Asymptotes[1]

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What is an asymptote? There is considerable disagreement on the subject. Some elementary calculus textbooks, (*e.g.,* Foerster's book and Leithold's book) do not allow curves to cross their asymptotes in every neighborhood of ∞. Thus, they deny that the curve $y = (\sin x)/x$ can have the x-axis as a horizontal asymptote because the curve crosses the x-axis for arbitrarily large values of x.

If the standard calculus books that happened to sit on my shelf at the moment I started this note are any indication, then Foerster and Leithold are distinctly in the minority. Stewart (*Calculus*, 3rd Edition, Brookes/Cole, 1995, pg. 208), Swokowski (*Calculus with Analytic Geometry*, Alternate Edition, Prindle, Weber & Schmidt, 1983, pg. 171), Thomas & Finney (*Calculus with Analytic Geometry*, 9th Edition, Addison-Wesley, 1996, pg. 224), Edwards & Penney (*Calculus with Analytic Geometry*, 5th Edition, Prentice-Hall, 1998, pg. 254), and Larson, Hofstetler & Edwards (*Calculus (Early Transcendental Functions)*, Houghton Mifflin Co., 1999, pg. 228) are the books in question, and all agree that a horizontal line $y = L$ is an asymptote for the function $y = f[x]$ if $\lim_{x\to\infty} f[x] = L$.

James & James (*Mathematics Dictionary*, Multilingual Edition, D. Van Nostrand Co., Inc., 1959, p.22) say that an asymptote is "A line such that a point, tracing a given curve and simultaneously receding to an infinite distance from the origin, approaches indefinitely near to the line; a line such that the perpendicular distance from a moving point on a curve to the line approaches zero as the point moves off an infinite distance from the origin. *Tech.* An asymptote is a tangent at infinity, *i.e.*, a line tangent to (touching) the curve at an ideal point." However, the fourth edition (1976, pg. 21) and the fifth edition (1992, p. 22) of the same work say "For a plane curve, an asymptote is a line which has the property that the distance from a point P on the curve to the line approaches zero as the distance from P to the origin increases without bound and P is on a suitable piece of the curve. Often it is required that the curve not oscillate about the line." This is a shame,

[1]Some time after I wrote this note, David Renfro pointed out that much of what I have to say had appeared in P. J. Giblin, "What is an asymptote," *The Mathematical Gazette,* **56**(1972), 274–284.

because the "tangent at infinity" idea, which somehow got lost between 1959 and 1976 and stayed lost until at least 1992, has great merit.

Even after we accept the "tangent at infinity" idea, the issue is still somewhat clouded, for there is no standard definition for this phrase. Let us adopt the perspective of projective geometry, which adjoins to the Euclidean plane an ideal point for every family of parallel lines—it being supposed that every member of a given such family passes through the ideal point associated with it. The collection of all such ideal points constitutes the "line at infinity"; it behaves, in projective geometry, just like all other lines.

One way to effect this extension of the Euclidean plane is by introducing homogeneous coordinates for the projective plane. We identify the points of the projective plane with equivalence classes $[u_1, u_2, u_3]$ of ordered triples of real numbers, not all zero, where two ordered triples (u_1, u_2, u_3) and (v_1, v_2, v_3) are considered equivalent if there is a non-zero real number k such that $(u_1, u_2, u_3) = (kv_1, kv_2, kv_3)$. Thus, if at least one of the numbers u_1, u_2, u_3 is non-zero, the equivalence class $[u_1, u_2, u_3]$ of the triple (u_1, u_2, u_3) is given by

$$[u_1, u_2, u_3] \;=\; \{\, (ku_1, ku_2, ku_3) : k \in \mathbb{R} \text{ and } k \neq 0 \,\}. \tag{1}$$

The *real projective plane*, $\mathbb{PR}^2$, consists of all such equivalence classes:

$$\mathbb{PR}^2 \;=\; \{\, [u_1, u_2, u_3] : u_1, u_2, u_3 \in \mathbb{R} \,\} \tag{2}$$

