AP Calculus 1998 AB FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

September 19, 2017

1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg|_0^4 = \frac{16}{3}. \tag{1}$$

1.2 Part b

If the vertical line $x = h$ divides R into two regions of equal area, then

$$\int_0^h \sqrt{x}\,dx = \frac{8}{3}, \text{ or} \tag{2}$$

$$\frac{2}{3}h^{3/2} = \frac{8}{3}. \tag{3}$$

Solving for h gives $h = 4^{2/3} = 2^{4/3} = 2\sqrt[3]{2}$.

1.3 Part c

The volume generated by revolving R about the x-axis is

$$\pi\int_0^4 x\,dx = \frac{\pi}{2}x^2\bigg|_0^4 = 8\pi. \tag{4}$$

1.4 Part d

Here we must solve for k in the equation

$$\pi \int_0^k x\,dx = 4\pi. \tag{5}$$

Integration gives

$$\frac{\pi}{2}k^2 = 4\pi, \text{ whence} \tag{6}$$

$$k = 2\sqrt{2}. \tag{7}$$

2 Problem 2

2.1 Part a

We observe first that

$$\lim_{x\to-\infty} 2xe^{2x} = \lim_{x\to-\infty} \frac{2x}{e^{-2x}}. \tag{8}$$

Numerator and denominator of this last fraction both become infinite as $x \to -\infty$, so we may attempt l'Hôpital's Rule. This gives

$$\lim_{x\to-\infty} \frac{2x}{e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-2e^{-2x}} = 0, \tag{9}$$

and we conclude that

$$\lim_{x\to-\infty} 2xe^{2x} = 0. \tag{10}$$

2.2 Part b

If $f(x) = 2xe^{2x}$, then $f'(x) = (2+4x)e^{2x}$, which is defined for all real x. Thus, $f'(x) = 0$ only when $x = -1/2$, so f has just one critical point—which lies at $x = -1/2$. But $e^{2x} > 0$ for all x, while $2 + 4x < 0$ on $(-\infty, -1/2)$ but $2x + 4 > 0$ on $(-1/2, \infty)$. So $f'(x) < 0$ on $(-\infty, -1/2)$, and $f'(x) > 0$ on $(-1/2, \infty)$. Because f is everywhere continuous[1], it follows that f is a strictly decreasing function on $(-\infty, -1/2]$, but that f is a strictly increasing function on $[-1/2, \infty)$. That is, if $x < -1/2$ then $f(x) > f(-1/2)$ while if $x > -1/2$ then $f(x) > f(-1/2)$. Consequently, $f(-1/2) = -e^{-1}$ is an absolute minimum for $f(x)$.

[1]Continuity allows us to extend our conclusions of monotonicity to the finite endpoints of both intervals

2.3 Part c

By our conclusion in Part b, above, the observation that $\lim_{x\to\infty} 2xe^{2x} = \infty$, and the continuity of f, we see that the range of f is $[-e^{-1}, \infty)$.

2.4 Part d

We put $f_b(x) = bxe^{bx}$, and we find that $f_b{}'(x) = (b + b^2x)e^{bx}$. If $b > 0$, we argue as in Parts a and b, above, and we find that f_b has an absolute minimum at $x = -1/b$. This minimum is $f_b(-1/b) = -e^{-1}$, which does not depend on b. If $b < 0$, we obtain the same result after the change of variables $u = -x$, which amounts to a reflection about the y-axis.

3 Problem 3

3.1 Part a

Acceleration is the derivative, taken with respect to time, of velocity, so acceleration is positive at each point where the tangent line to the graph of the velocity function has positive slope. From the picture given, we find that acceleration is positive on the intervals $[0, 35)$ and $(45, 50]$.

3.2 Part b

Taking $a(t)$ to be the acceleration at time t, average acceleration is

$$\frac{1}{50}\int_0^{50} a(t)\,dt = \frac{v(50) - v(0)}{50} = \frac{72 - 0}{50} = \frac{36}{25}\ \text{ft/sec}^2. \tag{11}$$

3.3 Part c

We have

$$a(40) \sim \frac{v(45) - v(35)}{45 - 35} = -\frac{21}{10}\ \text{ft/sec}^2. \tag{12}$$

3.4 Part d

$$\int_0^{50} v(t)\,dt \sim v(5)(10-0) + v(15)(20-10) + v(25)(30-20) + \\ v(35)(40-30) + v(45)(50-40) \quad (13)$$

$$\sim 4810 \text{ feet.} \quad (14)$$

The integral measures, in feet, the distance traveled during the time interval $0 \leq t \leq 50$.

4 Problem 4

4.1 Part a

At the point $(1, f(1))$, the slope is

$$\frac{3(1)^2+1}{2f(1)} = \frac{1}{2}. \quad (15)$$

4.2 Part b

The line tangent to the curve $y = f(x)$ has equation $y = 4 + \frac{1}{2}(x-1)$. Consequently, $f(1.2) \sim 4 + \frac{1}{2}(0.2) = 4.1$.

