The Definite Integral as an Accumulator

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In the last few years, the committee that writes the AP Calculus exams has placed a number of problems on those exams that require candidates to think of the definite integral as an accumulator. Thus, Jennifer Nichols asks

> "Have any of you found good resources on practice problems for the students on the topic of accumulated change? I created my own a couple of years ago, but they aren't realistic and they honestly aren't very good. With my created questions, I have been teaching my students the equivalent of a half chapter on 'accumulation functions', but I need better questions. I know there are many collegeboard samples on old exams, but I like to save those for review time, so I need outside resources."

Properly read, the standard proof of the Fundamental Theorem of Calculus that appears in most elementary calculus texts gives us a good clue as to how to answer this question. That argument suggests that many of the settings in which we use definite integrals can be thought of as settings in which we use those integrals as accumulators. The accumulator approach is best seen by approaching standard definite integral problems—problems we usually solve by thinking of the definite integral as a limit of Riemann sums—by way of accumulation instead. I would like to suggest that we should teach students this approach to setting up definite integral problems *in addition to* the standard Riemann sum approach. Here are some examples.

Example 1

Find the area inside the first quadrant lobe of the polar curve $f(\theta) = \sin 2\theta$.

Solution: Let θ_0 be any first-quadrant angle in standard position, and let $A(\theta_0)$ be the area that lies inside the first-quadrant lobe of $f(\theta) = \sin 2\theta$ and between the rays $\theta = 0$ and $\theta = \theta_0$. (See Figure 1, where we have chosen a specific value for θ_0 and colored $A(\theta_0)$ turquoise.) The function A accumulates area inside the lobe as we vary θ_0 from 0 to $\pi/2$.

Select a small positive number $\Delta\theta$. The region R cut off from the lobe by the rays $\theta = \theta_0$ and $\theta = \theta_0 + \Delta\theta$ is $R = \{\, (r, \theta) : 0 \le r \le f(\theta),\ \theta_0 \le \theta \le \theta_0 + \Delta\theta \,\}$, and the area of R is given by $A(\theta_0 + \Delta\theta) - A(\theta_0)$. (This region lies inside the lobe and between the two blue rays shown in Figure 1.)

Now $\theta \mapsto f(\theta)$ is a continuous function on the interval $[\theta_0, \theta_0 + \Delta\theta]$, so $r(\theta)$ takes on both a maximum value, r_M and a minimum value, r_m in that interval. The sector $\{\,(r,\theta) : 0 \le r \le r_m,\ \theta_0 \le \theta \le \theta_0 + \Delta\theta\,\}$ is completely contained by the region R, and the sector $\{\,(r,\theta) : 0 \le r \le r_M,\ \theta_0 \le \theta \le \theta_0 + \Delta\theta\,\}$ completely contains R. Hence

$$\frac{1}{2} r_m^2 \Delta\theta \quad \le \quad A(\theta_0 + \Delta\theta) - A(\theta_0) \quad \le \quad \frac{1}{2} r_M^2 \Delta\theta. \qquad (1)$$

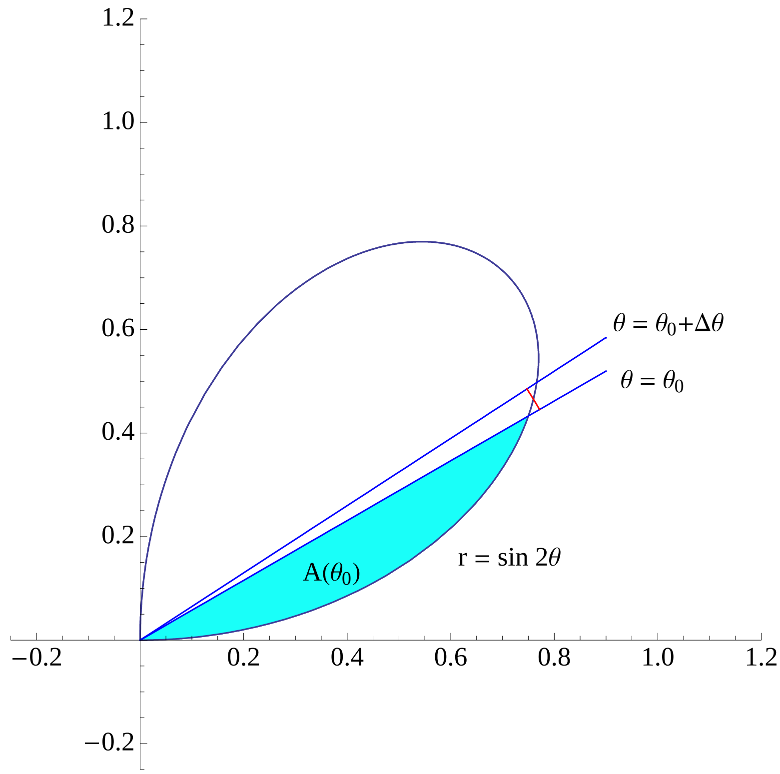

Figure 1: $r = \sin 2\theta$

Appealing once more to the continuity of f, we see that there must be a number θ^* somewhere in $[\theta_0, \theta_0 + \Delta\theta]$ such that

