Error in Linearization

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Michael Swatek wrote:

> *We stumbled onto an interesting (I think) problem while discussing linearization, and I want to test my analysis:*
>
> *About what value of x will a tangent-line-based linear approximation be the worst such approximation on that function?*
>
> *I would think it's where the slope of the function changes the most rapidly.*
>
> *In other words, at the maximum rate of change of the slope.*
>
> *So to find that value of x, one would take the third derivative and set it equal to zero (and test for that point being a maximum, possibly using the fourth derivative!)*

Michael has raised a very interesting question—and it is interesting not least because of the imprecise way in which it is phrased. In order to get a handle on things, let's consider the error function E for the approximation to a function f by its linearization at the point a. We shall assume that f has a continuous second derivative throughout the domain in which we consider the error function. The latter is given by

$$E[x] \quad = \quad f[x] - f[a] - f'[a](x-a). \tag{1}$$

By the Fundamental Theorem of Calculus, we may write

$$E[x] \quad = \quad E[a] + \int_a^x E'[t]\,dt. \tag{2}$$

In the latter integral, we put $u = E'[t]$, $v = -(x-t)$. Then $du = E''[t]\,dt$ and $dv = -(-dt) = dt$, so an integration by parts leads to

$$E[x] \quad = \quad E[a] - E'[t](x-t)\Big|_a^x + \int_a^x E''[t](x-t)\,dt \tag{3}$$

$$\quad = \quad E[a] + E'[a](x-a) + \int_a^x E''[t](x-t)\,dt. \tag{4}$$

Now $E[a] = 0$. Moreover, $E'[x] = f'[x] - f'[a]$, so that $E'[a] = 0$ also. Consequently,

$$E[x] \quad = \quad \int_a^x E''[t](x-t)\,dt, \text{ or} \tag{5}$$

$$E[x] \quad = \quad \int_a^x f''[t](x-t)\,dt, \tag{6}$$

where we have made use of the fact that $E''[x] = f''[x]$.

Equation (6) shows that the error we are examining does indeed depend on the second derivative of the function f, but in subtle fashion. *The error depends not just on the single value $f''[a]$, but on the way in which $f''[t]$ varies on the entire interval from a to x.* This dependence finds its expression as a weighted average of values of $f''[t]$ in that interval, the heaviest weighting being granted to those values of $f''[t]$ for which t is nearest a and the lightest weighting to those values of $f''[t]$ for which t is nearest x.

Thus, any answer to Michael's original question must take into account not only the values of the second derivative, but the intervals on which we want the answers. This is implicit in his wording: *"About what values..."*. However, we can't be more precise about our answer unless we make the question more precise.

If we are willing to throw away some of the information in the integral of equation (6), we can apply the First Mean Value Theorem for Integrals. We have assumed that f'' is continuous on the interval of integration, and the function $t \mapsto (x - t)$ does not change sign in that interval. So there is a number ξ in the interval of integration such that

$$\int_a^x f''[t](x-t)\,dt = f''[\xi]\int_a^x (x-t)\,dt \tag{7}$$

$$= f''[\xi]\frac{(x-a)^2}{2}. \tag{8}$$

This leads us directly to a weak form of the classical statement, given in many elementary textbooks, concerning the error in linearization approximations: *Let f be twice continuously differentiable on the interval whose endpoints are a and x. If M is a bound for $|f''[t]|$ throughout that interval, then $E[x]$, the error in replacing $f[x]$ by the linearization approximation $f[a] + f'[a](x - a)$, satisfies*

$$\Big|E[x]\Big| \leq \frac{M}{2}(x-a)^2. \tag{9}$$

(It is possible to strengthen the statement by eliminating the requirement that the second derivative be continuous, but one must use an argument that doesn't depend on the integral to do so.)

It is not difficult to give a similar analysis of the quadratic approximation

$$f[x] \sim f[a] + f'[a](x-a) + \frac{1}{2}f''[a](x-a)^2. \tag{10}$$

All that we need do is carry out another integration by parts. Of course, this will require that we bring $f^{(3)}$ into the picture.