ Y_{22n}\end{array}\right\} \sim N(\mu_{22},\sigma)$	$\cdots$	$\left.\begin{array}{c}Y_{2b1}\\ Y_{2b2}\\ \vdots\\ Y_{2bn}\end{array}\right\} \sim N(\mu_{2b},\sigma)$	$\mu_{2\cdot}$
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
	Level $i=a$	$\left.\begin{array}{c}Y_{a11}\\ Y_{a12}\\ \vdots\\ Y_{a1n}\end{array}\right\} \sim N(\mu_{a1},\sigma)$	$\left.\begin{array}{c}Y_{a21}\\ Y_{a22}\\ \vdots\\ Y_{a2n}\end{array}\right\} \sim N(\mu_{a2},\sigma)$	$\cdots$	$\left.\begin{array}{c}Y_{ab1}\\ Y_{ab2}\\ \vdots\\ Y_{abn}\end{array}\right\} \sim N(\mu_{ab},\sigma)$	$\mu_{a\cdot}$
		$\mu_{\cdot 1}$	$\mu_{\cdot 2}$	$\cdots$	$\mu_{\cdot b}$	μ

(Optional for Spring 2020)

- The value of $\boldsymbol{\mu_{ij}}$ will depend on the level i of factor A and the level j of factor B.

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- We can describe the data using the **statistical model** called the ***additive effects two-factor ANOVA model***.

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- The value of $\boldsymbol{\mu_{ij}}$ will depend on the level i of factor A and the level j of factor B.

- We can describe the data using the **statistical model** called the ***additive effects two-factor ANOVA model***.

  In the model, each $\boldsymbol{\mu_{ij}}$ is written in terms of an **effect** of level i of factor A and an **effect** of level j of factor B.

Additive Two-Factor ANOVA Model (Optional for Spring 2020)

Additive Two-Factor ANOVA Model: One statistical model for describing data in a two-factor study is:

$$Y_{ijk} = \underbrace{\mu + \alpha_i + \beta_j}_{\text{This is } \mu_{ij}} + \epsilon_{ijk} ,$$

where

$\boldsymbol{Y_{ijk}}$ is the kth observation at the ith level of factor A and jth level of factor B.

$\boldsymbol{\mu}$ is a constant called the ***overall true mean***.

$\boldsymbol{\alpha_i}$ is the ***effect*** of the ith level of factor A.

$\boldsymbol{\beta_j}$ is the ***effect*** of the jth level of factor B.

$\boldsymbol{\epsilon_{ijk}}$ is a $\mathrm{N}(0, \sigma)$ ***random error*** term.

(Optional for Spring 2020)

Example

For the soil phosphorus study, the **additive effects model** for describing a phosphorus concentration Y is of the form

$$Y = \text{Overall Mean} + \text{Soil Type Effect} + \text{Topography Effect} + \text{Error}$$

- But the **additive effects model** requires that the **effects** of the two factors be ***additive***.

(Optional for Spring 2020)

Example

For the soil phosphorus study, the **additive effects model** for describing a phosphorus concentration Y is of the form

$$Y = \text{Overall Mean} + \text{Soil Type Effect} + \text{Topography Effect} + \text{Error}$$

- But the **additive effects model** requires that the **effects** of the two factors be ***additive***.
- They're ***additive*** when the effect of **factor** A is the **same regardless** of the **level** of **factor** B, and the effect of factor B is the **same regardless** of the **level** of **factor** A.

Interactions

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- When the effects of two factors **aren't** additive, we say that there's an ***interaction effect*** between them.
- An ***interaction effect*** exists when the effect of **factor** A is **different** depending on the **level** of **factor** B, and the effect of **factor** B is **different** depending on the **level** of **factor** A.
- We can check for an **interaction effect** by graphing the **group means** either in a **bar plot** or in an ***interaction plot***.