$$A(\theta_0 + \Delta\theta) - A(\theta_0) \quad = \quad \frac{1}{2} \left[f(\theta^*)\right]^2 \Delta\theta. \qquad (2)$$

In Figure 1, $r = f(\theta^*)$ is shown as a short red arc. It is placed so that the area contained in the sector that it defines together with the rays $\theta = \theta_0$ and $\theta = \theta_0 + \Delta\theta$ is the same as the area of the "wedge" cut from the larger region inside the curve $r = f(\theta)$ by the same two rays.

We have therefore shown that for every sufficiently small $\Delta\theta > 0$ there is θ^* in the interval $[\theta_0, \theta_0 + \Delta\theta]$ such that

$$\frac{A(\theta_0 + \Delta\theta) - A(\theta_0)}{\Delta\theta} \quad = \quad \frac{1}{2} \left[f(\theta^*)\right]^2. \qquad (3)$$

An altogether similar argument establishes that a corresponding statement is true if $\Delta\theta < 0$ is small enough.

We now pass to the limit as $\Delta\theta \to 0$ in equation (3). Because θ^* is constrained to lie between θ_0 and $\theta_0 + \Delta\theta$, we conclude that $\theta^* \to \theta_0$ as $\Delta\theta \to 0$. Therefore, by the continuity of f, $\lim_{\Delta\theta\to 0} f(\theta^*) = f(\theta_0)$. It then follows that $A'(\theta) = [f(\theta)]^2/2$ for every θ in the interval $[0, \pi/2]$.

Finally, we apply the Fundamental Theorem of Calculus to conclude that

$$\begin{aligned}
A\left(\frac{\pi}{2}\right) &= A\left(\frac{\pi}{2}\right) - A(0) && (4)\\
&= \int_0^{\pi/2} A'(\theta)\,d\theta && (5)\\
&= \frac{1}{2}\int_0^{\pi/2} [f(\theta)]^2\,d\theta && (6)\\
&= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta && (7)\\
&= \frac{\pi}{8}. && (8)
\end{aligned}$$

Example 2

Find the amount of work done in stretching a spring (of spring constant K) from equilibrium to a point 2 units from equilibrium. (Assume that Hooke's Law is valid for the spring throughout the interval in question.)

Solution: According to Hooke's Law, the restoring force F exerted by the spring when it is stretched to a point x units from its equilibrium is given by $F = -Kx$, the minus sign arising from the fact that the force acts in the direction opposite the displacement. To stretch the spring, we must exert a force f equal in magnitude to F and in the opposite direction. Thus, the stretching force is given by $f = Kx$.

For each x_0, $0 \le x_0 \le 2$, let $W(x_0)$ be the amount of work we must do to stretch the spring from equilibrium at $x = 0$ to $x = x_0$. (We can think of $W(x_0)$ as the energy we have stored in the spring when it is stretched to $x = x_0$, so that W accumulates energy.) For $x_0 < 2$ we choose $\Delta x > 0$ and we suppose that $x_0 + \Delta x \le 2$.

Now $W(x_0 + \Delta x) - W(x_0)$ is the amount of work done in streching the spring from $x = x_0$ to $x = x_0 + \Delta x$. At each point x of this interval, the force we must exert on the spring is given by $f = Kx$. Moreover, $x_0 \le x \le x_0 + \Delta x$ implies that $Kx_0 \le Kx \le K(x_0 + \Delta x)$. Hence

$$Kx_0\Delta x \quad \le \quad W(x_0 + \Delta x) - W(x_0) \quad \le \quad K(x_0 + \Delta x)\Delta x$$

Consequently, there is a number x^* in the interval $[x_0, x_0 + \Delta x]$ such that

$$\begin{aligned}
W(x_0 + \Delta x) - W(x_0) &= Kx^*\Delta x, \text{ or}\\
\frac{W(x_0 + \Delta x) - W(x_0)}{\Delta x} &= Kx^*.
\end{aligned}$$

Because $x_0 \le x^* \le x_0 + \Delta x$, we have $\lim_{\Delta x \to 0} Kx^* = Kx_0$. Hence, $W'[x_0] = Kx_0$. it now follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} W(2) &= W(2) - W(0) && (9)\\ &= \int_0^2 W'(x)\,dx && (10)\\ &= K\int_0^2 x\,dx && (11)\\ &= 2K. && (12) \end{aligned}$$

Example 3

The base of a certain solid is the unit circle in the xy-plane. Every vertical cross-section of this solid perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid.

