Simpson's Rule Is Exact for Quintics

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1. INTRODUCTION. Everyone who has taught the error estimate for Simpson's rule in a few freshman calculus courses knows that the chances are very good that someone who hasn't quite thought things through will look at the error estimate and then ask "Yes, but how can we find the error *exactly*?" The questioner is usually thunderstruck by the observation that if we knew how to do so in general, then we wouldn't speak of "Simpson's rule for *approximating* integrals," but of "Simpson's rule for *evaluating* integrals." Nevertheless, under certain circumstances we *can* find exact error. It is well known, for instance, that Simpson's rule error is zero whenever the integrand is a polynomial of degree three or less. What is less well known is the fact that we have adduced in our title: as we show in this article, we can easily find the exact error (see equation (5)) for Simpson's rule approximations to integrals of polynomial functions of degree four or five. We accomplish some other things as well, including an extension of the theorem generally known as the "First Mean Value Theorem for Integrals." We begin by reviewing some background.

We all learned in our elementary calculus courses that if K_2 is a bound for $|f^{(2)}(x)|$ on the interval $[a, b]$, then the error E_n^T in replacing $\int_a^b f(t)\,dt$ with its n-subdivision trapezoidal rule approximation

$$T_n = \frac{b-a}{2n} \sum_{k=1}^{n} [f(x_k) + f(x_{k-1})],$$

with $x_k = a + k(b-a)/n$ for $k = 1, \ldots, n$, satisfies the inequality

$$\left|E_n^T\right| \le K_2 \frac{(b-a)^3}{12n^2}. \tag{1}$$

We also learned that if K_4 is a bound for $|f^{(4)}(x)|$ on $[a, b]$, then the error E_{2n}^S in replacing $\int_a^b f(t)\,dt$ with its $2n$-subdivision Simpson's rule approximation

$$S_{2n} = \frac{b-a}{6n} \sum_{k=1}^{n} \left[f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right], \tag{2}$$

where $x_k = a + k(b-a)/(2n)$ for $k = 0, 1, \ldots, 2n$, satisfies the inequality

$$|E_{2n}^S| \le K_4 \frac{(b-a)^5}{180(2n)^4}. \tag{3}$$

Most of us didn't learn then, however, why these things are so. If our calculus books said anything at all about why these inequalities work, they probably just referred us to a numerical analysis book. Those of us who pursued the matter found that the numerical analysis books couched the proofs in terms of Lagrange interpolation in the context of general Newton-Cotes quadratures (see, for example, [**6**] or [**9**]). Most who had gotten that far threw up their hands at that point: the arguments are not accessible to freshmen.

In October 2003, D. Cruz-Uribe and C. J. Neugebauer [**2**] gave an elementary argument that establishes (1). They based their note in *Mathematics Magazine* upon a more advanced—and more thorough—treatment that they had given in [**3**], and in both papers they also discussed error estimates for the trapezoidal rule applied to functions that might not possess bounded second derivatives. While their methods extend to the cases of Simpson's rule in which the integrand possesses just one or two derivatives, Cruz-Uribe and Neugebauer were not able to apply them to obtain higher order estimates for smoother functions. In particular, they couldn't use their methods to derive (3) for Simpson's rule. Moreover, they explicitly asked whether it was possible to give an elementary derivation of error estimates for application of Simpson's rule to functions possessing either four or, even better, just three derivatives. Other authors (see [**1**, pp. 607–609], [**5**, p. 571, ex. 8], [**7**, p. 146, ex. 23], [**8**, p. 119, ex. 17], or [**10**, pp. 501–2]) give elementary arguments that establish (3). But while the argument in [**1**] avoids the general context of Newton-Cotes, it still relies on Lagrange interpolation. Furthermore, the arguments given in [**1**], [**5**], [**7**], [**8**], and [**10**] break down when we try to apply them to functions that do not possess fourth derivatives.