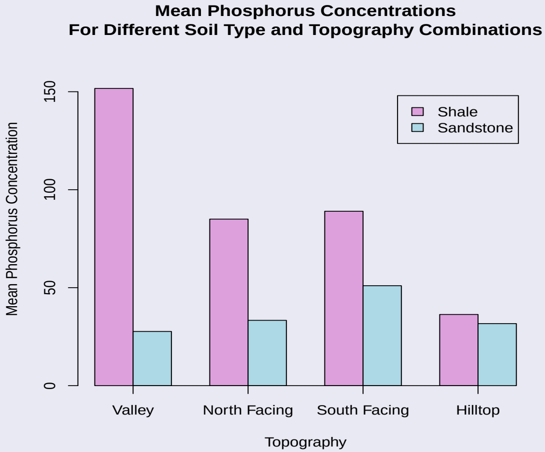
Example

For the soil phosphorus study, the eight **group means** are shown in the **body** of the table below.

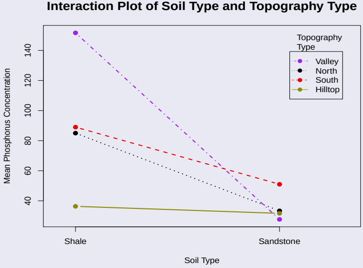
Factor B: Topography

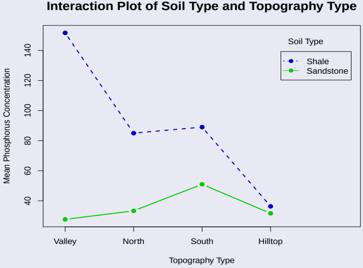
		Valley (j=1)	North-Facing (j=2)	South-Facing (j=3)	Hilltop (j=4)	
Factor A: Soil Type	Shale (i=1)	$\bar{Y}_{11} = 151.7$	$\bar{Y}_{12} = 85.0$	$\bar{Y}_{13} = 89.0$	$\bar{Y}_{14} = 36.3$	$\bar{Y}_{1\cdot} = 90.5$
	Sandstone (i=2)	$\bar{Y}_{21} = 27.7$	$\bar{Y}_{22} = 33.3$	$\bar{Y}_{23} = 51.0$	$\bar{Y}_{24} = 31.7$	$\bar{Y}_{2\cdot} = 35.9$
		$\bar{Y}_{\cdot 1} = 89.7$	$\bar{Y}_{\cdot 2} = 59.2$	$\bar{Y}_{\cdot 3} = 70.0$	$\bar{Y}_{\cdot 4} = 34.0$	$\bar{Y} = 63.2$

A **bar plot** of the **group means** is shown (again) below.



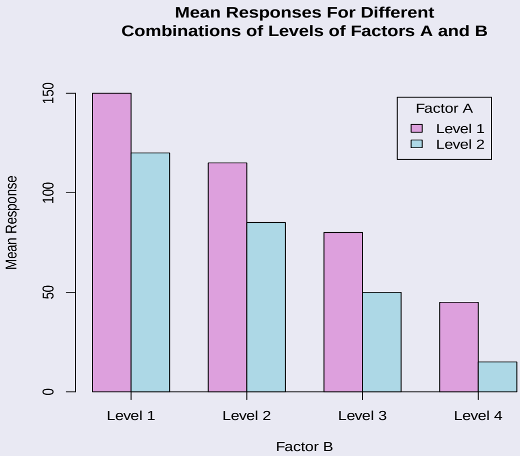
Here are **interaction plots** of the **group means.**



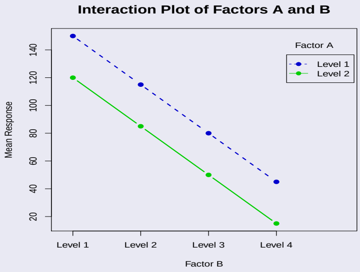


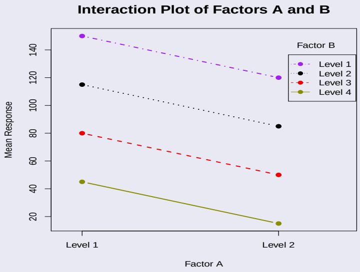
Example

Here's **bar plot** showing group means for two factors whose effects are **additive.**



Here are **interaction plots** of those same **group means** (for two factors whose effects are **additive**).





