Defining the Logarithm Function

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Roy Gover asks:

> In my Calculus text, the following is presented as a definition:
>
> $$\int_1^e \frac{dx}{x} = 1. \tag{1}$$
>
> This fact is not at all obvious to me. Can anyone help me understand why this is so?

Roy doesn't say what equation (1) is supposed to define; I presume that it is the number e, in which case the naive answer to the question he asks is that

$$\int_1^e \frac{dx}{x} = \ln x\Big|_1^e \tag{2}$$
$$= \ln e - \ln 1 = 1, \tag{3}$$

so e is precisely the upper limit that makes the integral be one.

But the naive answer above begs an important question in that it assumes that we already know what natural logarithms are and what the number e is. Thus, it isn't a fully satisfactory answer because we are used to expecting that a definition will *tell* us what its definiend is—not make use of our existing knowledge of what it is. So the real question that underlies the question that Roy actually asked may be "What in the world is going on here?"

In order to understand the answer to this latter question, we must understand the nature and purpose of mathematical definition. In mathematics, definitions supply us with the raw materials we need to state and prove theorems. Unlike dictionary definitions (which are connotative), mathematical definitions are denotative. The definitions we find in a dictionary tell us what a word means in the context of an existing language—by virtue, in fact, of that existing language. Mathematical definition functions in quite the opposite way; mathematical definition actively assigns meaning to terms that don't yet have meaning in the context of the logical structure of the mathematics we are developing. Thus, a mathematical definition doesn't describe the meaning of its definiend in the way we might expect; it tells us instead how we are to use that definiend in the logical structure we are building. For this reason, mathematical definition *always* draws on pre-existing knowledge of that structure. The person who writes a definition does so with a purpose in mind; she knows what theorems she wants to prove and establishes her definitions so that her theorems will work as she wants them to. And, in fact, a rather large fraction of research at the boundaries of our mathematical understanding revolves about deciding which of several variant definitions it would be best for the mathematical community to adopt.

The definition given via equation (1) fits nicely into a context where we have taken the definition of the natural logarithm function to be given by

$$\ln t = \int_1^t \frac{dx}{x}. \tag{4}$$

As we all know (because we have already studied calculus) it is a fact that the number e defined by equation (1) is so related to the function ln defined by equation (4) that

$$\ln e^x = x \text{ for all real } x \tag{5}$$

and

$$e^{\ln x} = x \text{ for all positive real } x. \tag{6}$$

In fact, statements (5) and (6) are among the most important theorems that we will prove using our definitions. But in order to prove these things, we must know what the meanings of the symbols are. In particular, we have to know what $\ln x$ means for all positive values of x and we have to know what e^t means for all real values of t—especially irrational values of t. Our proofs can be only as careful and rigorous as our definitions are.

Once we have developed the elementary theory that underlies the definite integral, we can use the integral to define the elementary transcendental functions. I don't advocate this approach in elementary calculus, and I'm not alone. In his "To the Instructor" remarks prefacing the fourth edition of his *Calculus and Analytic Geometry,* Sherman K. Stein wrote:

> "...since students have trouble grasping that a logarithm is an exponent, I do not feel that telling them that it is an area will help them. The suggestion that logarithms be presented as integrals goes back to J. W. Bradshaw's 'The logarithm as a direct function,' [*Annals of Mathematics,* **4**(1903), 51–62]. However, we should keep in mind Osgood's introduction to this article, in which

> he remarks, 'How simple the analysis is appears from a casual glance at the following pages, in which Mr. Bradshaw has carried through all the details in a rigourous development of the Logarithm. ... It is hoped that this presentation may prove attractive to students who have *finished a thorough course in elementary calculus*' [emphasis added by Stein]. The integral approach is [included in an appendix to Stein's book] for the convenience of instructors who wish to present it to students who are prepared to grasp it."

I could not agree more (except that I think he was fudging in the last statement quoted above; I think he included it because he knew that doing so would sell more books). Moreover, because I believe that students at the level of beginning calculus have, at best, a tenuous grasp of the ideas that underpin the definite integral, the approach seems to me to smack of false rigor.

This doesn't obviate the need for a definition, though. Careful thought should convince the reader that there is a serious gap in even the best beginning calculus students' understanding of logarithms and exponentials. After all, precalculus courses *necessarily* define exponentiation only for rational numbers. Because they don't yet have access to the ideas of calculus, students at this level can know what a^r means only for those r of the form p/q where p and $q \neq 0$ are integers; we have given them no meaning for, say, $3^{\sqrt{2}}$ or 7^π. And because we define logarithms as inverse functions of exponentials in precalculus, the meaning of a logarithm to beginning calculus students is somewhat more problematic. A logarithmic function, according to the definitions of precalculus, can have as its domain only the rather curious subset of the real numbers obtained as the image, under the appropriate exponential function, of the rational numbers. (The domain of, say, $\log_2$ has to be $\{x \in \mathbb{R} : x = 2^r, \text{ where } r \in \mathbb{Q}\}$ until we have access to ideas of calculus.) It is therefore no wonder that authors of calculus books think that they need to put exponentials and logarithms on a firmer ground than that established in their students' backgrounds.

Personally, I think that the way out of this difficulty is somewhat simpler than that of defining logarithms to be integrals—though, mathematically speaking, that is a perfectly sound approach. (It is the pedagogical approach with which I take issue.) I think that if you ask a well-equipped student of calculus what 7^π means, or, better, how to find 7^π, that student will tell you that you can get as close as you want by finding 7^3, $7^{31/10}$, $7^{314/100}$, $7^{3141/1000}$, $7^{31415/10000}$, $7^{314159/100000}$, etc. It is certainly the answer I would have given when I was studying calculus for the first time, and I think that today's use of calculators to find approximate values makes the answer even more likely.

And this answer is a correct answer; we can approach exponentials in precisely this way. We should tell our calculus students that when r is irrational we take a^r to mean the limiting value of the numbers a^x as x approaches r through the rational numbers. In keeping with the dictum "Tell them the truth, but not the *whole* truth," we don't need to tell them how to deal with all of the issues that this definition raises. Among those issues:

1. Does the limit in question always exist?

2. Once we've made the definition, is it true that $\lim_{x \to r} a^x = a^r$, where now x is allowed to approach r through all real numbers instead of just the rational numbers?

3. Does the algebra of exponents still work correctly under this definition? (That is, is it true that $a^{x+y} = a^x a^y$, etc., for all x, y?)

4. Is the exponential function $x \mapsto a^x$, so defined, a one-to-one function?

We need only tell them the answers to the questions (all of which are "Yes.") and that mathematicians have figured out why those answers are correct. But we should also promise them that if they become math majors they will see why for themselves, and, of course, we should keep that promise. (If you, yourself, want to know why the answers to these questions are what they are, see Chapter III of *The Teacher's Guide to Calculus*, a draft of which is available on my web-site—the URL is at the head of this note.)

The advantage of this approach is that it builds on what our students (should) already know. In doing so, it avoids the pedagogical trap that leads them to ask the very question that Roy asked.