In this article, we show first how to exploit Taylor's theorem with integral remainder to represent the error in an elementary numerical quadrature as a certain integral. This part of our argument does not depart substantively from an argument given in [**2**], though the authors do not explicitly mention Taylor's theorem. We then show how to evaluate such an integral by using a symmetric derivative. Taylor's theorem with Lagrange remainder also plays a role in some of these latter calculations. Our methods, which are accessible to freshmen, allow us to achieve two kinds of results. For starters, we demonstrate how to establish the results that the numerical analysis books give for the numerical quadratures of elementary calculus:

Theorem. *If f is a continuously differentiable function on $[a, b]$ for which $f^{(2)}(u)$ exists at each point u of (a, b) and if T_n is the n-subdivision trapezoidal rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that*

$$\int_a^b f(t)\,dt = T_n - f^{(2)}(\xi)\frac{(b-a)^3}{12n^2}.$$

Theorem. *If f is a continuously differentiable function on $[a, b]$ for which $f^{(2)}(u)$ exists at each point u of (a, b) and if M_n is the n-subdivision midpoint rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that*

$$\int_a^b f(t)\,dt = M_n + f^{(2)}(\xi)\frac{(b-a)^3}{24n^2}.$$

Theorem. *If f is a thrice continuously differentiable function on $[a, b]$ for which $f^{(4)}(u)$ exists at each point u of (a, b) and if S_{2n} is the $2n$-subdivision Simpson's rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that*

$$\int_a^b f(t)\,dt = S_{2n} - f^{(4)}(\xi)\frac{(b-a)^5}{180(2n)^4}. \tag{4}$$

From (4), we will see in section 4 that if $q(x) = \sum_{k=0}^{5} a_k x^k$ is a fifth degree polynomial, then

$$\int_a^b q(t)\,dt = \frac{\{q(a)+4q[(a+b)/2]+q(b)\}(b-a)}{6} - \frac{1}{120}\left[5a_5\left(\frac{a+b}{2}\right)+a_4\right](b-a)^5. \tag{5}$$

This is the sense in which Simpson's rule is exact for quintic polynomials.

We also illustrate how one can use our technique to establish error estimates for the integral approximation schemes of elementary calculus, including not just the trapezoidal rule, but the midpoint rule and Simpson's rule as well, even when the integrands are not as smooth as the standard results require. We do not pursue these latter estimates in any depth, though we identify an underlying principle and give an example to point the reader in what we think is a fruitful direction.

2. SOME PRELIMINARY FACTS.

We will make use of some facts that, while generally known and elementary in nature, do not ordinarily appear in the standard calculus sequence. Because they do not ordinarily appear, we provide proofs.

First, we note that the mean value theorem takes on a special form for quadratic functions.

Theorem 1 (Quadratic Mean Value Theorem). *If q is a polynomial of degree at most two, then*

$$q(b) - q(a) = q'(\xi)(b-a)$$

for all real numbers a and b, where $\xi = (a+b)/2$.

Proof. Write $q(x) = \alpha x^2 + \beta x + \gamma$. Then, with ξ as in the statement of the theorem, we have

$$\begin{aligned} q'(\xi)(b-a) &= \left[2\alpha \cdot \tfrac{1}{2}(a+b) + \beta\right](b-a) = \alpha(b^2-a^2) + \beta(b-a) \\ &= (\alpha b^2 + \beta b + \gamma) - (\alpha a^2 + \beta a + \gamma) = q(b) - q(a). \end{aligned}$$ ∎

Everybody knows that continuous functions have the intermediate value property. It may surprise some readers to learn that derivatives—which need not be continuous functions—also have this property (see [**4**]).

Theorem 2 (Darboux). *Suppose that a function f is differentiable on (a, b) and that $f'(x_1) = \alpha < \lambda < \beta = f'(x_2)$ for points x_1 and x_2 of (a, b). There is a number ξ between x_1 and x_2 such that $f'(\xi) = \lambda$.*

Proof. In order to be definite, we assume that $x_1 < x_2$. Consider the function g defined on the interval (a, b) by $g(x) = f(x) - \lambda x$. Then $g'(x) = f'(x) - \lambda$, and $g'(x_1) < 0 < g'(x_2)$. Because of this latter pair of inequalities, neither $g(x_1)$ nor $g(x_2)$ can be the minimum value taken on by g in the interval $[x_1, x_2]$. On the other hand, as a continuous function on $[x_1, x_2]$, g must assume a minimum value somewhere in the interval, so must achieve it at some ξ in (x_1, x_2), at which point $g'(\xi) = 0$. ∎

Theorem 2 has an important consequence that is not as well known as it should be.

