A Discontinuous Derivative

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The following example is the standard example of a function with a discontinuous derivative, and none of the thoughts below are new or original to me. However, questions about this example arise so frequently that I present a complete discussion here.

Let f be the function given by

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{when } x \neq 0; \\ 0, & \text{when } x = 0. \end{cases} \tag{1}$$

Then the usual rules of differentiation assure us that when $x \neq 0$ we must have

$$\begin{align} f'(x) &= 2x \sin(1/x) + x^2 \cos(1/x) \cdot (-1/x^2) \tag{2} \\ &= 2x \sin(1/x) - \cos(1/x). \tag{3} \end{align}$$

But the expression on the right side of (3) is meaningless when $x = 0$. What does this mean for the existence of $f'(0)$?

We might be tempted to think that the answer would be that $f'(0)$ doesn't exist. But let's consider what the Product Rule really says:

> **Theorem A:** *If $F(x) = u(x)v(x)$, and if $u'(x_0)$ and $v'(x_0)$ both exist, then $F'(x_0) = u'(x_0)v(x_0) + u(x_0)v'(x_0)$.*

The statement says that if both u' and v' exist, then F' exists and can be expressed in terms of u, v, u', and v'. But it *does not* say that F' doesn't exist when one or both of u' and v' fails to exist. And that's precisely the situation in which we find ourselves above.

We can see how Theorem A fails rather spectacularly to show that derivatives don't exist by considering the functions

$$u(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ -x, & \text{if } x \text{ is not rational;} \end{cases} \tag{4}$$

and

$$v(x) = \begin{cases} -x, & \text{if } x \text{ is rational;} \\ x, & \text{if } x \text{ is not rational.} \end{cases} \tag{5}$$

Both functions are nowhere differentiable (though both are continuous at $x = 0$). But $u(x)v(x) = -x^2$ has a derivative everywhere. As a rule, theorems tell us what processes preserve good behavior—not what processes preserve bad behavior.

The function $g : x \mapsto \sin(1/x)$ isn't continuous at $x = 0$, and the discontinuity is one that isn't even removable. This means that there's no hope of assigning a value to $g(0)$ in such a way as to make the extension of g differentiable at $x = 0$. Consequently, Theorem A is *inapplicable* to f at $x = 0$. This means that we can't even use it to support the negative conclusion that $f'(0)$ doesn't exist.

Here is a theorem that we can sometimes apply when the usual differentiation rules break down:

Theorem B: *Let F be continuous on $[a, b]$, and suppose that $x_0 \in (a, b)$ is such that $F'(x)$ exists for all $x \in (a, x_0) \cup (x_0, b)$. If $\lim_{x \to x_0} F'(x) = L$, then $F'(x_0) = L$.*

Proof: If $h > 0$ is sufficiently small, F is continuous on $[x_0, x_0 + h]$ and $F'(x)$ exists for all $x \in (x_0, x_0 + h)$. By the Mean Value Theorem, there is a number $\xi_h \in (x_0, x_0 + h)$ such that $F(x_0 + h) - F(x_0) = F'(\xi_h)h$. Hence

$$F'_+(x_0) = \lim_{h \to 0^+} \frac{F(x_0 + h) - F(x_0)}{h} \tag{6}$$

$$= \lim_{h \to 0^+} \frac{F'(\xi_h)h}{h} \tag{7}$$

$$= \lim_{h \to 0^+} F'(\xi_h) = L. \tag{8}$$

Similarly, $F'_-(x_0) = L$.•

But this theorem, too, is inapplicable to our function f because

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right] \tag{9}$$

doesn't exist. And, once again, we must take care to read what the theorem doesn't say as well as what it does say. It *does* tell us how to find the derivative when certain circumstances apply. But it *doesn't* say that the derivative doesn't exist when those circumstances don't apply. So this theorem too gives us no useful information about $f'(0)$.

One means of finding a derivatives alway applies: The definition of the derivative. (To be sure, we may not always be able to carry out the requisite calculations. But that would be our fault, and not the method's.) So we'll take the bull by the horns and apply the definition to our function f. If the limit exists, we must have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}. \tag{10}$$

Equation (10) is definitive for the existence of $f'(0)$. The latter exists if and only if the limit exists; if we can complete the calculation correctly, there can be no appeal from our conclusion.

Using (1), we find that when $h \neq 0$, we have $f(0+h) = f(h) = h^2 \sin(1/h)$—while $f(0) = 0$. Hence

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \left[\frac{h^2 \sin(1/h) - 0}{h} \right] \tag{11}$$

$$= \lim_{h \to 0} \left[h \sin\left(\frac{1}{h}\right) \right]. \tag{12}$$

Now whatever $h \neq 0$ may be, $-1 \leq \sin(1/h) \leq 1$. Hence $-h \leq h \sin(1/h) \leq h$ when $h > 0$ and $h \leq h \sin(1/h) \leq -h$ when $h < 0$. And, of course, $\lim_{h \to 0} \pm h = 0$. Consequently, by the Flyswatter Principle (alias the Squeeze Theorem, alias the Sandwich Theorem), $\lim_{h \to 0} [h \sin(1/h)] = 0$. It therefore follows that $f'(0)$ exists and is zero.

We have already noticed that $\lim_{x \to 0} f'(x)$ doesn't exist, and the fact that we've now found that $f'(0) = 0$ doesn't change that earlier observation. We have to conclude that this function is differentiable at $x = 0$, but that the derivative is discontinuous there.