A Question About Increasing Functions

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The question

> "Does anyone have a logical explanation of why we say a function is increasing or decreasing on a closed interval? How can we say a function is increasing or decreasing at an x value which makes the derivative equal zero?"

is a perennial favorite on the Advanced Placement Calculus listserve. From a logical point of view, the question is very much like asking "How can we say that a 90°–45°–45° triangle is isosceles, when it isn't equilateral?" After all, we know that every equilateral triangle is isoceles—just as we know that every function which has a positive derivative on an interval is increasing on that interval. The answer to the question about triangles is clear: Being equilateral isn't the criterion that *determines* whether or not a triangle is isosceles; being equilateral is merely a condition that happens to *imply* equality of at least two sides—and it is the latter condition that determines the isosceles property. Having at least two equal sides is, in fact, the definition of "isosceles", and the criterion given by the definition is always the one that *determines* whether or not an object has a particular property.

So it is with monotonicity. Having a positive derivative on an interval isn't the criterion that *determines* whether or not a function is increasing on that interval; it's merely a condition that happens to *imply* monotonicity. The determining condition is, as always, the definition that we adopt for the term in question. Here is a generally accepted definition for the phrase "f is increasing on the interval I". (Some authors insist on the term "strictly increasing", and use "increasing" for what we might call "non-decreasing" functions. We will not address such quibbles beyond noting that they exist.)

Definition: *A function f is said to be* increasing *on an interval I when $f(u) < f(v)$ for every pair u, v of points in I for which $u < v$.*

The reader should note that this definition is for *intervals*, which it implicitly takes to be non-degenerate. If we allow degenerate intervals, of the form $[a, a]$, we find ourselves in a rather unsatisfactory situation in which *every* function defined on such an interval is both increasing *and* decreasing. After all, the statement "$f(u) < f(v)$" is true whenever u, v is a pair of points in $[a, a]$ for which $u < v$. *There are no such pairs,* so $f(u) < f(v)$ is true for every one of them. And so is the statement "$f(v) > f(u)$", which makes f a decreasing function. We therefore had better not talk about functions which are "increasing at a point" unless we somehow take into account the behavior of f in some neighborhood of the point in question. But that's another discussion. For the rest of this discussion, we will assume that our intervals are non-degenerate.

Let us illustrate how we can apply this definition directly—without any mention of derivatives—to show that the function given by $g(x) = x^3$ is increasing on any interval I whatsoever. To this end, let $u < v$ be two points of I. Then

$$\begin{aligned} u^3 - v^3 &= (u-v)(u^2+uv+v^2) && (1) \\ &= (u-v)\left[\left(u^2+uv+\frac{v^2}{4}\right)+\frac{3v^2}{4}\right] && (2) \\ &= (u-v)\left[\left(u+\frac{v}{2}\right)^2+\left(\frac{\sqrt{3}}{2}v\right)^2\right] && (3) \end{aligned}$$

Now $u < v$, so $u - v < 0$. The quantity in the square brackets on the right side of (3) is positive because it is the sum of two squares which can't both be zero. Thus, the product on the right side of (3) is negative, and from this it follows that $u^3 < v^3$. Our appeal to the definition shows that the cubing function is increasing on I, where I may be any interval at all.

In particular, we may take I to be $[0, 1]$. Thus, the function given by $g(x) = x^3$ is increasing on $[0, 1]$—in spite of the fact that $g'(x) = 3x^2$ is zero when $x = 0$. In fact, the function g is increasing on $[-1, 1]$ as well, and it doesn't matter that the derivative vanishes at $x = 0$. The fact that the derivative happens to vanish once in a while just isn't relevant to the decision as to whether or not the function is increasing; what is relevant is the fact that no matter which u and v we choose with $u < v$, we always find that $u^3 < v^3$.

Let us go a little bit further. We can show that if f is a function given continuous on a *closed* interval $[a, b]$ and increasing in the *open* interval (a, b), then f *must* be increasing on the closed interval $[a, b]$. Let us show first that f is increasing on $[a, b)$. In order to accomplish this, we must verify that if u and v are points of $[a, b)$ for which $u < v$, then it follows that $f(u) < f(v)$.

Proceeding by contradiction, we suppose that f is *not* increasing on $[a, b)$. Then we can find points u, v in $[a, b)$, with $u < v$ and $f(u) \geq f(v)$. Because f was given as increasing on (a, b), $u \neq a$ is impossible. Thus, there is $v \in (a, b)$ such that $f(a) \geq f(v)$.

Let $t = (a + v)/2$, so that $a < t < v$. Because f is increasing on (a, b), we know that $f(t) < f(v) \leq f(a)$. Put $s = (a + t)/2$. Then $a < s < t$, so $f(s) < f(t) < f(v) \leq f(a)$.

But then $f(t)$ is a number that lies strictly between $f(s)$ and $f(a)$, and f is continuous on $[a, s]$. So by the intermediate value property of continuous functions there is a number w strictly between a and s such that $f(w) = f(t)$. This can't be, because we now have $w < s$, but $f(w) = f(t) > f(s)$—which violates the fact that f is increasing on (a, b). We conclude that f must be increasing on $[a, b)$.

A similar argument then shows that f must also be increasing on $[a, b]$.

Notice that f' didn't enter the picture here; indeed, we don't even know if f' exists anywhere in $[a, b]$.