- When the lines in an **interaction plot** are **parallel**, the effects of the two factors are **additive.**

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  When the lines **aren't parallel**, the effects **aren't additive**, i.e. there's an **interaction effect.**

- In practice:
  - We only have to look at one of the **interaction plots** or the other (not both).
  - The lines in an **interaction plot** won't ever be **exactly** parallel because of random ***sampling error***.

Two-Factor ANOVA Model With Interaction (Optional for Spring 2020)

- The model we'll use when the effects of two factors **aren't** additive is the ***two-factor ANOVA model with interaction effect***.

Two-Factor ANOVA Model With Interaction Effect: Another statistical model for describing data in a two-factor study is:

$$Y_{ijk} = \underbrace{\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}}_{\text{This is } \mu_{ij}} + \epsilon_{ijk} \,,$$

(Optional for Spring 2020)

where

$\boldsymbol{Y_{ijk}}$ is the kth observation at the ith level of factor A and jth level of factor B.

$\boldsymbol{\mu}$ is a constant called the ***overall true mean***.

$\boldsymbol{\alpha_i}$ is the ***main effect*** of the ith level of factor A.

$\boldsymbol{\beta_j}$ is the ***main effect*** of the jth level of factor B.

$\boldsymbol{(\alpha\beta)_{ij}}$ is an ***interaction effect*** representing **non-additivity** in the effects of the two factors.

$\boldsymbol{\epsilon_{ijk}}$ is a N$(0, \sigma)$ ***random error*** term.

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  So it's **"safer"** to **include** the **interaction effect** in the **model** when carrying out a **two-factor ANOVA.**

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  ... and it can still be used even if they ***are* additive.**

  So it's **"safer"** to **include** the **interaction effect** in the **model** when carrying out a **two-factor ANOVA.**

  From here on, we'll ***always* include** the **interaction effect** in the model.

Example

For the soil phosphorus study, the interaction plots suggest that the effects of **soil type** and **topography** ***aren't*** **additive**, so the we'd definitely use the **ANOVA model** ***with*** the **interaction effect.**

For a phosphorus concentration Y, the model has the form:

$$\begin{aligned} Y &= \text{Overall Mean} + \text{Soil Type Effect} + \text{Topography Effect} \\ &\quad + \text{Interaction Effect} + \text{Error} \end{aligned}$$

Fitted Values and Residuals

- The **group means** $\bar{Y}_{11}, \bar{Y}_{12}, \ldots, \bar{Y}_{ab}$ are sometimes called ***fitted values***.

Fitted Values:

$$\text{Fitted Value for } ij\text{th Group} \;=\; \bar{Y}_{ij}$$

- **Comments**:
  - Statistical software reports $\mathbf{n}$ **duplicates** of the **fitted value** for **each** of the ab **groups**, one duplicate for each of the n individuals in the group.

Residuals

- A ***residual***, denoted $\boldsymbol{e_{ijk}}$, is the **deviation** of an individual's observed $\boldsymbol{Y_{ijk}}$ value away from the **fitted value** for that individual.

Residuals:

$$e_{ijk} = Y_{ijk} - \bar{Y}_{ij}$$

- Statistical software reports the values of all N **residuals**, one for each individual in the study.

Sums of Squares

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  (We'll also test for an **interaction effect** by comparing **variation due to the interaction** to **within-groups** variation.)

- We measure the different types of variation using ***sums of squares*** (and later ***mean squares***).

