Using Definite Integrals to Solve a Separable Initial Value Problem

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1 The Plan

Given the separable differential equation $\frac{dy}{dx} = f(x)g(y)$, with initial condition $y = y_0$ when $x = x_0$, we begin by supposing that $y = \varphi(x)$ gives the solution.

We note that if $g(y_1) = 0$, then the constant function $\varphi(y) \equiv y_1$ is a solution.

Thus, we may suppose that $g[\varphi(t)]$ doesn't vanish when t is near x_0, so that

$$\frac{\varphi'(t)}{g[\varphi(t)]} = f(t), \tag{1}$$

at least when t is close enough to x_0. Thus, if we choose x close enough to x_0, we have

$$\int_{x_0}^{x} \frac{1}{g[\varphi(t)]}\varphi'(t)\,dt = \int_{x_0}^{x} f(t)\,dt. \tag{2}$$

The substitution theorem for definite integrals allows us to make the substitution $u = \varphi(t)$ on the left side of equation (2); this gives

$$\int_{x_0}^{x} \frac{1}{g[\varphi(t)]}\varphi'(t)\,dt = \int_{\varphi(x_0)}^{\varphi(x)} \frac{1}{g(u)}\,du. \tag{3}$$

Using the initial value and the fact that $y = \varphi(x)$, we see that we can now rewrite (2) as

$$\int_{y_0}^{y} \frac{du}{g(u)} = \int_{x_0}^{x} f(t)\,dt. \tag{4}$$

If we know functions G and F for which $G'(u) = \dfrac{1}{g(u)}$ and $F'(t) = f(t)$, we may now write

$$G(u)\Big|_{y_0}^{y} = F(t)\Big|_{x_0}^{x}, \text{ or} \tag{5}$$

$$G(y) = F(x) + [G(y_0) - F(x_0)]\,. \tag{6}$$

It remains only to solve for $y = \varphi(x)$ in terms of x. Finishing the solution depends, of course, on our ability to carry out whatever algebra is needed to invert the function G.

2 An Example

Consider the initial value problem

$$\frac{dy}{dx} = yx^2 + yx; \tag{7}$$

$$y(-3) = -e^{-9/2}. \tag{8}$$

We use the standard formal calculation (that is, a calculation that is correct in *form*, though perhaps not clearly so in content) to avoid a surfeit of notation and explicit use of the substitution theorem. First, we rewrite equation (7):

$$\frac{dy}{dx} = yx^2 + yx; \tag{9}$$

becomes

$$\frac{dy}{y} = (x^2 + x)\,dx. \tag{10}$$

Also, because we are going to want the upper limits of our definite integrals to be the variables, y and x, but it's unwise to use one symbol with two different meanings in the same expression, we rewrite (10) as

$$\frac{du}{u} = (t^2 + t)\,dt. \tag{11}$$

Thus,

$$\int_{-e^{-9/2}}^{y} \frac{du}{u} = \int_{-3}^{x} (t^2 + t)\,dt. \tag{12}$$

Now turn the crank:

$$\ln|u|\Big|_{-e^{-9/2}}^{y} = \left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_{-3}^{x}, \tag{13}$$

$$\ln\left|\frac{y}{-e^{-9/2}}\right| = \frac{x^3}{3} + \frac{x^2}{2} + \frac{9}{2}, \tag{14}$$

$$\left|\frac{y}{-e^{-9/2}}\right| = \exp\left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{9}{2}\right], \tag{15}$$

where we have used the standard notation $\exp u = e^u$.

At this point, we recall that, on account of our intitial condition, we know that y must be negative for values of x near -3. For such values of x, then, the fraction on the left side of (15) is positive. We may therefore write

$$\frac{y}{-e^{-9/2}} = \exp\left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{9}{2}\right], \tag{16}$$

which leads to the solution

$$y = -\exp\left[\frac{x^3}{3} + \frac{x^2}{2}\right]. \tag{17}$$

It is easily checked that (17) does indeed give the solution to our initial value problem.