A line in $\mathbb{PR}^2$ is then any set of points $[u_1, u_2, u_3]$ such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \;=\; 0, \tag{3}$$

where α_1, α_2, and α_3 are fixed real numbers, not all zero. Indeed, if a projective point $[u_1, u_2, u_3] \in \mathbb{PR}^2$ satisfies equation (3) for such a triple $(\alpha_1, \alpha_2, \alpha_3)$, and κ is any non-zero real number, then $[u_1, u_2, u_3]$ also satisfies the equation

$$\kappa\alpha_1 u_1 + \kappa\alpha_2 u_2 + \kappa\alpha_3 u_3 \;=\; 0, \tag{4}$$

so that the lines in $\mathbb{PR}^2$ can themselves be represented by homogeneous equivalence classes of the form

$$[\alpha_1, \alpha_2, \alpha_3]^{tr} \;=\; \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{5}$$

The reader should note that equation (4) guarantees that our definition of projective lines, which we phrased in terms of single representatives of the homogeneous equivalence classes that represent points, doesn't depend on the choices we make of those representatives. This issue arises again and again, but the arguments that resolve the difficulties are all very simple. Henceforth we will treat the issue with benign neglect.

It is now natural to write equation (3) as

$$[u_1, u_2, u_3][\alpha_1, \alpha_2, \alpha_3]^{tr} \;=\; 0, \tag{6}$$

the multiplication on the left being matrix multiplication of the underlying three-dimensional vectors.

The map $I : (x, y) \mapsto [x, y, 1]$ gives an embedding of $\mathbb{R}^2$ into $\mathbb{PR}^2$. If $[u_1, u_2, u_3]$ is any point of $\mathbb{PR}^2$ for which $u_3 \neq 0$, then $[u_1, u_2, u_3]$ lies in the image of I, for then

$$[u_1, u_2, u_3] \;=\; \left[\frac{u_1}{u_3}, \frac{u_2}{u_3}, 1\right] \tag{7}$$

$$\;=\; I\left[\left(\frac{u_1}{u_3}, \frac{u_2}{u_3}, 1\right)\right]. \tag{8}$$

Consequently, the complement in $\mathbb{PR}^2$ of $I[\mathbb{R}^2]$, the image under I of the Euclidean plane, consists of the set of all projective points of the form $[x, y, 0]$, where x and y are not both zero. Such points are points of "the line at infinity", and we may legitimately think of them as "ideal points" that have been adjoined to the Euclidean plane in such a way that each such point is the single point common to a family of all lines parallel to one particular line. The "line at infinity" is thus the projective line whose equation is $u_3 = 0$. This is the projective line which corresponds to the homogeneous equivalence class $[0, 0, 1]^{tr}$.

Consider now a line in the Euclidean plane whose equation is

$$ax + by + c \;=\; 0. \tag{9}$$

Under the embedding I, the points (x, y) that lie on this line become the points $[x, y, 1]$ of $\mathbb{PR}^2$. Hence, the image under I of the Euclidean line given by equation (9) is a subset of the line given by the homogeneous equivalence class $[a, b, c]^{tr}$. In particular, the image under I of the Euclidean line whose slope-intercept equation is $y = mx + b$ is a subset of the line given by $[m, -1, b]^{tr}$

and the image of the vertical line $x = h$ is a subset of the line given by $[1, 0, h]^{tr}$.

We want to consider the projective transformation

$$T \;:\; [u_1, u_2, u_3] \mapsto [u_3, u_2, u_1]. \tag{10}$$

It is clear that T carries the family of all lines in $\mathbb{PR}^2$ onto itself. Indeed, the image of the line given by $[a_1, a_2, a_3]^{tr}$ is the line given by $[a_3, a_2, a_1]^{tr}$. We note that T exchanges almost all of the points of the "line at infinity" with those of the image of the y-axis under the embedding I. (Only the point $[0, w, 0]$—where, of course, $w \neq 0$—remains on the "line at infinity".)

Finally, we observe that T is its own inverse mapping: $T^2 = \mathrm{id}_{\mathbb{PR}^2}$.

