AP Calculus 1999 AB FRQ Solutions

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1 Problem 1

1.1 Part a

If $v(t) = t \sin t^2$, then $v(1.5) \sim 1.16711 > 0$. Motion is upward because $v(1.5) > 0$.

1.2 Part b

From $v(t) = t \sin t^2$, we find $v'(t) = 2t^2 \cos t^2 + \sin t^2$, when $a(1.5) = v'(1.5) \sim -2.04871$. Acceleration is negative, so velocity is decreasing.

1.3 Part c

By the Fundamental Theorem of Calculus,

$$\begin{aligned} y(t) &= y(0) + \int_0^t v(\tau)\, d\tau &(1)\\ &= y(0) + \int_0^t \tau \sin \tau^2 \, d\tau &(2)\\ &= y(0) - \frac{1}{2}\cos \tau^2 \bigg|_0^t = 3 + \frac{1}{2}(1 - \cos t^2). &(3) \end{aligned}$$

Thus, $y(2) \sim 3 + \frac{1}{2}(1 - \cos 4) \sim 3.82682$.

1.4 Part d

Total distance traveled during the interval $0 \le t \le 2$ is

$$\begin{aligned}
\int_0^2 |v(t)|\, dt &= \int_0^2 |t \sin t^2|\, dt && (4)\\
&= \int_0^{\sqrt{\pi}} t \sin t^2\, dt - \int_{\sqrt{\pi}}^2 t \sin t^2\, dt && (5)\\
&= -\frac{1}{2} \cos t^2 \Big|_0^{\sqrt{\pi}} + \frac{1}{2} \cos t^2 \Big|_{\sqrt{\pi}}^2 && (6)\\
&= -\frac{1}{2}(-1-1) + \frac{1}{2}(\cos 4 + 1) && (7)\\
&= \frac{3}{2} + \frac{1}{2} \cos 4 \sim 1.17318. && (8)
\end{aligned}$$

2 Problem 2

2.1 Part a

The area of the pictured region is

$$\int_{-2}^2 \left(4 - x^2\right)\, dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^2 = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}. \qquad (9)$$

2.2 Part b

Revolving the pictured region about the x-axis produces a solid whose volume is

$$\begin{aligned}
\pi \int_{-2}^2 \left(16 - x^4\right)\, dx &= \pi \left(16x - \frac{1}{5}x^5\right)\Big|_{-2}^2 && (10)\\
&= \pi\left(32 - \frac{32}{5}\right) - \pi\left(-32 + \frac{32}{5}\right) = \frac{256}{5}\pi && (11)
\end{aligned}$$

2.3 Part c

The required equation is

$$\pi \int_{-2}^2 \left[\left(k - x^2\right)^2 - (k-4)^2\right]\, dx = \frac{256}{5}\pi. \qquad (12)$$

Solution is not required. However, a tedious integration reduces the equation to

$$\frac{64}{3}k - \frac{256}{5} = \frac{256}{5}, \tag{13}$$

which easily gives $k = 24/5$.

3 Problem 3

3.1 Part a

The midpoint Riemann sum with 4 subdivisions of equal length gives

$$\int_0^{24} R(t)\, dt \sim 10.4 \times 6 + 11.2 \times 6 + 11.3 \times 6 + 10.2 \times 6 = 258.6. \tag{14}$$

This means that approximately 258.6 gallons of water flows out of the pipe during the time interval $0 \le t \le 24$.

3.2 Part b

The function R is given differentiable on $[0, 24]$, so it must also be continuous on that interval. Moreover, $R(0) = 9.6 = R(24)$. Thus, R meets the requirements of Rolle's Theorem, and we may conclude that there must be a time, t, with $0 < t < 24$, such that $R'(t) = 0$.

3.3 Part c

The average rate of flow is approximately

$$\frac{1}{24}\int_0^{24} \frac{1}{79}\left(768 + 23t - t^2\right)\, dt = \frac{1}{24} \cdot \frac{1}{79}\left(768t + \frac{23}{2}t^2 - \frac{1}{3}t^3\right)\Bigg|_0^{24} = \frac{852}{79} \text{ gallons/hour.} \tag{15}$$

4 Problem 4

4.1 Part a

An equation for the line tangent to the graph of f at the point where $x = 2$ is

$$y = f(0) + f'(0)(x - 0), \text{ or} \tag{16}$$
$$y = 2 - 3x. \tag{17}$$

4.2 Part b

We don't have enough information to decide whether f has an inflection point at $x = 0$. If $f(x) = 2 - 5x - 2\sin x$, then f satisfies the given conditions with $f''(x) = -2\sin x$, so that $f''(x)$ undergoes a sign change at $x = 0$, meaning that f has an inflection point at $x = 0$. However, if $f(x) = 3 - 3x - \frac{1}{2}x^2 - \cos x$, then this f also satisfies the given conditions. However $f''(x) = \cos x - 1$ vanishes when $x = 0$, but does not undergo a sign change at $x = 0$—meaning that f has no inflection point at $x = 0$.