Corollary (Positive Linear Combination Property of Derivatives). *Suppose that f is a function differentiable on (a, b) and that n is a positive integer. For $k = 1, 2, \ldots, n$ let θ_k belong to (a, b), and let α_k be a positive real number. Then there exists ξ in (a, b) such that*

$$\sum_{k=1}^{n} \alpha_k f'(\theta_k) = f'(\xi) \sum_{k=1}^{n} \alpha_k.$$

In particular, we can replace any convex combination (i.e., all $\alpha_k > 0$ and $\sum \alpha_k = 1$) of values that f' assumes in (a, b) with its value at some particular (albeit unspecified) point in the interval.

Proof. For $k = 1, 2, \ldots, n$ put

$$\lambda_k = \frac{\alpha_k}{\sum_{j=1}^{n} \alpha_j}.$$

Take M to be the largest of the numbers $f'(\theta_1), f'(\theta_2), \ldots, f'(x_n)$ and m to be the smallest. Then $M = f'(\theta_{k_1})$ and $m = f'(\theta_{k_2})$ for certain k_1 and k_2. The sum $\sum_{k=1}^{n} \lambda_k f'(\theta_k)$ is a weighted average of the numbers $f'(\theta_1), f'(\theta_2), \ldots, f'(\theta_n)$, and as such must lie between M and m. Darboux's theorem now yields a number ξ in (a, b) with the desired property. ∎

We will also have need of Taylor's theorem with remainder, including two standard forms of the remainder sometimes found in calculus books.

Theorem 3 (Taylor's Theorem with Integral Remainder). *If f is a function whose derivative of order $n + 1$ is continuous throughout some open interval I centered at the origin and if h is a number in I, then*

$$f(h) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} h^k + \frac{1}{n!} \int_0^h f^{(n+1)}(t)\,(h - t)^n\,dt. \tag{6}$$

Proof. By the fundamental theorem of calculus,

$$f(h) = f(0) + \int_0^h f'(t)\,dt,$$

which is (6) for $n = 0$. Expand the integral using integration by parts, taking $u = f'(t)$, $du = f''(t)\,dt$, $dv = dt$, $v = -(h - t)$. Repeat inductively. ∎

Theorem 4 (Taylor's Theorem with Lagrange Remainder). *If f is a function whose derivative of order $n + 1$ is defined throughout some open interval I centered at the origin and if h is a number in I, then there exists a point ξ lying between 0 and h such that*

$$f(h) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!} h^{n+1}.$$

We could derive the latter theorem as a corollary to Theorem 3 if we made the stronger assumption that $f^{(n+1)}$ were continuous on I. The proof relies on the first mean value theorem for integrals, which we state presently. However, we require the theorem in the stronger form that we have given. We omit the proof (the usual one that appears in calculus texts is based upon Rolle's theorem).

Finally, we appeal to (and eventually extend) the first mean value theorem for integrals:

Theorem 5 (First Mean Value Theorem for Integrals). *If f and φ are both continuous on the interval $[a, b]$ and if $\varphi(t) \geq 0$ for all t in $[a, b]$, then there exists ξ in (a, b) such that*

$$\int_a^b f(t)\varphi(t)\,dt = f(\xi)\int_a^b \varphi(t)\,dt.$$

Proof. If $\varphi(t)$ vanishes for all t in $[a, b]$, the conclusion is trivially true, so we may assume that φ takes on a positive value somewhere in the interval, from which it follows that $\int_a^b \varphi(t)\,dt > 0$. Let m be the minimum value taken on by the continuous function f in the interval $[a, b]$, and let M be its maximum value. Then $m\,\varphi(t) \leq f(t)\varphi(t) \leq M\,\varphi(t)$ holds for all t in $[a, b]$, implying that

$$m\int_a^b \varphi(t)\,dt \leq \int_a^b f(t)\varphi(t)\,dt \leq M\int_a^b \varphi(t)\,dt.$$

Consequently,

$$m \leq \frac{\int_a^b f(t)\varphi(t)\,dt}{\int_a^b \varphi(t)\,dt} \leq M.$$

Because the quotient of the integrals is a value that lies between the minimum and maximum values taken on by f in the interval $[a, b]$, the intermediate value theorem implies that there must be ξ in (a, b) such that

$$f(\xi) = \frac{\int_a^b f(t)\varphi(t)\,dt}{\int_a^b \varphi(t)\,dt}.$$

This is the conclusion we sought in the nontrivial case. ∎

It should be clear that the conclusion of the first mean value theorem for integrals still follows if we replace the nonnegativity hypothesis on φ with the more general condition that φ not change sign in $[a, b]$.

