# Implicit Differentiation

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Questions regarding implicit differentiation come to the AP Calculus list in a variety of flavors. Most of them reflect confusion that is almost certainly due to the poor job that mainstream elementary calculus textbooks do with this topic. Typical problems are mechanical exercises of the form "Here is a relation $F(x, y) = 0$; find $y'$." Such problems completely ignore several issues: What does the equation $F(x, y) = 0$ mean? What does $y'$ mean under such circumstances? What does it mean to find $y'$ under these circumstances? Is there any reason to believe that we can actually do so?

There are certainly other issues—but they are not appropriate for most students at this level. I am not about to suggest that we teach beginning calculus students the Implicit Function Theorem. But that theorem *is* the foundation upon which the matter rests, so it ought to occupy a prominent place in *our* thinking when we deal with implicit differentiation in our beginning courses. Here is a statement of that theorem; the symbols $F_1$ and $F_2$ denote partial derivatives:

**Theorem:** *Let $F$ be a function that possesses partial derivatives continuous in some neighborhood of some solution, $(x_0, y_0)$, of the equation $F(x, y) = 0$. If $F_2(x_0, y_0) \neq 0$, there are an $\epsilon > 0$ and a unique continuously differentiable function $\varphi$ such that $\varphi(x_0) = y_0$ and $F[x, \varphi(x)] = 0$ for $|x - x_0| < \epsilon$. Moreover, when $|x - x_0| < \epsilon$, we have*

$$\varphi'(x) = -\frac{F_1[x, \varphi(x)]}{F_2[x, \varphi(x)]}.$$

The reader who wants to see a proof of this theorem can consult David V. Widder's *Advanced Calculus*, originally published by Prentice-Hall in 1947, but more recently reissued in a cost-effective paperback by Dover.

Note that, as problems in freshman calculus are all too frequently cast, the point $(x_0, y_0)$ is missing. The notion that we are finding $\varphi'(x)$ for a certain (locally defined!) function for which $F[x, \varphi(x)] = 0$ is there only, ahem, implicitly. But the formal treatment that our textbooks suggest, and that we allow students to give these problems, has a tendency to remove it from their thinking (and ours, too?). Because of this formal treatment, students will happily process a problem like "Find $y'$ given that $x^2 + y^2 = -1$" to obtain the nonsensical answer $y' = -x/y$.

When I deal with this topic, I try to emphasize the local quality of what I am asking them about. I try to give problems in which the point $(x_0, y_0)$ appears explicitly, and I usually ask for $y'$ at or near the point in question in ways that will focus their attention of the underlying implicitly defined function. For example, I might ask them something like this: "Find an equation for the line tangent to the curve $x^3 - 4xy + 2y^3 = 2$ at the point $(2, 1)$. Use your result to estimate the value of $y$ for the point on the curve near $(2, 1)$ where $x = 1.997$."

## 1. Failure of the hypotheses

Here is a question that is very similar to one that appeared on the AP Calculus list in October, 2004 (the original questioner wanted to know about a curve of the same shape oriented somewhat differently):

> The following curve $\sqrt{2}x^3 + 3\sqrt{2}xy^2 - 9x^2 + 9y^2 = 0$ has two tangents at the origin. However, implicit differentiation doesn't seem to reveal this as $y'$ ends up giving you $0/0$ at the origin. What is going on here?

In this case, $F(x, y) = \sqrt{2}x^3 + 3\sqrt{2}xy^2 - 9x^2 + 9y^2$, so $F_2(x, y) = 6\sqrt{2}xy + 18y$. This is zero at $(0, 0)$. Consequently, the Implicit Function Theorem is inapplicable at $(0, 0)$. There could be several reasons underlying the failure of the theorem, however. It could be that the function $F$ does define $y$ as a differentiable function of $x$ near $(x_0, y_0)$, but that $F$ is just too convoluted for the implicit function theorem to apply; this is what happens when $G(x, y) = x^4 + 2x^2y^2 + y^4 - 50x^2 - 50y^2 + 625$ and we try to apply the implicit function theorem at $(3, 4)$ (discussion of this example begins on page 4 of this note). It could be that there is no derivative to be found; this is the case for $H(x, y) = y^3 - x^2$ at the origin. But the difficulty with the function $F$ is more subtle.