- We measure the **between-rows variation** by the ***factor A sum of squares***, denoted **SSA.**

Factor A Sum of Squares:

$$\text{SSA} \;=\; nb\sum_{i=1}^{a}(\bar{Y}_{i\cdot} - \bar{Y})^2.$$

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Factor A Sum of Squares:

$$\text{SSA} \;=\; nb\sum_{i=1}^{a}(\bar{Y}_{i\cdot} - \bar{Y})^2.$$

SSA will be **large** when there's substantial variation among **row means** $\bar{Y}_{1\cdot}, \bar{Y}_{2\cdot}, \ldots, \bar{Y}_{a\cdot}$, which would suggest **factor A** has an **effect.**

- We measure the **between-columns variation** by the ***factor B sum of squares***, denoted **SSB**.

Factor B Sum of Squares:

$$\text{SSB} \;=\; na\sum_{j=1}^{b}(\bar{Y}_{\cdot j} - \bar{Y})^2.$$

- We measure the **between-columns variation** by the ***factor B sum of squares***, denoted **SSB**.

Factor B Sum of Squares:

$$\text{SSB} \;=\; na\sum_{j=1}^{b}(\bar{Y}_{\cdot j} - \bar{Y})^2.$$

SSB will be **large** when there's substantial variation among **column means** $\bar{Y}_{\cdot 1}, \bar{Y}_{\cdot 2}, \ldots, \bar{Y}_{\cdot b}$, which would suggest **factor B** has an **effect.**

- We measure **variation** due to an **interaction effect** by the AB ***interaction sum of squares***, denoted **SSAB.**

AB **Interaction Sum of Squares**:

$$\text{SSAB} \ = \ n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y})^2.$$

- We measure **variation** due to an **interaction effect** by the AB ***interaction sum of squares***, denoted **SSAB.**

AB **Interaction Sum of Squares**:

$$\text{SSAB} \; = \; n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y})^2.$$

It can be shown that SSAB will be **large** when the **group means** $\bar{Y}_{11}, \bar{Y}_{12}, \ldots, \bar{Y}_{ab}$ aren't consistent with an additive effects model, which would suggest there's an **interaction effect** between **factors A** and **B** (i.e. their effects aren't additive).

- We measure **within-groups variation** by the ***error sum of squares***, denoted **SSE.**

Error Sum of Squares:

$$\text{SSE} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk} - \bar{Y}_{ij})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} e_{ijk}^2 \,.$$

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The **error sum of squares is just the *sum of squared* *residuals***, and reflects variation due to **random error.**

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The **error sum of squares is just the *sum of squared residuals***, and reflects variation due to **random error.**

SSE will be **large** if there's **substantial** variation among individual observations (Y_{ijk}'s) **within** groups, and **small** otherwise.

The ANOVA Partition

- The ***total sum of squares***, denoted **SSTo**, measures **total variation** in the data.

Total Sum of Squares:

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Total Sum of Squares:

$$\text{SSTo} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk} - \bar{Y})^2 .$$

SSTo reflects **total variation** due both to effects of the factors (if they have any effects) **and** random variation among individual observations within groups.

- It can be shown that

Two-Factor ANOVA Partition:

$$\text{SSTo} = \text{SSA} + \text{SSB} + \text{SSAB} + \text{SSE}.$$

This splits the total variation in the data as:

$$\begin{aligned}\textbf{Total Variation} &= \textbf{Between-Rows Variation} \\ &\quad + \textbf{Between-Columns Variation} \\ &\quad + \textbf{Variation Due to Interaction} \\ &\quad + \textbf{Within-Groups Variation}\end{aligned}$$

Example

For the soil phosphorus study, statistical software reports the following **sums of squares.**

$$\begin{aligned} \text{SSTo} &= 51406.0 && \text{(Total variation)} \\ \text{SSA} &= 17876.0 && \text{(Variation due to soil type)} \\ \text{SSB} &= 9693.8 && \text{(Variation due to topography)} \\ \text{SSAB} &= 11390.8 && \text{(Variation due to interaction)} \\ \text{SSE} &= 12445.3 && \text{(Variation due to random error)} \end{aligned}$$

We see that the **two-factor ANOVA partition** holds since

$$51406.0 = 17876.0 + 9693.8 + 11390.8 + 12445.3.$$

This indicates that a large portion of the total variation in phosphorus measurements (**17876.0** out of **51406.0**, or **35%**) is due to the difference between the row means (soil types).

