On Functions "Increasing at a Point"

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This one seems to arise every April:

> "$y = x^2$ is increasing on $(0, 1)$ but not on $[0, 1]$ since it is not increasing at $x = 0$."

The notion of a function that is "increasing at $x = 0$" is an extraordinarily difficult one. Any definition of this notion must necessarily be a local one. Most books that attempt it make the definition of increasing at a point simply take it to be positivity of the derivative at the point in question. There are important deficiencies in that definition, as I will show shortly.

The standard definition of "increasing" is global—not local: f is increasing on the interval I if $f(u) < f(v)$ whenever u and v are points of I for which $u < v$. We can establish a very nice theorem connecting the positivity of the derivative of f on I with its increasingness there. We usually do so in our elementary calculus courses, and that theorem (and its cousins) are on the AP syllabus. But the reason that this theorem is of mathematical interest is that it connects an essentially *local* hypothesis ($f'(x) > 0$ at each point of I) with a *global* conclusion (f is increasing on the interval I). We can establish this connection using any of several devices; most people choose the Mean Value Theorem, but there are others.

One reason for not introducing the notion of functions increasing at a point is the loss of this mathematically interesting step from the local to the global. If we simply *define* the increasing property by the positivity of the derivative, we are in effect ignoring this interesting connection.

But there are deeper reasons. Whatever the local notion of increasing may be, we should certainly like it to have some connection with the standard global notion. The obvious, and very desirable, connection would be this: If f is increasing at $x = 0$, then f is increasing on some open interval centered at $x = 0$.

It is an unfortunate fact that there is no such connection.

Consider the function f given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + \frac{x}{2} & ; \quad x \neq 0; \\ 0 & ; \quad x = 0. \end{cases} \tag{1}$$

We have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} \tag{2}$$

$$= \lim_{h\to 0} \frac{[h^2 \sin(1/h) + (h/2)] - 0}{h} \tag{3}$$

$$= \lim_{h\to 0} [h \sin(1/h) + (1/2)] \tag{4}$$

$$= 1/2, \tag{5}$$

because

$$|h \sin(1/h)| \leq |h| \to 0 \tag{6}$$

as $h \to 0$. Hence $f'(0) = 1/2 > 0$.

But when $x \neq 0$, we have

$$f'(x) = 2x \sin(1/x) - \cos(1/x) + 1/2. \tag{7}$$

Consequently, $f'(x) = -1/2 < 0$ whenever $x = 1/(2k\pi)$ for any non-zero integer k, while $f'(x) = 3/2 > 0$ whenever $x = 1/[(2k+1)\pi]$ for any integer k. Thus, f is neither increasing nor decreasing on any open interval centered at 0. This is because f' is continuous at every $x \neq 0$, and so every one of the points $x = 1/(2k\pi)$ is the center of a small interval where f' is negative, while every point $x = 1/[(2k+1)\pi]$ is the center of a small interval where f' is positive. Every interval centered at zero contains points of both kinds, and so contains small intervals on which f is decreasing at the same time that it contains other small intervals on which f is increasing.

There is another difficulty. We probably shouldn't extend the global definition given above in order to acquire a definition for the phrase "increasing at a point". The global definition implicitly assumes that the interval I is non-degenerate, and eliminating this assumption gets us into trouble. For if we allow the definition "f is increasing on the interval I if $f(u) < f(v)$ whenever u and v are points of I for which $u < v$" to apply to an interval of the form $[a, a]$, we find that *every* function is increasing on $[a, a]$. After all, there are no pairs, u, v, with $u < v$ in $[a, a]$, and so $f(u) < f(v)$ *for every such pair*, no matter what f is. (Even if the domain of f doesn't include a.) Notice that every function f is also decreasing on $[a, a]$ under this convention. We probably don't want that to happen.

One workable idea is this: Agree that f *is increasing through* a if the domain of f includes an interval $(a-\delta, a+\delta)$ for some $\delta > 0$ and if $a-\delta < u < a < v < a+\delta$ implies that $f(u) < f(a) < f(v)$. (I want to avoid "increasing at a" because of potential confusion it might cause.)

Now suppose that we are given a function f whose domain contains an interval centered at a, and that $f'(a) > 0$. Choose δ so that $[f(a+h) - f(a)]/h > f'(a)/2$ whenever $|h| < \delta$, and suppose that $a-\delta < u < a < v < a+\delta$. Taking $h = u - a$, we have $[f(u) - f(a)]/(u-a) = [f(a+h) - f(a)]/h > 0$, and because $h = u - a < 0$, this means that $f(u) < f(a)$. Similarly, $f(a) < f(v)$, and it follows that f is increasing through a.

Some of the subscribers to the AP Calculus List (Doug Kuhlman in particular) like this idea. I'm not too fond of it myself—at least partly because I don't believe I've ever seen it in print anywhere. It is not a definition that's widely accepted by the mathematical community, and so I think it's not an idea we should introduce in our elementary calculus courses. I might change my mind if I ever saw the definition in a major elementary calculus text.