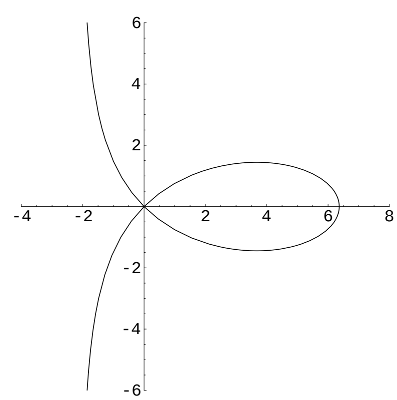


Figure 1: $\sqrt{2}x^3 + 3\sqrt{2}xy^2 - 9x^2 + 9y^2 = 0$

The graph of $F(x, y) = 0$ (see Figure 1) near the origin consists of two curves that cross at the origin, each having a different tangent line than the other. The implicit function theorem fails here because the equation $F(x, y) = 0$ doesn't have unique solutions for $y$ as a function of $x$ in any neighborhood of $(0, 0)$.

It might be hoped that we can find the slopes of the two curves passing through the origin from the quotient $-F_1(x, y)/F_2(x, y)$ through some kind of limiting process. This hope turns out to be vain. First, consider that

$$-\frac{F_1(x,y)}{F_2(x,y)} = -\frac{6x - \sqrt{2}x^2 - \sqrt{2}y^2}{6y + 2\sqrt{2}xy}$$

doesn't approach a limit as $(x, y) \to (0, 0)$. Try letting, say, $y = mx$, and see what happens as $x \to 0$. You will find that the limit depends upon $m$. Indeed, we see in retrospect, this *has* to be the case if the quotient is to be related to the slopes of tangent lines, because there are *different* tangents to the original, non-simple, curve at the origin. The limit, as $(x, y) \to (0, 0)$ of $-F_1(x, y)/F_2(x, y)$ does not exist in general, but is *curve-dependent.* This means that in order to find the value of the $y'(0)$ we seek, we must know the curve $y(x)$ explicitly—that is, we must actually solve $F(x, y) = 0$ for $y$ in terms of $x$ near $x = 0$, $y = 0$. In other words, we must find $\epsilon > 0$ and we must find a differentiable *function* $y = \varphi(x)$ that satisfies $F[x, \varphi(x)] = 0$ for $-\epsilon < x < \epsilon$, with $\varphi(0) = 0$. But, of course, the whole point of the implicit differentiation was that we didn't know how to do this very thing. And, in this example, there isn't a *unique* smooth solution curve $y = \varphi(x)$ passing through $(0, 0)$ in any interval $(-\epsilon, \epsilon)$, no matter how small $\epsilon > 0$ may be.

If we really must have the slopes of the tangents at such a crossing point, we will generally need to be able to find a useful parametrization of the curve in question. This example succumbs to the use of polar coordinates: We can explicitly solve the equation $F(r\cos\theta, r\sin\theta) = 0$ for $r$ in terms of $\theta$. Standard techniques then give the desired slopes.

Here is another example in which things are a little more subtle. This one appeared on the list in March, 2005:

> The problem in question is in the new AB version (D&S Marketing), test 6, #32. The problem asks for vertical tangents of $x^3 - y^2 + x^2 = 0$. Upon using implicit differentiation, you get $y' = (3x^2 + 2x)/(2y)$. Obviously, the only possibility of having a vertical tangent, means that $y = 0$. Substitute this back into the original relationship, [and this] yields $x = 0$ or $x = -1$. I say there is a vertical tangent at the point $(-1, 0)$ only. The book says there's also one at the origin. My reasoning is two-fold, one reason which is sure to raise some eyebrows. If $x$ and $y$ approach $0$ at the same rate, $y$ does not approach infinity (I don't think). And, for the reason many will not like ... I graphed the relation (this was the calculator portion) and there certainly doesn't appear to be a vertical tangent at the origin ... any help???