Degrees of Freedom

- Here are the ***degrees of freedom*** associated with the sums of squares in two-factor ANOVA. (These will determine which F distributions our p-values come from.)

Degrees of Freedom: For two-factor ANOVA, the degrees of freedom are:

$$
\begin{aligned}
df \text{ for SSTo} &= N-1 \\
df \text{ for SSA} &= a-1 \\
df \text{ for SSB} &= b-1 \\
df \text{ for SSAB} &= (a-1)(b-1) \\
df \text{ for SSE} &= ab(n-1) \quad = \quad N-ab
\end{aligned}
$$

- The degrees of freedom, like the associated sums of squares, are additive in the following sense.

Additivity of Degrees of Freedom:

df for SSTo $=$ df for SSA$+df$ for SSB$+df$ for SSAB$+df$ for SSE.

Example

For the soil phosphorus study, we have $a = 2$ soil types, $b = 4$ topographies, and $n = 3$ phosphorus observations per group. Thus the total number of phosphorus observations (overall sample size) is $N = 24$, and

$$\begin{aligned} df \text{ for SSTo} &= 23 \\ df \text{ for SSA} &= 1 \\ df \text{ for SSB} &= 3 \\ df \text{ for SSAB} &= 3 \\ df \text{ for SSE} &= 16. \end{aligned}$$

As expected,

$$23 = 1 + 3 + 3 + 16.$$

Mean Squares

- A ***mean square*** is a **sum of squares** divided by its **degrees of freedom.**
- The ***factor A mean square***, ***factor B mean square***, ***AB interaction mean square***, and ***mean squared error*** are below.

Mean Squares: For two-factor ANOVA, the mean squares are

$$\text{MSA} = \frac{\text{SSA}}{a-1} \qquad \text{MSB} = \frac{\text{SSB}}{b-1}$$

$$\text{MSAB} = \frac{\text{SSAB}}{(a-1)(b-1)} \qquad \text{MSE} = \frac{\text{SSE}}{ab(n-1)}$$

The ANOVA F Tests

- There are **three** sets of hypotheses for the ***two-factor ANOVA*** F ***tests*** tests.

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- There are **three** sets of hypotheses for the ***two-factor ANOVA F tests*** tests.

  In each case, H_0 says there's **no effect**, and H_a say's there's **an effect.**

1. Test for a ***factor A main effect***:

$$\begin{aligned} H_{OA}: &\quad \text{There's no factor A effect} \\ H_{aA}: &\quad \text{There is a factor A effect} \end{aligned}$$

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2. Test for a ***factor B main effect***:

$$H_{OB}: \quad \text{There's no factor B effect}$$
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2. Test for a ***factor B main effect***:

$$\begin{aligned} H_{OB}: &\quad \text{There's no factor B effect} \\ H_{aB}: &\quad \text{There is a factor B effect} \end{aligned}$$

3. Test for an AB ***interaction effect***:

$$\begin{aligned} H_{OAB}: &\quad \text{There's no factor A and B interaction effect} \\ H_{aAB}: &\quad \text{There is a factor A and B interaction effect} \end{aligned}$$

(Optional for Spring 2020)

In terms of the **ANOVA model parameters** these are written as:

1. Test for a ***factor A main effect***:

$$
\begin{aligned}
H_{OA}: &\quad \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \\
H_{aA}: &\quad \text{The } \alpha_i\text{'s don't all equal } 0
\end{aligned}
$$

(Optional for Spring 2020)

In terms of the **ANOVA model parameters** these are written as:

1. Test for a ***factor*** A ***main effect***:

$$\begin{aligned} H_{OA}: &\quad \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \\ H_{aA}: &\quad \text{The } \alpha_i\text{'s don't all equal 0} \end{aligned}$$