Solution: If $-1 \le t \le 1$, let $V(t)$ denote the volume of that portion of this solid that lies between the planes $x = -1$ and $x = t$. The function V accumulates volume as we increase t through the interval $[-1, 1]$. Choose t_0 in $[-1, 1)$, and let $\Delta t > 0$ be small enough that $t_0 + \Delta t$ lies in $[-1, 1]$ as well. We consider the difference $V(t_0 + \Delta t) - V(t_0)$, which is the volume of the solid S cut off by the two planes $x = t_0$ and $x = t_0 + \Delta t$.

If x lies in the interval $[t_0, t_0 + \Delta t]$, then the upper half of the curve $x^2 + y^2 = 1$ (which is the boundary of the base of S) is given by $y = \sqrt{1 - x^2}$ and the lower half is given by $y = -\sqrt{1 - x^2}$. The left face of S is the equilateral triangle perpendicular to the x-axis, whose base is the interval connecting $(t_0, -\sqrt{1 - t_0^2}, 0)$ to $(t_0, \sqrt{1 - t_0^2}, 0)$, and whose vertex is at $(t_0, 0, \sqrt{3}\sqrt{1 - t_0^2})$. The right face of S is the equilateral triangle perpendicular to the x-axis, whose base is the interval connecting $(t_0 + \Delta t, -\sqrt{1 - (t_0 + \Delta t)^2}, 0)$ to $(t_0 + \Delta t, \sqrt{1 - (t_0 + \Delta t)^2}, 0)$, and whose vertex is at $(t_0 + \Delta t, 0, \sqrt{3}\sqrt{1 - (t_0 + \Delta t)^2})$.

Now the function $G : t \mapsto \sqrt{1 - t^2}$ is continuous on the interval $[t_0, t_0 + \Delta t]$, so it takes on a minimum value b_m and a maximum value b_M in that interval. The cylinder whose height is Δt and whose base is an equilateral triangle of base $2b_m$ will fit entirely inside of S, while the cylinder whose height is Δt and whose base is an equilateral triangle of base $2b_M$ will entirely contain S. Consequently,

$$b_m^2\sqrt{3}\Delta t \quad \le \quad V(t_0 + \Delta t) - V(t_0) \quad \le \quad b_M^2\sqrt{3}\Delta t. \qquad (13)$$

Continuity of the function G now guarantees that there is a number t^* in the interval $[t_0, t_0 + \Delta t]$ such that

$$\begin{aligned} V(t_0 + \Delta t) - V(t_0) &= [G(t^*)]^2\sqrt{3}\Delta t && (14)\\ &= \sqrt{3}[1 - (t^*)^2]\Delta t. && (15) \end{aligned}$$

We have thus shown that whatever $t_0 \in [-1, 1)$ we may choose, and whatever $\Delta t > 0$ may be, there is $t^* \in [t_0, t_0 + \Delta t]$ such that

$$\frac{V(t_0 + \Delta t) - V(t_0)}{\Delta t} \quad = \quad \sqrt{3}\left[1 - (t^*)^2\right].$$

An entirely similar argument shows that we can write a similar equation when $\Delta t < 0$, and it follows, again from the continuity of G, that

$$\begin{aligned} V'(t_0) &= \lim_{\Delta t \to 0} \frac{V(t_0 + \Delta t) - V(t_0)}{\Delta t} \\ &= \sqrt{3}\,(1 - t_0^2). \end{aligned}$$

The Fundamental Theorem of Calculus now assures us that the required volume, which is $V(1)$, is given by

$$\begin{aligned} V(1) &= V(1) - V(-1) \\ &= \int_{-1}^{1} V'(t)\,dt \\ &= \sqrt{3}\int_{-1}^{1} (1 - t^2)\,dt \\ &= \frac{4}{\sqrt{3}}. \end{aligned}$$

Example 4

The region bounded by the x-axis, the lines $x = 1$ and $x = 4$, and the curve $y = 5/4 + \sin(\pi x/2)$ is revolved about the y-axis. Find the resulting volume.

Solution: Let V be the volume accumulation function: For each choice of x_0 in $[1, 4]$, $V(x_0)$ is the volume generated by revolving the region bounded by the x-axis, the lines $x = 1$ and $x = x_0$, and the curve $y = 5/4 + \sin(\pi x/2)$ about the y-axis. (See Figure 2.)