4.3 Part c

An equation for the line tangent to the graph of g at the point where $x = 0$ is

$$y = g(0) + g'(0)(x - 0) = 4 + \left[e^0(3 \cdot 2 + 2(-3)\right](x - 0), \text{ or} \tag{18}$$
$$y = 4. \tag{19}$$

4.4 Part d

From $g'(x) = e^{-2x}[3f(x) + 2f'(x)]$, we have

$$g''(x) = e^{-2x}[3f'(x) + 2f''(x)] - 2e^{-2x}[3f(x) + 2f'(x)] \tag{20}$$
$$= 3e^{-2x}f'(x) + 2e^{-2x}f''(x) - 6e^{-2x}f(x) - 4e^{-2x}f'(x) \tag{21}$$
$$= e^{-2x}\left[-6f(x) - f'(x) + 2f''(x)\right]. \tag{22}$$

From what we are given regarding $g'(x)$ we see that $g'(0) = 3 \cdot f(0) + 2f'(0) = 3 \cdot 2 + 2 \cdot (-3) = 0$.

From (22), we see that $g''(0) = (-6) \cdot 2 - (-3) + 2 \cdot 0 = -9 < 0$.

By the Second Derivative Test, f has a local maximum at $x = 0$.

5 Problem 5

5.1 Part a

On the interval $[2, 4]$, the graph is symmetric about the point $(3, 0)$, so the integral over $[2, 4]$ is zero. Consequently,

$$\int_1^4 f(t)\,dt = \int_1^2 f(t)\,dt + \int_2^4 f(t)\,dt = \int_1^2 f(t)\,dt, \tag{23}$$

and the latter integral is the area of the trapezoid whose corners are $(1, 0)$, $(2, 0)$, $(2, 1)$, and $(1, 4)$, or

$$\frac{4+1}{2} \cdot 1 = \frac{5}{2}. \tag{24}$$

Thus,

$$g(4) = \frac{5}{2}, \text{ and} \tag{25}$$

$$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt \tag{26}$$

is the negative of the area of a triangle of base 3, height 4, or -6.

5.2 Part b

By the Fundamental Theorem of Calculus,

$$g'(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x). \tag{27}$$

Hence $g'(1) = f(1) = 4$.

5.3 Part c

The absolute minimum of $g(x)$ for $-2 \leq x \leq 4$ is to be found either, on the one hand, at one of the points $x = -2$ or $x = 4$, or, on the other hand, at a value of x where $-2 < x < 4$ and $g'(x) = 0$. As we have seen in Part a, above, $g(-2) = -6$, and $g(4) = 5/2$. If $g'(x) = 0$, then by our first observation in Part b, above, $f(x) = 0$. This happens only at $x = 3$. But f, and therefore g' undergoes a change of sign from positive to negative at $x = 3$, so, by the First Derivative Test, g must have a local maximum—which, because f is not a constant function cannot be an absolute minimum for f—at $x = 3$. We see, thus, that the absolute minimum for g on $[-2, 4]$ is $g(-2) = -6$.

5.4 Part d

If g is to have an inflection point somewhere, then g' must change from increasing to decreasing or from decreasing to increasing at that point. This happens when $x = 1$, but not when $x = 2$. So g has an inflection point at just one of the two points in question.

6 Problem 6

6.1 Part a

If $y = x^{-2}$, then $y' = -2x^{-3}$. So if w is any constant, the equation of the line tangent to the curve $y = x^{-2}$ at the point (w, w^{-2}) is

$$y = \frac{1}{w^2} - \frac{2}{w^3}(x - w) = -\frac{2}{w^3}x + \frac{3}{w^2}. \tag{28}$$

If $R\colon (k, 0)$ is the point where the tangent line at (w, w^{-2}) crosses the x-axis, then

$$0 = -\frac{2}{w^3}k + \frac{3}{w^2}, \text{ or} \tag{29}$$

$$k = \frac{3}{2}w. \tag{30}$$

Thus, $w = 3 \Rightarrow k = \dfrac{9}{2}$.

6.2 Part b

We saw in Part a, above, that, in general, $k = \dfrac{3}{2}w$.

6.3 Part c

Treating w and k as functions of time, t, we differentiate (30) implicitly with respect to t, and we obtain

$$\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt}. \tag{31}$$

Substituting $\dfrac{dw}{dt} = 7$ units per second and $w = 5$ into this equation, we obtain

$$\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2} \cdot 7 = \frac{21}{2} \text{ units/second.} \tag{32}$$

6.4 Part d

We have seen in Part b, above, that the tangent line to the curve at the point $P : (w, w^{-2})$ corresponding to $x = w$ crosses the x-axis at the point $R : (3w/2, 0)$. Because Q has coordinates $(w, 0)$, the area , A, of $\triangle PQR$ is

$$A = \frac{1}{2}\left(\frac{3}{2}w - w\right) \cdot \frac{1}{w^2} = \frac{1}{4w}. \tag{33}$$

Another implicit differentiation yields

$$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt}, \tag{34}$$

Substituting the given data for w and for $\frac{dw}{dt}$ into this equation then gives

$$\frac{dA}{dt} = -\frac{1}{4 \cdot 25} \cdot 7 = -\frac{7}{100} < 0, \tag{35}$$

and we can conclude that A, having a negative derivative at the moment in question, is decreasing at that time.

Note: The final numeric computation is not necessary. Because of the way the question is worded, it suffices to note that equation (34) guarantees that the derivatives $\frac{dA}{dt}$ and $\frac{dw}{dt}$ have opposite signs.