A Remarkable Concurrence

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Lemma: *Let A, B, and C be three noncollinear points labeled so that $\angle ABC$ is less than $180°$, and let D, E be, respectively, the midpoints of the segments $\overline{AB}$ and $\overline{BC}$. Let α_1 and α_2 be parallel lines passing through, respectively, D and E. Let β_1 be the line determined by the reflections of the points A and B about the line α_1, and let β_2 be the line determined by the reflections of the points B and C about the line α_2. Then β_1 and β_2 meet at a unique point G and the angle at G from β_2 to β_1 (which we shall write as $\angle[\beta_2, \beta_1]$ is congruent with $\angle ABC$.*

Proof: We may assume, without loss of generality, that $\angle ABC = \angle[\overline{BA}, \overline{BC}]$ opens counter-clockwise. We impose a standard Cartesian coordinate system in such a way that its origin lies at the point B, while the point A lies on the positive half of the x-axis at, say, $(2a, 0)$ for a certain $a > 0$. Then the coordinates of D are $(a, 0)$.

We suppose that C has coordinates $(2c, 2mc)$ for a certain $c > 0$ so that the terminal ray of $\angle[\overline{BA}, \overline{BC}]$ is the line through the origin with slope m. Moreover, the point E then has coordinates (c, mc).

Finally, we suppose further that the parallel lines α_1 and α_2, passing through D and E respectively, have slope M.

The line β_1, which is the reflection of the x-axis about the line α_1 is then the line through $(a, 0)$ with slope $2M/(1 - M^2)$, or the line whose equation is

$$y = \frac{2M}{1 - M^2}(x - a), \tag{1}$$

which can be rewritten

$$2Mx + (M^2 - 1)y = 2Ma. \tag{2}$$

The angle $\angle[\overline{BC}, \alpha_2]$ satisfies

$$\tan \angle[\overline{BC}, \alpha_2] = \frac{M - m}{1 + mM}. \tag{3}$$

and so the line β_2 must have slope given by

$$\text{slope}[\beta_2] = \frac{M + \frac{M-m}{1+mM}}{1 - M\frac{M-m}{1+mM}} \tag{4}$$

$$= -\frac{mM^2 + 2M - m}{M^2 - 2mM - 1}. \tag{5}$$

Consequently, an equation for the line β_2 is

$$y = mc - \frac{mM^2 + 2M - m}{M^2 - 2mM - 1}(x - c), \tag{6}$$

and this can be rewritten as

$$(mM^2 + 2M - m)x + (M^2 - 2mM - 1)y = 2c(mM - 1)(M - m). \tag{7}$$

A straightforward (but tedious) calculation shows that equations (2) and (7) are independent unless $m = 0$—which we have ruled out by our requirement that A, B, and C be non-collinear. This assures that the lines β_1 and β_2 meet in a unique point G, whose coordinates we could calculate if we were interested.

Moreover, we have

$$\tan\angle[\beta_2, \beta_1] = \frac{\text{slope}[\beta_1] - \text{slope}[\beta_2]}{1 + \text{slope}[\beta_1]\text{slope}[\beta_2]} \tag{8}$$

$$= \frac{\frac{2M}{1-M^2} + \frac{mM^2+2M-m}{M^2-2mM-1}}{1 - \left(\frac{2M}{1-M^2}\right)\left(\frac{mM^2+2M-m}{M^2-2mM-1}\right)} \tag{9}$$

$$= m, \tag{10}$$

and it follows that $\angle[\beta_2, \beta_1] \cong \angle ABC$.•

Corollary: *Let A, B, C, D, E, and G be as in the Lemma. If G and B lie on the same side of $\overleftrightarrow{DE}$, then $\angle ABC$ and $\angle DGE$ are supplementary. If G and B lie on opposite sides of $\overleftrightarrow{DE}$, then $\angle ABC$ and $\angle DGE$ are congruent.*•

Remark: It is obvious that G cannot lie *on* $\overleftrightarrow{DE}$ unless G coincides with one of the points D or E.

Theorem: *Let D, E, and F be the midpoints, respectively, of the sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ of the triangle $\triangle ABC$. Let α_1, α_2, and α_3 be parallel lines with α_1 passing through D, α_2 passing through E, α_3 passing through F. If β_1 is the line determined by reflecting A and B about α_1, β_2 is the line determined by reflecting B and C about α_2, and β_3 is the line determined by reflecting A and C about α_3, then β_1, β_2, and β_3 are concurrent at a point G which lies on the Nine-Point Circle of $\triangle ABC$.*

Proof: Consider $\triangle DEF$, which is the medial triangle of $\triangle ABC$. Thus, $\angle EFD \cong \angle ABC$. Moreover, F and B lie on opposite sides of the line $\overleftrightarrow{DE}$. Taking the Corollary into account, we see, as a consequence of the Two-Chord Angle Theorem and its relatives (the Two-Secant Angle Theorem, etc.) that the point G, where the lines β_1 and β_2 meet according to the Lemma, lies on the circumcircle of the medial triangle, which is the Nine-Point Circle for $\triangle ABC$.

Let G' be the point where the lines β_2 and β_3 meet according to the Lemma. Then, as above, G' also lies on the Nine-Point Circle of $\triangle ABC$. But β_2 meets the Nine-Point Circle only at E and G, whereas β_3 meets the Nine-Point Circle only at F and G'. It follows that $G' = G$ so that the lines β_1, β_2, and β_3 are concurrent at G.•