Evaluating Limits

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

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Here's a good question from Ronald Brown, posted on Oct. 4, 2006:

> I feel like my brain has frozen. I know the derivative of a constant is 0. But we are still at the point of only knowing the limit definition of a derivative (i.e., $\lim_{h\to 0}(f(x+h)-f(x))/h$). I have a function, $f(x)=3$. When I plug into the definition of a derivative I get $\lim_{h\to 0}(3-3)/h = \lim_{h\to 0} 0/h$ which I think gives the indeterminate $0/0$ when I do the substitution of 0 for h. I don't see how I can manipulate the expression to get the h out of the denominator to avoid the indeterminate form of $0/0$. The solution manual for our text (Larson 7th Edition) just goes from above to the conclusion that the derivative is 0.

On Oct. 5, 2006, at 6:27 AM, Dave Slomer wrote in reply:

> But a limit is "a process"... and the process will often involve transforming $f(h)$ into $g(h)$ where $g=f$ everywhere but at 0 and g *is* continuous at 0, whereupon, limit is $g(0)$.

Dave put his finger squarely on the issue here. One of our strategies for evaluating $\lim_{h\to 0} f(h)$ for a function f for which $f(0)$ is undefined is to replace f with a different function g which is continuous at 0 but which has the property that $g(h)=f(h)$ for all h in some deleted neighborhood of 0. Let's examine the reasoning we use to evaluate $\lim_{h\to 0} f(h)$, where $f(h)=(h^2+4h)/h$. The domain of this function f doesn't contain 0, because calculating $f(0)$ would require division by zero. But we note that when $h\neq 0$, we have $f(h)=g(h)$, where $g(h)=h+4$. Moreover, the function g not only has $h=0$ in its domain, but also is continuous at $h=0$. Thus, $\lim_{h\to 0} f(h)=g(0)=4$.

In fact, we are using a theorem here—actually a corollary to a slightly more general theorem.

Theorem: *Let g be a function whose domain contains a deleted neighborhood of $x=a$ and suppose that $\lim_{x\to a} g(x)=L$. If f is a function whose domain contains a deleted neighborhood of $x=a$ throughout which $f(x)=g(x)$, then $\lim_{x\to a} f(x)=L$.*

Proof: Let $\epsilon > 0$ be given. By our hypothesis that $\lim_{x \to a} g(x) = L$, we can find $\delta_1 > 0$ such that $|g(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_1$. Because $g(x) = f(x)$ for all x in some deleted neighborhood of $x = a$, there is a $\delta_2 > 0$ such that $f(x) = g(x)$ whenever $0 < |x - a| < \delta_2$. It follows that if we take $\delta = \min\{\delta_1, \delta_2\}$ and require that x satisfy $0 < |x - a| < \delta$, we will then have $|f(x) - L| < \epsilon$. Thus, $\lim_{x \to a} f(x) = L$.•

We apply this theorem to find the derivative of the constant function $F(x) = k$ in the following way: We want

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \tag{1}$$

$$= \lim_{h \to 0} \frac{k - k}{h} \tag{2}$$

$$= \lim_{h \to 0} \frac{0}{h}. \tag{3}$$

Now the function f, given by $f(h) = 0/h$, is 0 for all $h \neq 0$, and is undefined when $h = 0$. We may therefore not allow $h = 0$ in $f(h)$. However, if $g(h) = 0$ for all h, then $f(h) = g(h)$ for all $h \neq 0$, and, of course, $\lim_{h \to 0} g(h) = 0$. Hence, by the theorem, $\lim_{h \to 0} f(h) = 0$.

Effectively, we have evaluated $\lim_{h \to 0} f(h)$ by finding $g(0)$—that is, setting $h = 0$ as input to the function g. We can evaluate $\lim_{h \to 0} g(h)$ in this way because we know that g is continuous at the origin. This can be a tricky point for beginners—who may not yet have studied continuity. It is probably therefore better to appeal to one of the standard theorems regarding the algebra of limits: the limit of a constant function is the constant. (In our earlier example, we need also to appeal to the theorems that tell us that the limit of a sum is the sum of the limits and that $\lim_{x \to a} x = a$, rather than appeal to the continuity of the function $h \mapsto h + 4$.)

The confusing issue here is that we often are tempted to tell our students that we never let h be 0 when we evaluate $\lim_{h \to 0} f(h)$. This is not quite correct. In the example above, we *do* let h be 0—*but not as an input to the original function* f. We let h be zero only when we are thinking about the function g, which we have replaced f with. This is a difficult point; Newton, Leibniz, and Bishop Berkeley all missed it. That we understand it today doesn't mean that we're smarter or better than they were; only that we have had the privilege of examining centuries of thinking that had not yet been done in their time. Among the things that we have access to, and that they did not, are the modern definition of *limit* and, especially, the modern definition of *function*. Without a clear understanding of the latter concept, it is very difficult to grasp reasoning that ultimately depends upon the fact that the function $f : h \mapsto 0/h$ isn't the same function as $g : h \mapsto 0$.