Continuity & Differentiability of Inverse Functions

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0.1 Theorem: Continuous Inverse

Suppose that $f : (a, b) \to (\alpha, \beta)$ is strictly monotonic and onto. Then f is continuous on (a, b) and has an inverse $f^{-1} : (\alpha, \beta) \to (a, b)$ which is also strictly monotonic, onto, and continuous.

Proof: We will assume, without loss of generality, that f is increasing; the proof is very similar for decreasing f. Choose $x_0 \in (a, b)$, and let $\epsilon > 0$ be given. We may assume that ϵ is so small that $f(x_0) \pm \epsilon \in (\alpha, \beta)$, and thus that $\alpha < f(x_0) - \epsilon < f(x_0) < f(x_0) + \epsilon < \beta$. Because f is onto, we can find x_0^- and x_0^+, both in (a, b) so that $f(x_0^-) = f(x_0) - \epsilon$ and $f(x_0^+) = f(x_0) + \epsilon$. We put $\delta = \min\{x_0 - x_0^-, x_0^+ - x_0\}$. Now if x is chosen so that $|x - x_0| < \delta$, it follows that $x_0^- < x < x_0^+$, so that, by monotonicity, $f(x_0) - \epsilon = f(x_0^-) < f(x) < f(x_0^+) = f(x_0) + \epsilon$, or, equivalently, $|f(x) - f(x_0)| < \epsilon$. Hence f is continuous on (a, b).

That f^{-1} exists is immediate from the strict monotonicity of f (which guarantees that f is one-to-one); that the domain of f^{-1} is (α, β) follows from the fact that f maps (a, b) onto (α, β). Moreover, if $a < x < b$, then $f(x) \in (\alpha, \beta)$ so that $x = f^{-1}[f(x)]$. This means that f^{-1} maps (α, β) onto (a, b).

Suppose that $\alpha < y_1 < y_2 < \beta$. We have $f\left[f^{-1}(y_1)\right] = y_1 < y_2 = f\left[f^{-1}(y_2)\right]$, and from the monotonicity of f, we find that $f^{-1}(y_1) \geq f^{-1}[y_2]$ is not possible. Hence f^{-1} is a strictly monotonic function mapping (α, β) onto (a, b), and it follows from what we have already established that f^{-1} is continuous.•

0.2 Theorem: Inverse Function Rule

Let $\delta > 0$, and suppose that f is a function carrying $(x_0 - \delta, x_0 + \delta)$ onto another interval (α, β) in strictly monotone fashion. Let $y_0 = f(x_0)$. If $f'(x_0)$ exists and is non-zero, then $(f^{-1})'(y_0)$ exists and is given by

$$(f^{-1})'(y_0) = \frac{1}{f'[f^{-1}(y_0)]}. \tag{1}$$

Proof: Theorem 0.1, Continuous Inverse, guarantees us not only that f is continuous on the interval $(x_0 - \delta, x_0 + \delta)$ but also that f^{-1} is defined on (α, β), is strictly monotonic on (α, β), is onto $(x_0 - \delta, x_0 + \delta)$, and is continuous on (α, β). To show that $(f^{-1})'$ exists and has the necessary properties, we consider the difference quotient:

$$\frac{f^{-1}(y_0 + \Delta y) - f^{-1}(y_0)}{\Delta y} = \frac{f^{-1}[f(x_0 + \Delta x)] - f^{-1}[f(x_0)]}{f(x_0 + \Delta x) - f(x_0)} \tag{2}$$

$$= \frac{(x_0 + \Delta x) - x_0}{f(x_0 + \Delta x) - f(x_0)} \tag{3}$$

$$= \frac{1}{\left[\dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right]}. \tag{4}$$

We may write this for Δy sufficiently small, because for such Δy we know that $f^{-1}(y_0 + \Delta y)$ lies in $(x_0 - \delta, x_0 + \delta)$ and is distinct from x_0; thus, $f^{-1}(y_0 + \Delta y) = x_0 + \Delta x$, or, equivalently, $y_0 + \Delta y = f(x_0 + \Delta x)$. Moreover, the continuity of f^{-1} guarantees that $\Delta x \to 0$ when $\Delta y \to 0$. Consequently,

$$(f^{-1})'(y_0) = \lim_{\Delta y \to 0} \frac{f^{-1}(y_0 + \Delta y) - f^{-1}(y_0)}{\Delta y} \tag{5}$$

$$= \frac{1}{\lim_{\Delta x \to 0} \left[\dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right]} \tag{6}$$

$$= \frac{1}{f'(x_0)} \tag{7}$$

$$= \frac{1}{f'[f^{-1}(y_0)]}. \bullet \tag{8}$$