There is not a vertical tangent at $x = 0$. We can see this analytically by transforming to polar coordinates, which is how I produced the following picture:

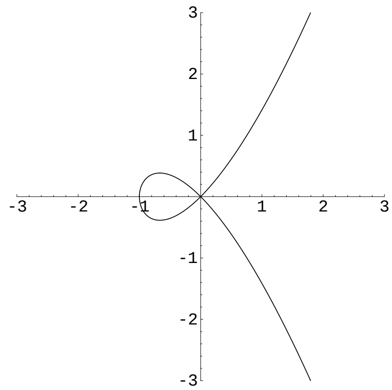

Figure 2: $x^3 - y^2 + x^2 = 0$

Use of implicit differentiation to find $dy/dx$ in a problem like this one is questionable *at best*, because that technique depends upon the implicit function theorem. One of the hypotheses for that theorem requires that the partial derivative of (in this case) $x^3 - y^2 + x^2$ taken with respect to $y$ be non-zero at a point of interest before we can draw any conclusions at all about the derivative of $y$ with respect to $x$ at such a point of interest. That hypothesis fails here if we set $y = 0$, as the problem invites us to do. We can therefore draw no conclusions from implicit differentiation about the existence of the derivative at points on this curve where $y = 0$. In particular, we may not conclude that $dy/dx$ fails to exist when $y = 0$.

This problem, however, naively assumes that this partial derivative's vanishing is tantamount to non-existence of $dy/dx$; this is incorrect reasoning. To see this consider the function $G(x, y) = x^4 + 2x^2y^2 + y^4 - 50x^2 - 50y^2 + 625$, and consider the curve $G(x, y) = 0$ at the point $(3, 4)$. (Or, indeed, at any point on the curve where $y$ is not 0.) Here we have $G_2(x, y) = 4y(x^2 + y^2 - 25)$, and this vanishes if $x^2 + y^2 = 25$—which, it turns out, is precisely the curve given by the equation $G(x, y) = 0$. The difficulty is the old one of confusing a statement with its converse. In this case, non-vanishing of the partial derivative insures existence of $dy/dx$; existence of $dy/dx$ does not insure non-vanishing of the partial.

In the $x^3 - y^2 + x^2 = 0$ problem, as the graph and the analysis suggested above both show, there are two non-vertical tangents to the curve at the origin. But we may not use the implicit function theorem, or its consequence, implicit differentiation, to find them.

The only way I can see to effect a partial rescue of this problem is to reverse the roles of $x$ and $y$. Use of implicit differentiation to find vertical tangents while thinking of $y$ as a function of $x$ *can't succeed in general*. The logic is flawed: $p \Rightarrow q$ just isn't equivalent to $q \Rightarrow p$, and any technique based upon that equivalence is bad mathematics. Even reversing the roles of $x$ and $y$, so that $y$ becomes the independent variable, doesn't allow us to determine what happens at the origin for this problem, because, again, the denominator vanishes—making application of implicit differentiation invalid.

## 2. Too many derivatives

Here is another question taken from the list (October, 2003):

> Can you simplify (or eliminate) fractions before finding derivatives?
>
> Consider the example: $4x^2y - \dfrac{3}{y} = 0$. The derivative implicitly is
>
> $$-\frac{8xy^3}{4x^2y^2+3}.$$
>
> If you multiply by $y$ first the derivative is $-y/x$.
>
> For $(xy - y^2)/(y - x) = 1$ the derivative is $(y + 1)/(-x + 2y + 1)$ however if you factor out $y - x$ and then cancel the derivative is $0$.

More recently (October, 2005), another poster asked:

> Two students solved this problem two different ways and obtained two different answers and I'm not sure why. The problem was to find $y'$ given $x^2 + y^2 = y^2/x^2$ The first student used the quotient rule on the right side and obtained the answer in the book $(x^4 + y^2)/(xy - x^3y)$. The second student multiplied both sides by $x^2$, and then differentiated and obtained $(2x^3 + xy^2)/(y - x^2y)$. I'm guessing multipling both sides by $x^2$ somehow changes the derivative.

These are nice examples of some of the confusion that can arise when we give too much attention to the algebraic forms that result from implicit differentiation at the expense of meaning. What we must always remember in problems such as these is that we are to keep the local nature of the implicit function in mind. That implicitly defined function is a *local extension of some isolated solution* $(x_0, y_0)$ of the equation $F(x, y) = 0$ we started with.

