More on Improper Integrals

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Here is a frequently asked question about improper integrals:

> When we teach improper integrals in elementary calculus, why don't we take
>
> $$\int_{-1}^{1} \frac{2x}{1-x^2}\,dx = \lim_{t\to 1^-} \int_{-t}^{t} \frac{2x}{1-x^2}\,dx \tag{1}$$
>
> $$= -\lim_{t\to 1^-} \ln(1-x^2)\Big|_{-t}^{t} \tag{2}$$
>
> $$= -\lim_{t\to 1^-} \{\ln(1-t^2) - \ln[1-(-t)^2]\} = 0 \tag{3}$$
>
> for the definition of this improper integral (and others like it)? It would make life so much easier for our students.

The calculation (1)–(3) gives us something called the "Cauchy Principal Value" (CPV) of the improper integral. The CPV is frequently written

$$\mathsf{PV}\int_{-1}^{1} \frac{2x}{1-x^2}\,dx,$$

and it does find application in mathematics more advanced than elementary calculus. But there are very good reasons not to introduce this notion in elementary calculus. Here's one.

Let P be the polynomial function given by

$$P(u) = (u-1)(u+1)(u+3)(u+5) \tag{4}$$

$$= u^4 + 8u^3 + 14u^2 - 8u - 15, \tag{5}$$

so that

$$P'(u) = 4u^3 + 24u^2 + 28u - 8. \tag{6}$$

We also put

$$f(u) = -\frac{P'(u)}{P(u)}. \tag{7}$$

Then, taking $|u| < 1$ (so that $P(u)$ does not vanish) and putting

$$F(u) = \int f(u)\, du = -\ln|P(u)| \tag{8}$$

$$= -\ln(1-u) - \ln(1+u) - \ln(3+u) - \ln(5+u), \tag{9}$$

we easily calculate that

$$\mathsf{PV}\int_{-1}^{1} f(u)\, du = \lim_{t\to 1^-}\left[F(u)\Big|_{-t}^{t} \right] \tag{10}$$

$$= \lim_{t\to 1^-}\, [F(t) - F(-t)] \tag{11}$$

$$= \lim_{t\to 1^-}\left(\ln\left[\frac{(3-t)(5-t)}{(3+t)(5+t)}\right]\right) = -\ln 3. \tag{12}$$

With this preliminary calculation out of the way, we turn our attention to the improper integral with which we began this discussion: $\int_{-1}^{1}[2x/(1-x^2)]\, dx$. As we have seen, the CPV of this integral is 0.

But now let us make the substitution $4x = u^2+4u-1$ in this integral. The Substitution Theorem for Definite Integrals then tells us that we must then transform the integral $\int_{-1}^{1}[2x/(1-x^2)]\, dx$ according to:

$$x = \frac{u^2+4u-1}{4}; \tag{13}$$

$$dx = \frac{u+2}{2}\, du; \tag{14}$$

$$x = -1 \Rightarrow u = -1; \tag{15}$$

$$x = 1 \Rightarrow u = 1. \tag{16}$$

We thus obtain

$$\int_{-1}^{1} \frac{2x}{1-x^2}\, dx = \int_{-1}^{1} x \cdot \frac{1}{1-x^2} \cdot 2\, dx \tag{17}$$

$$= \int_{-1}^{1} \frac{u^2+4u-1}{4} \cdot \frac{16}{16-(u^2+4u-1)^2} \cdot (u+2)\, du \tag{18}$$

$$= -\int_{-1}^{1} \frac{4u^3+24u^2+28u-8}{u^4+8u^3+14u^2-8u-15}\, du. \tag{19}$$

Comparing numerator and denominator of this latter integral with (5) and (6), we find that our transformation has produced the integral whose CPV we found in the calculation (10)–(12)—and which is $-\ln 3$.

We are forced to the conclusion that *adopting (1) as the definition of the improper integral*

$$\int_{-1}^{1} \frac{2x\,dx}{1-x^2}$$

breaks the Substitution Theorem for Definite Integrals. That theorem is too valuable to give up (in elementary calculus), so we adopt the more complicated definition—which doesn't break the theorem.

Remark: There is a whole family of substitutions similar to the one we chose. Let B be any real number for which $|B| > 1$, and take

$$\begin{aligned} P_B(u) &= (u-1)(u+1)(u+2B-1)(u+2B+1); \\ f_B(u) &= -\frac{P_B'(u)}{P_B(u)}. \end{aligned}$$

We can use any of the functions P_B and f_B (with the same B, of course) in the rôles of P and f in the calculation we have just examined—where we took $B = 2$. We find that

$$\mathsf{PV}\int_{-1}^{1} f_B(u)\,du = \ln\frac{B-1}{B+1}, \tag{20}$$

and it is worth noting that the only real value that this latter expression cannot take on is the value 0.

The rest of the calculation is left to the reader.