Let x_0 be in the interval $[1, 4)$, and choose $\Delta x > 0$ but small enough that $x_0 + \Delta x$ lies in $[1, 4]$. Then $V(x_0 + \Delta x) - V(x_0)$ gives the volume of the solid S generated by revolving the region bounded by the x-axis, the lines $x = x_0$ and $x = x_0 + \Delta x$, and the curve $y = 5/4 + \sin(\pi x/2)$ about the y-axis. The volume of S is very nearly the volume of a cylindrical shell.

Let $m_{\Delta x}$ denote the minimum value of $5/4 + \sin(\pi x/2)$ on the interval $[x_0, x_0 + \Delta x]$, and let $M_{\Delta x}$ denote the maximum value. (Continuity assures us that these values exist.) If we revolve the rectangular region bounded by the x-axis, the line $x = x_0$, the line $x = x_0 + \Delta x$, and the line $y = m_{\Delta x}$ about the y-axis, we obtain a cylindrical shell that lies entirely inside of the solid S. If we replace the line $y = m_{\Delta x}$ with the line $y = M_{\Delta x}$ and again revolve the rectangle about the y-axis, we obtain another cylindrical shell that entirely contains the solid W. Now the volume generated by revolving the smaller rectangle about the y-axis is $\pi m_{\Delta x}[(x_0 + \Delta x)^2 - x_0^2]$, and the volume generated by revolving the larger rectangle about the y-axis is $\pi M_{\Delta x}[(x_0 + \Delta x)^2 - x_0^2]$. Thus,

$$\pi m_{\Delta x}[(x_0 + \Delta x)^2 - x_0^2] \;\leq\; V(x_0 + \Delta x) - V(x_0) \;\leq\; \pi M_{\Delta x}[(x_0 + \Delta x)^2 - x_0^2]. \quad (16)$$

But $x \mapsto 5/4 + \sin(\pi x/2)$ is a continuous function, so there is a number x^* in the interval $[x_0, x_0 + \Delta x]$ which has the property that the volume of S is given exactly by

$$V(x_0 + \Delta x) - V(x_0) = \pi \left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right] [(x_0 + \Delta x)^2 - x_0^2] \quad (17)$$

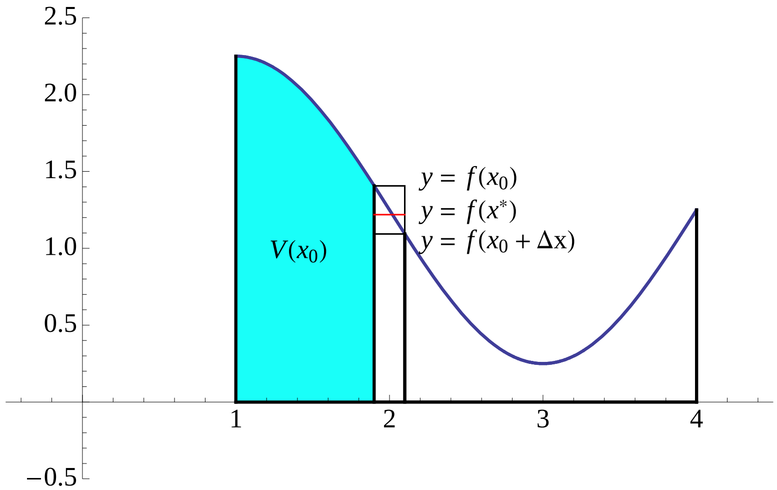

Figure 2: $y = 5/4 + \sin(\pi x/2)$

$$= \pi[2x_0\Delta x + (\Delta x)^2]\left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right]. \tag{18}$$

Thus, for each $\Delta x > 0$, there is a number x^* in the interval $[x_0, x_0 + \Delta x]$ such that

$$\frac{V(x_0 + \Delta x) - V(x_0)}{\Delta x} = \pi(2x_0 + \Delta x)\left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right]. \tag{19}$$

A similar argument establishes a similar equation for $\Delta x < 0$, and it now follows from continuity that

$$V'(x_0) = 2\pi x_0\left[\frac{5}{4} + \sin\left(\frac{\pi x_0}{2}\right)\right]. \tag{20}$$

We have now only to apply the Fundamental Theorem of Calculus to conclude that the volume we seek is

$$V(4) = V(4) - V(1) \tag{21}$$

$$= \int_1^4 V'(x)\,dx \tag{22}$$

$$= 2\pi\int_1^4 \left[\frac{5}{4}x + x\sin\left(\frac{\pi x}{2}\right)\right] dx \tag{23}$$

$$= \frac{75\pi}{4} - 16 - \frac{8}{\pi}. \tag{24}$$