Exponential Functions, Separation of Variables, etc.

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March 2, 2006

Jeanne Pohlman wrote:

> I am working on slope fields and I am not sure if the differential equation $dy/dx = y$ can exist below the x axis. Because it is an e^x equation it doesn't seem like it would but I am really unsure.

There are several questions, all bundled up into these two sentences.

Let's begin with the standard solution, by formal manipulation, of Jeanne's problem—but converted to an initial value problem so that we can talk about *the* solution to the problem rather than some nebulous "general" solution. The initial value problem associated with this differential equation is

$$\frac{dy}{dx} = y; \tag{1}$$

$$y(a) = \alpha. \tag{2}$$

The solution technique offered in many texts consists of formally separating the variables in the differential equation to obtain an equation of differentials:

$$\frac{dy}{y} = dx, \tag{3}$$

and then integrating both sides. In this case we obtain

$$\int \frac{dy}{y} = \int dx \tag{4}$$

or

$$\ln|y| = x + c, \tag{5}$$

where c designates an arbitrary constant. As we have seen elsewhere[1], equation (5) is problematic, but let us treat those issues as understood and not quite germane to the matters now of interest. What does matter—what is, in fact, essential here—is that the quantity on the left side of (5) is $\ln |y|$, and not $\ln y$. The footnoted reference explains why.

The next step in the formal solution of our initial value problem is to use elementary properties of logarithms and exponentials to rewrite equation (5) as

$$|y| = e^{x+c} \tag{6}$$

which can itself be rewritten as

$$|y| = Ce^{x}. \tag{7}$$

Because $C = e^c$, the quantity C must be a positive number, and this is consistent with the appearance of $|y|$ on the left side of equation (7). If we allow C to be negative, we may rewrite (7) as

$$y = Ce^{x}. \tag{8}$$

At this point, more careful treatments observe that when we wrote equation (3), we divided by y, and in so doing eliminated the constant solution $y = 0$. (Far too many treatments *don't* take enough care at this point; for a bad example, see Stewart, *Calculus: Concepts and Contexts.*) We reclaim it at this point by allowing the possibility that $C = 0$ in addition to the already permissible positive and negative values for C.

With these understandings, equation (8) gives the "general solution" to the differential equation. We obtain the solution to our initial value problem by making use of the initial condition $y(a) = \alpha$: substituting this latter into (8), we obtain

$$\alpha = Ce^{a}, \tag{9}$$

from which it follows that

$$C = \alpha e^{-a}. \tag{10}$$

At long last we find that the solution to the initial value problem (1–2) is

$$y = \alpha e^{x-a}. \tag{11}$$

This latter equation *does* give a solution to the original initial value problem, whatever α may be: We rigged it so that $y(a) = \alpha$ and it is clear that $dy/dx = y$. Note that whether the solution is always positive or always negative (or always neither) depends wholly on the nature of α, which was given in the initial condition. This answers the most obvious of the questions that are implicit in what Jeanne wrote: Solutions of the differential equation $y' = y$ can be negative.

But there are other questions implicit in the treatment we have just examined.

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[1] `http://clem.mscd.edu/~talmanl/PDFs/APCalculus/OnAnIntegral.pdf`

1. What does equation (3) mean? After all, the symbol dy/dx is only formally a quotient (that is, it is a quotient in form, but not in meaning). We have multiplied through by dx to clear the "quotient". Is this justifiable?

2. The rule for dealing with equations is that performing the same operation on equals yields equals. But we began with equation (3) and then performed *different* operations on the respective sides: integration with respect to y on the left and integration with respect to x on the right. How can we then assert equality of the results?

We can certainly argue that the method of separation of variables is justified by the correct results it produces, but to do so is to accept *a posteriori* justification for the procedure, and as mathematicians we should look for *a priori* justifications. In fact we can give an *a priori* justification for the technique used here, and it answers both of the questions listed above.

Let's rephrase the initial value problem (1–2) more precisely and more explicitly. We seek a function f, defined on some open interval I containing $x = a$ and satisfying two conditions:

1. $f'(x) = f(x)$ for all x in I;

2. $f(a) = \alpha$.

Let us note that a solution, if one exists, must be continuously differentiable near the point $x = a$: The first condition is meaningless unless f is differentiable, which implies that f is continuous, and because f' and f are the same, f' must be continuous.

We observe first that the constant function $f(x) = 0$ is defined on every open interval containing $x = a$ and certainly satisfies condition 1 there. If it also satisfies condition 2, we are done. If not, and if there is a solution f, then by continuity it must be non-zero on some open interval containing $x = a$. In this interval, division by $f(x)$ is legitimate and we may write

$$\frac{f'(x)}{f(x)} = 1. \tag{12}$$

Now if t is any number in this same interval, we may integrate both sides of (12) from a to t. By the principle that doing the same thing to equals yields equals, we obtain a new equation

$$\int_a^t \frac{f'(x)}{f(x)}\,dx = \int_a^t dx. \tag{13}$$

We apply the Substitution Theorem for Definite Integrals to the integral on the left side of (13), taking $y = f(x)$ and $dy = f'(x)\,dx$. We find then that $x = a \Rightarrow y = f(a) = \alpha$ and

$x = t \Rightarrow y = f(t)$; this leads us to the equation

$$\int_{\alpha}^{f(t)} \frac{dy}{y} = \int_{a}^{t} dx, \tag{14}$$

whence

$$\ln|f(t)| - \ln|\alpha| = t - a \tag{15}$$

or

$$\ln\left|\frac{f(t)}{\alpha}\right| = t - a. \tag{16}$$

But f doesn't vanish in the interval where we have allowed ourselves to choose t, so $f(t)$ has the same sign as $f(a) = \alpha$ throughout this interval. We conclude that $|f(t)/\alpha| = f(t)/\alpha$ and may now write

$$\frac{f(t)}{\alpha} = e^{t-a}, \tag{17}$$

from which it follows that

$$f(t) = \alpha e^{t-a}. \tag{18}$$

Finally, we remark that the argument we have just given for the initial value problem (1–2) works, with very slight changes, for any initial value problem of the form

$$\frac{dy}{dx} = f(y)g(x); \tag{19}$$

$$y(a) = \alpha. \tag{20}$$

We must assume that the functions f and g are continuous on appropriate intervals, and we must take the zeros of the function f into account. Isolated zeros of f yield constant solutions; zeros which are not isolated cause considerable difficulty—so we will disallow them.