3. ERROR IN SIMPSON'S RULE. We now turn to the error in a Simpson's rule approximation for the integral of a general (smooth) function. In what follows, we always think of the $2n$-subdivision Simpson's rule approximation S_{2n} for $\int_a^b f(t)\,dt$ in the form that we gave in (2). This form is not optimal for numerical computation, but it has the merit that each of its summands corresponds to the (signed) area over the interval $[x_{2k-2}, x_{2k}]$ between the x-axis and a certain parabola.

In this theorem and its proof, as well as elsewhere, we will have to consider derivatives of functions when the domains of those functions are closed intervals. Our arguments require continuity of those derivatives throughout those closed intervals. In this situation, we follow the convention that the derivative of such a function at an endpoint of its domain is the appropriate one-sided derivative.

Theorem 6 (Error in Simpson's Rule). *If f is a thrice continuously differentiable function on $[a, b]$ for which $f^{(4)}(u)$ exists at each u in (a, b), then there is a point ξ in (a, b) such that*

$$\int_a^b f(t)\,dt = S_{2n} - f^{(4)}(\xi)\frac{(b-a)^5}{180(2n)^4}. \tag{7}$$

Proof. We first consider the error in a single one of the n Simpson's rule summands. We can simplify matters appreciably by assuming that the interval associated with that summand is centered at the origin. We introduce the error function E given on $[0, h]$, where $h = (b-a)/(2n)$, by

$$E(u) = \int_{-u}^{u} f(t)\,dt - \frac{u}{3}\left[f(-u) + 4f(0) + f(u)\right],$$

and we examine E', E'', and $E^{(3)}$:

$$\begin{aligned}
E'(u) &= \frac{1}{3}\left[2f(-u) - 4f(0) + 2f(u)\right] + \frac{u}{3}\left[f'(-u) - f'(u)\right];\\
E''(u) &= -\frac{1}{3}\left[f'(-u) - f'(u)\right] - \frac{u}{3}\left[f''(-u) + f''(u)\right];\\
E^{(3)}(u) &= \frac{u}{3}\left[f^{(3)}(-u) - f^{(3)}(u)\right].
\end{aligned}$$

The function f and its derivatives are all continuous functions (at least in short intervals centered at the origin), so E and its derivatives are continuous on $[0, h]$ if h is sufficiently small. Moreover, $E(0) = E'(0) = E''(0) = 0$. Taylor's formula with integral remainder thus gives

$$\begin{aligned}
E(h) &= E(0) + E'(0)h + \frac{1}{2}E''(0)h^2 + \frac{1}{2}\int_0^h E^{(3)}(t)(h-t)^2\,dt\\
&= \frac{1}{6}\int_0^h \left[f^{(3)}(-t) - f^{(3)}(t)\right] t(h-t)^2\,dt.
\end{aligned} \tag{8}$$

Next, we define a function F on $[0, h]$ by

$$F(t) = \begin{cases} \dfrac{f^{(3)}(-t) - f^{(3)}(t)}{t} & \text{if } t \neq 0,\\ -2f^{(4)}(0) & \text{if } t = 0. \end{cases}$$

We observe that

$$\frac{f^{(3)}(-t) - f^{(3)}(t)}{t} = -\frac{f^{(3)}(-t) - f^{(3)}(0)}{-t} - \frac{f^{(3)}(t) - f^{(3)}(0)}{t} \longrightarrow -2f^{(4)}(0) \tag{9}$$

as $t \to 0^+$. Combining (9) with the continuity of $f^{(3)}$, we find that F is continuous on $[0, h]$. This means that we can write

$$\frac{1}{6}\int_0^h \left[f^{(3)}(-t) - f^{(3)}(t)\right] t(h-t)^2\,dt = \frac{1}{6}\int_0^h F(t)t^2(h-t)^2\,dt,$$

and we can invoke the first mean value theorem for integrals to assert the existence of η in $(0, h)$ such that

$$E(h) = \frac{F(\eta)}{6} \int_0^h t^2(h-t)^2\,dt = \frac{f^{(3)}(-\eta) - f^{(3)}(\eta)}{180\eta} h^5. \tag{10}$$