4.3 Part c

From the equation

$$f'(x) = \frac{3x^2+1}{2f(x)}, \quad (16)$$

together with $f(1) = 4$, we have $2f(x)f'(x) = 3x^2+1$, at least in some open interval I centered at $x = 1$, where, because f—as the solution to a differential equation—is continuous, we may assume that $f(x) > 0$. Thus, if x is any point of I, we have

$$2\int_1^x f(\xi)f'(\xi)\,d\xi = \int_1^x (3\xi^2+1)\,d\xi; \tag{17}$$

$$[f(\xi)]^2\Big|_1^x = \left(\xi^3+\xi\right)\Big|_1^x; \tag{18}$$

$$[f(x)]^2 - [f(1)]^2 = \left(x^3+x\right) - 2. \tag{19}$$

Replacing $f(1)$ with 4 and solving for $f(x)$, we find that

$$f(x) = \sqrt{x^3+x+14}, \tag{20}$$

where we have made use of the fact that we know that $f(x) > 0$.

4.4 Part d

$$f(1.2) = \sqrt{(1.2)^3+1.2+14} \sim 4.11436. \tag{21}$$

5 Problem 5

5.1 Part a

See Figure 1.

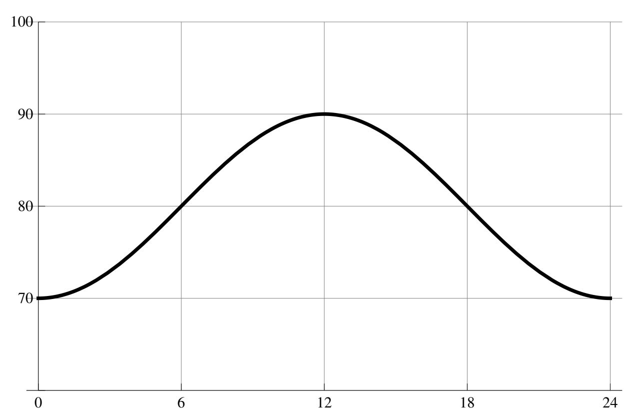

Figure 1: Problem 5, Part a

5.2 Part b

The average temperature is

$$\frac{1}{14-6}\int_6^{14}\left(80-10\cos\frac{\pi t}{12}\right)\,dt = \frac{1}{8}\left(80t-\frac{120}{\pi}\sin\frac{\pi t}{12}\right)\Bigg|_6^{14} \tag{22}$$

$$= \frac{1}{2}\left(160+\frac{45}{\pi}\right) \sim 87.16197^\circ. \tag{23}$$

To the nearest degree, this is 87°.

5.3 Part c

Examination of Figure 1 shows that an approximate answer is $5 \le t \le 19$. Numerical solution of the equation

$$78 = \left(80-10\cos\frac{\pi t}{12}\right) \tag{24}$$

yields $t \sim 5.24087$ for the left-hand solution, and $t \sim 18.76913$ for the right-hand solution. We conclude that the air condition ran when $5.24087 \le t \le 18.76913$. (Although, in this context, accuracy of more than a single digit to the right of the decimals is probably silly.)

5.4 Part d

The approximate total cost is

$$0.05\int_{5.24087}^{18.76913}\left(2-10\cos\frac{\pi t}{12}\right)\,dt \sim 5.09637. \tag{25}$$

We have evaluated the integral numerically, because we know the limits of integration only approximately and there is little point in trying for an "exact" integral. We note, for the record, that we can carry out the symbolic integration easily:

$$\int\left(2-10\cos\frac{\pi t}{12}\right)\,dt = 2t-\frac{120}{\pi}\sin\frac{\pi t}{12}. \tag{26}$$

6 Problem 6

6.1 Part a

From

$$2y^3 + 6x^2y - 12x^2 + 6y = 1, \tag{27}$$

by an implicit differentiation, treating y as a function of x, we obtain

$$6y^2y' + 12xy + 6x^2y' - 24x + 6y' = 0. \text{ Thus,} \tag{28}$$

$$\big(6y^2 + 6x^2 + 6)\big)y' = 24x - 12xy, \text{ or} \tag{29}$$

$$y' = \frac{4x - 2xy}{y^2 + x^2 + 1}, \tag{30}$$

it not being possible for the quantity $y^2 + x^2 + 1$ to vanish, because it is the sum of a positive number and two non-negative numbers.

6.2 Part b

Tangent lines are horizontal at points on a curve where $y' = 0$. For this curve, according to equation (30), that can happen only where $4x - 2xy = 0$, from which we conclude that $x = 0$ or $y = 2$.

If $x = 0$ then equation (27) becomes

$$2y^3 + 6y - 1 = 0, \tag{31}$$

and if y_0 is the only real solution to this equation, then $y_0 \sim 0.16590$. An equation for the tangent line through the point $(0, y_0)$ is then $y = y_0$.

If $y = 2$ then equation (27) becomes

$$16 + \cancel{12x^2} - \cancel{12x^2} + 12 = 1, \text{ or} \tag{32}$$

$$28 = 1, \tag{33}$$

which has no solution. We conclude that there are no horizontal tangent lines where $y = 2$.

6.3 Part c

If the line $y = -x$ is tangent to the curve, then, at the point of tangency, equation (27) becomes

$$-2x^3 - 6x^3 - 12x^2 - 6x = 1, \text{ or} \tag{34}$$
$$8x^3 + 12x^2 + 6x + 1 = 0, \text{ which, in turn, is} \tag{35}$$
$$(2x+1)^3 = 0. \tag{36}$$

There is but one solution: $x = -\frac{1}{2}$.

Because $y = -x$ at the point of tangency, we must then have $y = \frac{1}{2}$.

We conclude that the coordinates of the point of tangency are $\left(-\frac{1}{2}, \frac{1}{2}\right)$.