2. Test for a ***factor*** B ***main effect***:

$$\begin{aligned} H_{OB}: &\quad \beta_1 = \beta_2 = \cdots = \beta_b = 0 \\ H_{aB}: &\quad \text{The } \beta_j\text{'s don't all equal 0} \end{aligned}$$

(Optional for Spring 2020)

In terms of the **ANOVA model parameters** these are written as:

1. Test for a ***factor*** A ***main effect***:

$$\begin{aligned} H_{OA}: &\quad \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \\ H_{aA}: &\quad \text{The } \alpha_i\text{'s don't all equal } 0 \end{aligned}$$

2. Test for a ***factor*** B ***main effect***:

$$\begin{aligned} H_{OB}: &\quad \beta_1 = \beta_2 = \cdots = \beta_b = 0 \\ H_{aB}: &\quad \text{The } \beta_j\text{'s don't all equal } 0 \end{aligned}$$

3. Test for an AB ***interaction effect***:

$$\begin{aligned} H_{OAB}: &\quad (\alpha\beta)_{11} = (\alpha\beta)_{12} = \cdots = (\alpha\beta)_{ab} = 0 \\ H_{aAB}: &\quad \text{The } (\alpha\beta)_{ij}\text{'s don't all equal } 0 \end{aligned}$$

- Here are the corresponding ***two-factor ANOVA F test statistics***:

Two-Factor ANOVA F Test Statistics:

$$F_A = \frac{\text{MSA}}{\text{MSE}}$$

$$F_B = \frac{\text{MSB}}{\text{MSE}}$$

$$F_{AB} = \frac{\text{MSAB}}{\text{MSE}}$$

- Not that we can think of

$$F_A = \frac{\textbf{Between-Rows Variation}}{\textbf{Within-Groups Variation}}.$$

$$F_B = \frac{\textbf{Between-Columns Variation}}{\textbf{Within-Groups Variation}}.$$

$$F_{AB} = \frac{\textbf{Variation Due To Interaction}}{\textbf{Within-Groups Variation}}.$$

- In each case, **if H_0 was true**, it can be shown, F would be **approximately equal to 1**.

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In each case, *large* values of F (larger than about 1) provide evidence in favor of H_a.

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  In this case, the **null distributions** are as follows.

Sampling Distribution of $\boldsymbol{F}$ Under $\boldsymbol{H_0}$: If $\boldsymbol{F_A}$, $\boldsymbol{F_B}$, and $\boldsymbol{F_{AB}}$ are the two-factor ANOVA F test statistics, then:

1. When H_{OA} is true,

$$F_A = \frac{\text{MSA}}{\text{MSE}} \sim F\left(a-1,\ N-ab\right).$$

2. When H_{OB} is true,

$$F_B = \frac{\text{MSB}}{\text{MSE}} \sim F\left(b-1,\ N-ab\right).$$

3. When H_{OAB} is true,

$$F_{AB} = \frac{\text{MSAB}}{\text{MSE}} \sim F\left((a-1)(b-1),\ N-ab\right).$$

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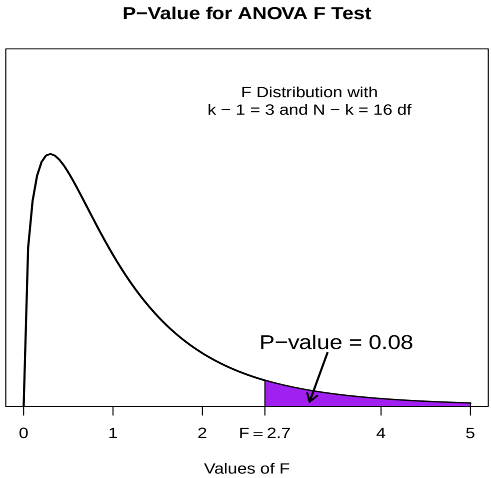
- The next slide shows the **p-value** when the observed test statistic value is $\boldsymbol{F = 2.7}$.