The existence of $f^{(4)}$ then allows us to apply the mean value theorem to the numerator of the right-most quotient in (10). This produces a number θ in $(-\eta, \eta)$ such that

$$E(h) = \frac{f^{(4)}(\theta)(-2\eta)}{180\eta} h^5 = -f^{(4)}(\theta)\frac{h^5}{90}.$$

Thus, the kth interval $[x_{2k-2}, x_{2k}]$ associated with Simpson's rule contains a point θ_k such that

$$\int_{x_{2k-2}}^{x_{2k}} f(t)\,dt - \frac{b-a}{6n}\left[f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right] = -f^{(4)}(\theta_k)\frac{(b-a)^5}{90(2n)^5}. \tag{11}$$

We can therefore obtain the error in the $2n$-subdivision Simpson's rule approximation by summing the quantities on the right-hand side of (11).

According to Darboux's theorem, we can find ξ in (a, b) such that

$$\sum_{k=1}^{n} f^{(4)}(\theta_k) = nf^{(4)}(\xi).$$

Consequently,

$$-\sum_{k=1}^{n} f^{(4)}(\theta_k)\frac{(b-a)^5}{90(2n)^5} = -\frac{(b-a)^5}{90(2n)^5}\cdot nf^{(4)}(\xi) = -\frac{f^{(4)}(\xi)(b-a)^5}{180(2n)^4}$$

for a certain ξ in (a, b). This establishes (7). ■

4. SIMPSON'S RULE FOR QUARTIC AND QUINTIC POLYNOMIALS. In this section we turn our attention to the behavior of the error term when the integrand is a polynomial of degree four or five. We demonstrate how to evaluate the error exactly in these cases.

Consider first that if $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is a fourth-degree polynomial, then $p^{(4)}(x) = 24a_4$ is constant. Thus, according to Theorem 6, the error in the $2n$-subdivision Simpson's rule approximation to $\int_a^b p(t)\,dt$ is

$$-\frac{p^{(4)}(\xi)(b-a)^5}{180(2n)^4} = -\frac{a_4(b-a)^5}{120n^4}.$$

It follows that for any positive integer n

$$\int_a^b p(t)\,dt = S_{2n} - \frac{a_4(b-a)^5}{120n^4}.$$

There is no need to take n to be anything other than 1, so

$$\int_a^b p(t)\,dt = S_2 - \frac{a_4(b-a)^5}{120}. \tag{12}$$

If $q(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is a fifth-degree polynomial, we must be just a little bit more clever. Recall from (10) that

$$E(h) = \frac{q^{(3)}(-\eta) - q^{(3)}(\eta)}{180\eta}h^5$$

for a certain η lying in $(0, h)$. If q is quintic, then $q^{(3)}$ is quadratic and Theorem 1 allows us to write

$$E(h) = \frac{q^{(4)}(0)(-2\eta)}{180\eta}h^5 = -\frac{q^{(4)}(0)}{90}h^5.$$

Translating this from the interval $[-h, h]$ to the interval $[a, b]$ and again noting that there is no reason to choose n different from 1, we conclude that

$$\int_a^b q(t)\,dt = S_2 - \frac{q^{(4)}[(a+b)/2]}{2880}(b-a)^5.$$

Because $q^{(4)}(x) = 120a_5x + 24a_4$, it follows that

$$\int_a^b q(t)\,dt = S_2 - \frac{1}{120}\left[5a_5\left(\frac{a+b}{2}\right) + a_4\right](b-a)^5. \tag{13}$$

Of course, we could have obtained equation (12) from equation (13) by simply putting $a_5 = 0$. However, we thought it more interesting to derive (12) independently.

5. SOME EXTENSIONS. Before proceeding with this final section of the paper, the reader may want to try to use the ideas that underlie the proof of Theorem 6 to derive the classical error expressions for the trapezoidal rule and midpoint rule approximations that we gave at the beginning of this paper. We begin the section with the core of just such an argument for each.

