# Why $\lim_{h\to 0}(1+h)^{1/h}$ Exists

Louis A. Talman

October 16, 2001

First, note that if $n$ is a natural number $> 1$, we have by the Binomial Theorem,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k && (1)\\
&= 1+\sum_{k=1}^{n}\left[\frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)\right] && (2)\\
&< 1+\sum_{k=1}^{n}\frac{1}{k!} && (3)\\
&< 1+\sum_{k=1}^{n}\frac{1}{2^{k-1}} && (4)\\
&< 1+\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}. && (5)
\end{aligned}$$

The latter sum is 3, and this shows that the sequence $\{(1+1/n)^n\}_{n=1}^{\infty}$ is bounded above.

Now, expanding

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+\sum_{k=1}^{n+1}\left[\frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n+1}\right)\right] \tag{6}$$

in like fashion to (2), and comparing each of the first $n$ terms of the sum on the right-hand side of (6) with the corresponding term of (2), we find, because each of the products in (6) is larger than the corresponding product in (2), and because (6) contains an additional term, that for each natural number $n > 1$ we must have

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}. \tag{7}$$

It now follows that the sequence $\{(1+1/n)^n\}_{n=1}^{\infty}$, being increasing and bounded above, must have a limit which is at most 3. Let us *define* $\lim_{n\to\infty}(1+1/n)^n = e$.

Now let $x$ be any positive real number larger than 2. Let $n$ be the greatest integer which does not exceed $x$. Then $n \le x < n+1$, so

$$\frac{1}{n+1} < \frac{1}{x} \le \frac{1}{n}. \tag{8}$$

Consequently,

$$1+\frac{1}{n+1} < 1+\frac{1}{x} \le 1+\frac{1}{n}, \tag{9}$$

so that

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}, \tag{10}$$

or

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)} \;<\; \left(1+\frac{1}{x}\right)^{x} \;\leq\; \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right). \tag{11}$$

But the limits on both the left side and the right side of the compound inequality (11) are $e$, and this shows that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \;=\; e. \tag{12}$$

Equivalently, we have now seen that

$$\lim_{h\to 0^+}(1+h)^{1/h} \;=\; e. \tag{13}$$

To study $\lim_{h\to 0^-}(1+h)^{1/h}$, we first note that

$$\lim_{h\to 0^-}(1+h)^{1/h} \;=\; \lim_{k\to 0^+}(1-k)^{-1/k}, \tag{14}$$

and that the latter limit is the same as $\lim_{x\to+\infty}(1-1/x)^{-x}$. But

$$\begin{aligned}\left(1-\frac{1}{x}\right)^{-x} &= \left(\frac{x-1}{x}\right)^{-x} && (15)\\ &= \left(\frac{x}{x-1}\right)^{x} && (16)\\ &= \left(1+\frac{1}{x-1}\right)^{x} && (17)\\ &= \left(1+\frac{1}{x-1}\right)^{x-1}\cdot\left(1+\frac{1}{x-1}\right). && (18)\end{aligned}$$

By what we have seen above, the limit, as $x\to\infty$, of the first factor on the right-hand side of (18) is $e$, while the limit of the second factor is 1. •