P−Value for ANOVA F Test
F Distribution with
k − 1 = 3 and N − k = 16 df
P−value = 0.08
0
1
2
F = 2.7
4
5
Values of F

The ANOVA Table

- The results of an **analysis of variance** are summarized in a ***two-factor ANOVA table*** having the form shown below.

Two-Factor ANOVA Table:

Source	DF	SS	MS	F	P-value
Factor A	$a-1$	SSA	MSA = SSA/$(a-1)$	MSA/MSE	p
Factor B	$b-1$	SSB	MSB = SSB/$(b-1)$	MSB/MSE	p
Interaction	$(a-1)(b-1)$	SSAB	MSAB = SSAB/$((a-1)(b-1))$	MSAB/MSE	p
Error	$ab(n-1)$	SSE	MSE = SSE/$(ab(n-1))$		
Total	$N-1$	SSTo			

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     (But the effect of each factor is different depending on the level of the other factor.)

  2. If the AB interaction ***isn't* significant**, **proceed** to the results of the tests for **factor** A and B **main effects**.

Example

For the soil phosphorus study, the **ANOVA table** (obtained using software) is below.

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Soil Type	1	17876.0	17876.0	22.98	0.000
Topography Type	3	9693.8	3231.3	4.15	0.024
Interaction	3	11390.8	3796.9	4.88	0.013
Error	16	12445.3	777.8		
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What **conclusions** are appropriate (using $\alpha = 0.05$)?

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Each factor has a **different effect** depending on the **level** of the **other factor.**

Checking the ANOVA Assumptions

- The **ANOVA F tests** require that the ab groups (samples) are from **normal** populations (or that their sample sizes are **large**) whose **standard deviations** are all **equal.**

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  - Make ab *separate* **histograms** or **normal probability plots**, one for each of the ab **groups**.
  - Make a *single* **histogram** or **normal probability plot** plot of the N **residuals** e_{ijk}.

- A few ways to check the **equal population standard deviation** assumption:

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  - A plot of the ***residuals*** (y-axis) versus **fitted values** (group means, x-axis).

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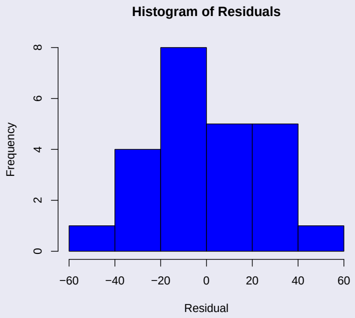
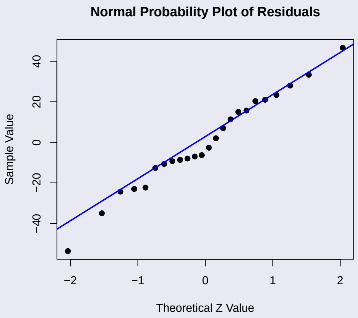
  - An ***individual value plot*** of the ab groups.
  - A plot of the ***residuals*** (y-axis) versus **fitted values** (group means, x-axis).
- In both plots, we look for roughly **equal amounts of within-group (vertical) spread** across the ab groups.

Example

For the soil phosphorus study, a **histogram** and **normal probability plot** of the **residuals** are below.





The plots show that the **normality assumption** appears to be met.

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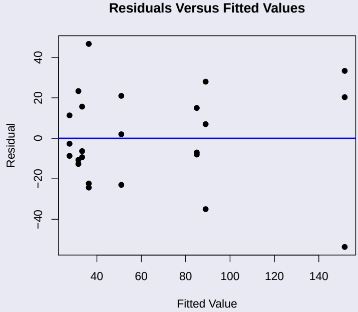
A plot the **residuals** versus the **fitted values** (**group means**) is below.



The amount of (vertical) spread of the points is roughly the same from left to right, so the **equal standard deviation assumption** appears to be met.

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Because the **normality** and **equal standard deviation assumptions** are met, the results of the F **tests** are **valid.**