The central issue in the proof of Theorem 6 is the evaluation of the integral that appears on the right-hand side of equation (8). If we write a zero-degree Taylor polynomial with integral remainder for the error in a single summand of a trapezoidal approximation,

$$E(h) = \int_{-h}^{h} f(t)\,dt - h[f(-h) + f(h)],$$

we learn that we must evaluate the integral $\int_0^h [f'(-t) - f'(t)]\,t\,dt$. (The latter is essentially the integral on the right-hand side of equation (3) of [**2**], though our route to it has been a little different from the one taken in that work.) Now we note that we can extend the quotient function $F(t) = [f'(-t) - f'(t)]/t$ continuously to the origin by defining $F(0) = -2f''(0)$. We can then see in much the same way as in the proof of Theorem 6 that there exist η in $(0, h)$ and ξ in $(-h, h)$ such that

$$\int_0^h \left[f'(-t) - f'(t)\right] t\,dt = \frac{f'(-\eta) - f'(\eta)}{\eta}\int_0^h t^2dt = -\frac{2f''(\xi)h^3}{3}.$$

A somewhat similar problem arises in calculating error in the midpoint rule, the integral in question in this case being

$$\int_0^h [f(-t) - 2f(0) + f(t)]\,dt.$$

Here we introduce the function

$$F(t) = \begin{cases} \dfrac{f(-t) - 2f(0) + f(t)}{t^2} & \text{if } t \neq 0, \\ f''(0) & \text{if } t = 0, \end{cases}$$

and—after establishing the continuity of F at the origin by using L'Hôpital's rule—we discover that

$$\begin{aligned} \int_0^h [f(-t) - 2f(0) + f(t)]\,dt &= \frac{f(-\eta) - 2f(0) + f(\eta)}{\eta^2} \int_0^h t^2\,dt \\ &= f''(\xi)\frac{h^3}{3} \end{aligned} \tag{14}$$

for certain η in $(0, h)$ and ξ in $(-h, h)$.[1]

Our results for Simpson's rule, the trapezoidal rule, and the midpoint rule differ in form, but there is a common thread in the arguments that establish them. We can isolate this thread by considering the functions that we designated F in each of the three arguments. In each case, we were able to define $F(0)$ as the limit of a certain quotient. The limiting values of these quotients are, except for a constant factor, what many call the *first*

$$D_1 f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

and the *second*

$$D_2 f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

symmetric derivatives of f at the point x_0. Some authors also call the second symmetric derivative the "Schwartz derivative" or the "Riemann derivative." It is a well-known fact, which we have reestablished for the sake of completeness, that if $f^{(k)}(x_0)$ exists for $k = 1$ or 2, then $D_k f(x_0)$ also exists and is equal to $f^{(k)}(x_0)$. The converse is not true (see [**11**, pp. 22–23]).

We could have attempted to apply the first mean value theorem for integrals (Theorem 5) to the integral that appears in (8). If we had done so, we would have obtained η in $(0, h)$ for which

$$E(h) = \frac{f^{(3)}(-\eta) - f^{(3)}(\eta)}{6} \int_0^h t(h - t)^2\,dt.$$

As in the argument we gave earlier, the form of the numerator that appears in the fraction here suggests that we apply the mean value theorem at the same time that we

[1] Equation (14) comes from the observation that, by Taylor's formula with Lagrange remainder, there exist ξ_1 and ξ_2 in $(0, h)$ for which

$$f(-\eta) = f(0) - f'(0)\eta + \tfrac{1}{2} f''(-\xi_1)\eta^2, \; f(\eta) = f(0) + f'(0)\eta + \tfrac{1}{2} f''(\xi_2)\eta^2,$$

so that

$$f(-\eta) - 2f(0) + f(\eta) = \tfrac{1}{2}\left[f''(-\xi_1) + f''(\xi_2)\right]\eta^2 = f''(\xi)\eta^2,$$

where ξ in $(-h, h)$ arises from the corollary to Theorem 2.

evaluate the integral. Doing these things results in a value θ in $(-\eta, \eta)$ for which

$$E(h) = \frac{f^{(4)}(\theta)(-2\eta)}{6} \cdot \frac{h^4}{12} = -f^{(4)}(\theta)\frac{\eta h^4}{36}.$$

At this point we realize that we have no way of eliminating the number η from the formula except by passing to magnitudes. Moreover, the denominator is only two-fifths of what we know it should be. This line of reasoning is therefore unsatisfactory—not just because it results in an estimate rather than an evaluation, but also because that estimate is not as tight as it ought to be.