The solution $(x_0, y_0)$ is missing from the problems above. When we remember that it belongs, and that we are treating a local extension of that solution, we can see, among other things, that the two seemingly different expressions we obtain for derivatives in each problem are really the same things. For example, in the first problem the implicit function $\varphi(x)$ extends some isolated solution of the equation $4x^2y - (3/y) = 0$, whence $4x^2\varphi(x)^2 = 3$. This means that the denominator of the first expression given for $y'$ can be rewritten: $4x^2[\varphi(x)]^2 + 3 = 4x^2[\varphi(x)]^2 + 4x^2[\varphi(x)]^2 = 8x^2[\varphi(x)]^2$. Thus, $\varphi'(x) = (-8x[\varphi(x)]^3)/(4x^2[\varphi(x)]^2+3) = (-8x[\varphi(x)]^3)/(8x^2[\varphi(x)]^2) = -\varphi(x)/x$. Of course, something similar happens in the other problems.

The moral of the story is that we really ought to be asking, not for $y'(x)$—which we can usually obtain only as a function of the two variables $x$ and $y$—but for $y'[x_0]$, or, perhaps

$y'(x)$ *for a specific value of* $x$ *that lies near* $x_0$ *and a specific value of* $y$ *that lies near* $y_0$. We must remember that the expression in two variables that we obtain for $y'$ can have meaning as a slope field in the context of the problem as given *only when* $(x, y)$ *lies on some local solution near* $(x_0, y_0)$ of the equation $F(x, y) = 0$, and *not* as a generic function of two variables.

If we specify $x_0$ and $y_0$ numerically in the statement of the problem, we get a numeric answer. And it will not depend upon what algebraic transformations we make in the problem as we solve it. (Provided those transformations are legal, of course.) Thus, if we ask for $y'(2)$ at the point $(2, -\sqrt{3}/4)$ in the first problem above, the seeming difficulty never arises.

The issue that arose in this problem is that we often forget that implicit differentiation is for solving a *local* problem; it isn't for finding a global derivative that depends, as a function of two variables, on both $x$ and $y$. One of the reasons why this happens to us is that our text books insist on phrasing these problems incorrectly, as in "$x^3 - y^3 = 0$; what is $y'$?". The underlying issue, not addressed in the problem as just given, is the question "How does $y$ depend on $x$?". In this example, that question has a good answer only along the curve $y = x$ (unless we want to step into the complex domain). So the slope field given by $y' = x^2/y^2$ really isn't of interest to us here. The question should have been phrased "$x^3 - y^3 = 0$; what is $y'$ *along the implicitly defined curve near the point* $(1, 1)$?" (Or pick your favorite other point on the curve $y = x$. But not the origin!) Notice that we could ask this very same question, which amounts to "What is $y'$ at $(1, 1)$ along the curve $y = x$?" in a lot of different ways, *e.g.*, "$x^n - y^n = 0$; what is $y'$ near $(1, 1)$?" The answer is the same for all of those questions. But the (almost irrelevant) slope fields for different values of $n$ look very different. They should be understood as complicated representations, valid away from the origin, of the derivative $y'[x] \equiv 1$ along the curve $y = x$.

## 3. Badly written problems

This one appeared in December 2004:

> A student of mine was working the following problem: If $g(x) + \sin g(x) = x^2$ and $g(1) = 0$, find $g'(1)$.
>
> I know that the correct way to work the problem is to find $g'(x)$ implicitly, plug in $1$ for $x$, and you have the answer. However, one of my students brought up an interesting fact. Plugging $x = 1$ into the given equation does not yield the same thing on both sides (equality does not hold). I can't think of a good reason why it would't make the original statement true. I know you don't have to know what $g(x)$ is to solve the problem, but I am curious as to why $x = 1$ won't work in the orginal equation. Can anyone offer a good explanation for this?

The substitution $x = 1$ doesn't work in the original equation because the problem was poorly thought out. The point $(1,0)$ doesn't lie on the curve $y + \sin y = x^2$, and the problem, as stated, is meaningless. The author was, at best, careless. At worst, s/he doesn't understand what implicit differentiation is or accomplishes.