Let us now consider the graph $\{\,(x, f[x]) : x \in \mathbb{R}\,\}$ of a function $f : \mathbb{R} \to \mathbb{R}$ in the Euclidean plane. The image of a point $(x, f[x])$ of this graph under the embedding I is the homogeneous equivalence class $[x, f[x], 1]$. We apply the transformation T to this image. If we rule out the possibility that $x = 0$ (a possibility that is of no interest to us), the result is

$$\begin{aligned} T[I[(x, f[x])]] &= T[[x, f[x], 1]] && (11)\\ &= [1, f[x], x] && (12)\\ &= [x^{-1}, f[x]x^{-1}, 1] && (13)\\ &= [t, tf[t^{-1}], 1], && (14) \end{aligned}$$

where $t = x^{-1} \neq 0$. Noting that the $[t, tf[t^{-1}], 1]$ lies in the image of $\mathbb{R}^2$ under I, we apply I^{-1}. We obtain

$$I^{-1}[T[I[(x, f[x])]]] \;=\; (t, tf[t^{-1}]), \tag{15}$$

where $t = x^{-1}$.

This analysis suggests that one way to investigate the behavior of $f[x]$ as $x \to \infty$ is to investigate, instead, the behavior of the function

$$F[t] \;=\; tf[t^{-1}] \tag{16}$$

as $t \to 0$. The analysis also makes it clear that we must now carefully distinguish between three cases:

1. $t \to 0$, (*i.e.*, $x \to \infty$),
2. $t \to 0^+$ ($x \to +\infty$), and

3. $t \to 0^-$ ($x \to -\infty$).

We will confine our attention to the first of these cases, leaving the other two to the reader. In the rest of this note, we will therefore interpret the statement $\lim_{u\to\infty} g[u] = L$ to mean that no matter what $\epsilon > 0$ may be given, there is a corresponding real number $M > 0$ such that $|g[u] - L| < \epsilon$ whenever $M < |u|$. We emphasize that in the sequel *we consider two-sided neighborhoods of* ∞ *only.*

As we have seen, the question of a "tangent at ∞" for the function f can reasonably be interpreted as being equivalent to that of a tangent at $t = 0$ for the function $F : t \mapsto tf[t^{-1}]$. We have not been given a meaning for $F[0]$ (because we have no meaning for $f[0^{-1}]$), but there is a standard way of deciding upon what $F[0]$ should mean—especially if we seek a line tangent to the curve $s = F[t]$ at $t = 0$. We must ask for F to have a continuous extension to $t = 0$; moreover, that continuous extension must be differentiable at the origin. Thus, we ask that there be a real number m such that

$$\lim_{t\to 0} F[t] \;=\; m. \tag{17}$$

We define $F[0] = m$, and require, in addition, that there be another real number b such that

$$\begin{aligned} b &= \lim_{h\to 0} \frac{F[h] - m}{h} && (18)\\ &= \lim_{h\to 0} \frac{F[0+h] - F[0]}{h}. && (19) \end{aligned}$$

When these conditions are met, it is customary to agree that the line tangent to the curve $s = F[t]$ at $t = 0$ is the line whose equation is $s = bt + m$.

Now we reverse the manipulations that got us from f to F. We need only note that $(I^{-1}TI)^{-1} = I^{-1}T^{-1}I = I^{-1}TI$. The image under I of the line $s = bt+m$ is the projective line $[b, -1, m]^{tr}$, and T carries this to $[m, -1, b]^{tr}$—which is the image under I of the Euclidean line whose equation is $y = mx+b$. This line is the "tangent at ∞" for the curve $y = f[x]$, and it is the line that we should call "the asymptote of the curve $y = f[x]$".

It remains to translate the conditions (17) and (18), which are conditions on F, into conditions on f. The key to this translation is, of course, equation (16).