We were able to obtain a stronger result because the difference $[f^{(3)}(-t) - f^{(3)}(t)]$ has a zero at $t = 0$. Introduction of the symmetric derivative allowed us to transfer this zero, in the form of a power of t, to the part of the integrand that we wanted to leave inside the integral when we applied Theorem 5. Similar considerations came into play in our other arguments. In point of fact, however, it is not the symmetric derivative as such that is crucial to our reasoning, but the observation that the correct portion of the integrand vanishes in a helpful way. We capture this thread in the arguments as a theorem:

Theorem 7 (Extended Mean Value Theorem for Integrals). *Let n be a positive integer. Suppose that g is $(n-1)$-times continuously differentiable on $[a, b]$, that t_0 lies in $[a, b]$, that $g^{(n)}(t)$ exists for every t in $\{t_0\} \cup (a, b)$, and that $g^{(k)}(t_0) = 0$ for $k = 0, 1, \ldots, n-1$. If a function φ is continuous on $[a, b]$ and if the function $t \mapsto \varphi(t)(t - t_0)^n$ does not change sign at any point of (a, b), then there exists η in (a, b) such that*

$$\int_a^b g(t)\varphi(t)\,dt = \frac{g^{(n)}(\eta)}{n!}\int_a^b \varphi(t)\,(t - t_0)^n\,dt.$$

Proof. By our hypotheses and $n - 1$ applications of L'Hôpital's rule,

$$\lim_{t\to t_0} \frac{g(t)}{(t-t_0)^n} = \lim_{t\to t_0} \frac{g^{(n-1)}(t)}{n!\,(t-t_0)} = \frac{1}{n!}\lim_{t\to t_0} \frac{g^{(n-1)}(t) - g^{(n-1)}(t_0)}{t - t_0} = \frac{g^{(n)}(t_0)}{n!}.$$

This means that the function $t \mapsto g(t)/(t-t_0)^n$ has a removable singularity at t_0, so the function $G : [a, b] \to \mathbb{R}$ given by

$$G(t) = \begin{cases} \dfrac{g(t)}{(t-t_0)^n} & \text{if } t \neq t_0, \\ \dfrac{1}{n!}g^{(n)}(t_0) & \text{if } t = t_0, \end{cases}$$

is continuous. Consequently, we can appeal to Theorem 5 to find a number ξ in (a, b) such that

$$\int_a^b g(t)\varphi(t)\,dt = \int_a^b G(t)(t-t_0)^n\varphi(t)\,dt = G(\xi)\int_a^b \varphi(t)\,(t-t_0)^n\,dt.$$

If $\xi = t_0$, we simply take $\eta = t_0 = \xi$, and we are done. If $\xi \neq t_0$, we apply Taylor's theorem with Lagrange remainder to $g(\xi)$ to find an η interior to the interval determined by t_0 and ξ (and a fortiori interior to $[a, b]$) such that

$$G(\xi) = \frac{1}{(\xi - t_0)^n} g(\xi) = \frac{1}{(\xi - t_0)^n} \left[\sum_{k=0}^{n-1} \frac{0}{k!} (\xi - t_0)^k + \frac{g^{(n)}(\eta)}{n!} (\xi - t_0)^n \right]$$

$$= \frac{1}{n!} g^{(n)}(\eta). \qquad \blacksquare$$

We invite the reader to apply Theorem 7 to derive the classical error expressions for the trapezoidal rule and the midpoint rule, as well as error expressions for the trapezoidal rule, midpoint rule, and Simpson's rule approximations to integrals whose integrands are not as smooth as the classical error expressions require. In each case, Taylor's formula with integral remainder provides an expression that we can use the extended mean value theorem for integrals to evaluate.

