Concerning $\int \frac{dx}{x}$

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Feb. 11, 2006

Brian Usselman asks:

> Why is absolute value needed when doing indefinite integration of $1/x$? Thus the answer is $\ln|x| + C$.
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> I am having a difficult time explaining why absolute value is needed with indefinite integration but not need with definite integration.
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> Is it a domain issue?

The trouble here arises from our habitual, and sometimes unthinking, inclination to give "the most general anti-derivative" for $\int f(x)\,dx$ whenever such a problem is put to us. Ironically, we fail in the case of $\int \frac{dx}{x}$.

The expression $\ln|x| + C$ is usually offered for the anti-derivative of $\frac{1}{x}$ because we observe that when we know that $0 < x$, then we also have

$$\frac{d}{dx}\ln x = \frac{1}{x}, \tag{1}$$

while when we know that $x < 0$ we have

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}. \tag{2}$$

Now, on account of the domain restrictions, the quantities whose derivatives we are taking on the left sides left-hand sides of these two equations are both $\ln|x|$, so we conclude that we may abbreviate the two statements (1) and (2) as one:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}. \tag{3}$$

at least as long as both sides of this equation are meaningful.

So far, so good.

But we are in the habit of reversing differentiation formulae in order to obtain integration formulae: Given that $F'(x) = f(x)$, we like to say that $\int f(x)\,dx = F(x) + C$, which we interpret as meaning "$F(x) + C$ is the most general anti-derivative for $f(x)$." But this interpretation is not always right. If f is a function that has a singularity somewhere in the interval we are discussing, then something else happens. And $x \mapsto 1/x$ *is* a function which has a singularity in the interval where the standard integration formula is supposed to hold.

If we were to restrict the domain of $x \mapsto 1/x$ to $x > 0$, the statement

$$\int \frac{dx}{x} = \ln|x| + C \tag{4}$$

would indeed give the most general anti-derivative for the function in question. If we were to restrict the domain to $x < 0$, the statement (4) would also give the most general anti-derivative for *that* function.

But—on account of the singularity at $x = 0$—we must be more careful in dealing with the function $x \mapsto \dfrac{1}{x}$ if we have in mind any domain that contains both positive numbers and negative numbers. On such a domain, we ought to write

$$\int \frac{dx}{x} = \begin{cases} \ln|x| + C_1 & \text{if } x > 0, \\ \ln|x| + C_2 & \text{if } x < 0, \end{cases} \tag{5}$$

or, perhaps better,

$$\int \frac{dx}{x} = \begin{cases} \ln x + C_1 & \text{if } x > 0, \\ \ln(-x) + C_2 & \text{if } x < 0, \end{cases} \tag{6}$$

if we *really* want the most general anti-derivative.

It is not quite correct to say that the absolute value is not required when we write out the antiderivative we plan to use in evaluating the definite integral $\displaystyle\int_a^b \frac{dx}{x}$. (But note: If this integral is to be meaningful, both a and b have to have the same sign.) For example, $\displaystyle\int_{-1}^{-2} \frac{dx}{x}$ is an integral where it would be wrong to omit the absolute value from the antiderivative (unless we choose (6) when we find the antiderivative)—because omitting the absolute value leads us to the expression $\ln(-2) - \ln(-1)$. In the context of freshman calculus, this expression is meaningless (though, given an appropriate choice of branch, it is perfectly correct in complex analysis).