In light of (16), condition (17) becomes

$$\lim_{t\to 0} tf[t^{-1}] \;=\; m, \tag{20}$$

or, substituting $x = t^{-1}$ and noting that $t \to 0$ is equivalent to $x \to \infty$,

$$\lim_{x\to\infty} \frac{f[x]}{x} \;=\; m. \tag{21}$$

This condition may be rewritten as

$$\lim_{x\to\infty} \left(\frac{f[x]}{x} - m\right) \;=\; 0, \tag{22}$$

or, finally, as

$$\lim_{x\to\infty} \left(\frac{f[x] - mx}{x}\right) \;=\; 0. \tag{23}$$

A function f for which there is a real number m satisfying condition (23) is said to be *asymptotically linear*, and the number m is called the *asymptotic derivative* of f. The condition is one that arises, for example, in the study of positive non-linear operators in ordered vector spaces. (See, *e.g.*, [M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd., Groningen, The Netherlands (1964)] or [Robert H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley & Sons, New York (1976)].)

As to condition (18), application of (16) yields

$$\lim_{h\to 0} \frac{hf[h^{-1}] - m}{h} \;=\; b, \tag{24}$$

which we can rewrite as

$$\lim_{x\to\infty} (f[x] - mx) \;=\; b. \tag{25}$$

This, in turn, may be rewritten as

$$\lim_{x\to\infty} [f[x] - (mx + b)] \;=\; 0. \tag{26}$$

Condition (26) is the condition that most of us have been taught is the condition that the line $y = mx + b$ must satisfy in order to be considered an asymptote to the curve $y = f[x]$. As we have seen, it arises out of the *tangency* condition that we have imposed on the auxiliary function F.

What about condition (23)? Well, let us suppose that f is a function for which there are numbers m and b such that (26), and therefore condition (25), holds. Let $\epsilon > 0$ be given. Find M so that

1. $M < |x|$ implies that $|f[x] - (mx + b)| < \frac{\epsilon}{2}$, and

2. $\min\left\{1, \frac{2|b|}{\epsilon}\right\} < M$.

Suppose that $M < |x|$. Then

$$\left|\frac{f[x] - mx}{x}\right| = \left|\frac{f[x] - mx - b}{x} + \frac{b}{x}\right| \tag{27}$$

$$\leq \left|\frac{f[x] - mx - b}{x}\right| + \left|\frac{b}{x}\right| \tag{28}$$

But $1 < M < |x|$, so

$$\left|\frac{f[x] - mx - b}{x}\right| < |f[x] - (mx + b)| \tag{29}$$

$$< \frac{\epsilon}{2}, \tag{30}$$

and $\frac{2|b|}{\epsilon} < M < |x|$, so that

$$\left|\frac{b}{x}\right| = |b| \cdot \frac{1}{|x|} \tag{31}$$

$$< |b| \cdot \frac{\epsilon}{2|b|} = \frac{\epsilon}{2}. \tag{32}$$

It follows now that condition (23) is satisfied. We see from this that condition (23) is actually a *consequence* of the existence of m and b satisfying condition (26). This should come as no surprise, because, after all, (23) is the requirement that the function F be *continuous* at the origin, while (26) is the requirement that F be *differentiable* there. It is well known that differentiability implies continuity.

It is also well known that continuity does not imply differentiability. In our context, this should mean that there are curves $y = f[x]$ for which condition (23) holds but condition (26) does not. In fact, this is the case, and $y = \sin x$ is such a curve, for taking $m = 0$, we have

$$\begin{aligned} \lim_{x\to\infty} \left(\frac{f[x] - mx}{x}\right) &= \lim_{x\to\infty} \left(\frac{\sin x}{x}\right) && (33) \\ &= 0. && (34) \end{aligned}$$

Let us remark that there is no argument about whether or not a curve $s = F[t]$ may cross its tangent line infinitely many times in every neighborhood of $t = 0$. If $F'[0]$ exists, then the tangent line is, by definition, the line whose equation is $s = F[0] + F'[0]t$; this definition is quite independent of all other behavior of the curve. We argue on this basis that whether condition (26) holds should be the only criterion for deciding whether the line whose equation is $y = mx + b$ is an asymptote for the curve $y = f[x]$.