For example, in the case of the midpoint rule and a twice-differentiable function f, Theorem 7 immediately gives an η in $(0, h)$ for which

$$\int_0^h [f(-t) - 2f(0) + f(t)]\, dt = \frac{1}{2} \left[f''(-\eta) + f''(\eta) \right] \int_0^h t^2\, dt,$$

whence by the corollary to Theorem 2 there is a ξ in $(-h, h)$ satisfying (14).

By way of a further example, if we know that f is a twice continuously differentiable function on $[a, b]$, we can write the Simpson's rule error function defined in the proof of Theorem 6 as a first-degree Taylor polynomial with integral remainder. Equivalently,

$$\begin{aligned} -3E(h) &= \int_0^h \left(\left[f'(-u) - f'(u) \right] + \left[f''(-u) + f''(u) \right] u \right) (h - u)\, du \\ &= \int_0^h \left[f'(-u) - f'(u) \right] (h - u)\, du + \int_0^h \left[f''(-u) + f''(u) \right] u(h - u)\, du. \end{aligned}$$

We can then invoke Theorem 7 in conjunction with the corollary to Darboux's theorem to reduce the first of these two integrals to $-f''(\theta)h^3/3$ for some θ in $(-h, h)$. We are then able to combine Theorem 5 with the aforementioned corollary to reduce the second integral to $f''(\eta)h^3/3$ for a certain η in $(-h, h)$. Dividing by -3, translating to each of the Simpson's rule subintervals of $[a, b]$, and summing, we infer that there exist ξ_1 and ξ_2 in (a, b) for which

$$E_{2n}^S = \left[f''(\xi_1) - f''(\xi_2) \right] \frac{(b - a)^3}{18(2n)^2}. \tag{15}$$

Equation (15), for twice continuously differentiable functions, is similar to the estimate (1.7) given for Lipschitz functions by Cruz-Uribe and Neugebauer in [**3**]. That (15) is an order two estimate, whereas (1.7) of [**3**] is order one, we attribute to the additional smoothness that we have imposed upon the integrand.

If, in addition, we know that $f^{(3)}(t)$ exists and satisfies $|f^{(3)}(t)| \le K_3$ for all t in (a, b), we can apply the mean value inequality immediately before summing and then conclude after the summation that

$$|E_{2n}^S| \le K_3 \frac{(b - a)^4}{9(2n)^3}.$$

This estimate is not the best available: its denominator is no more than one-eighth of what it should be (see Exercises 51 and 52 in [**6**, chap. 5]). Nevertheless, the estimate has the right order of decay, and it doesn't depend on the mysteries of an "influence function." We therefore have a partial answer to the question raised in [**2**] about using elementary techniques to estimate the errors that may arise when we apply Simpson's rule to functions whose fourth-order derivatives are not available.

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REFERENCES

1. T. M. Apostol, *Calculus*, vol. 2, 2nd ed., Blaisdell, Waltham, MA, 1969.
2. D. Cruz-Uribe and C. J. Neugebauer, An elementary proof of error estimates for the trapezoidal rule, *Math. Mag.* **76** (2003) 303–306.
3. ———, Sharp error bounds for the trapezoidal rule and Simpson's rule, *J. Ineq. Pure Appl. Math.* **3**(4) (2002), article 49; available at `http://jipam-old.vu.edu.au/v3n4/031_02.html`.
4. J. G. Darboux, Mémoire sur les fonctions discontinues, *Ann. Sci. École Norm. Sup.* Sér. 2 **4** (1875) 57–112.
5. M. P. Fobes and R. B. Smyth, *Calculus and Analytic Geometry*, vol. 1, Prentice-Hall, Englewood Cliffs, NJ, 1963.
6. F. B. Hildebrand, *Introduction to Numerical Analysis*, 2nd ed., McGraw-Hill, New York, 1974.
7. J. M. H. Olmstead, *Real Variables*, Appleton-Century-Crofts, New York, 1959.
8. ———, *Advanced Calculus*, Appleton-Century-Crofts, New York, 1961.
9. A. Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965.
10. R. E. Williamson, R. H. Crowell, and H. F. Trotter, *Calculus of Vector Functions*, 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1972.
11. A. Zygmund, *Trigonometric Series*, 2 vols., Cambridge University Press, Cambridge, 1977.

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