AP Calculus 2000 AB FRQ Solutions

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1 Problem 1

1.1 Part a

We must first locate the first intersection of the curves $y = e^{-x^2}$ and $y = 1 - \cos x$ to the right of the y-axis. Solving numerically for b, we find that the smallest positive value of b for which $e^{-b^2} = 1 - \cos b$ is $b \sim 0.94194$. A numerical integration then gives the area of the region R as

$$\int_0^b \left[e^{-x^2} - (1 - \cos x)\right] \, dx \sim 0.59096. \tag{1}$$

1.2 Part b

By the method of washers and a numerical integration, the volume generated when R is revolved about the x-axis is

$$\pi \int_0^b \left[e^{-2x^2} - (1 - \cos x)^2\right] \, dx \sim 1.74661. \tag{2}$$

1.3 Part c

Another numerical integration gives this volume as

$$\pi \int_0^b \left[e^{-x^2} - (1 - \cos x)\right]^2 \, dx \sim 0.46106. \tag{3}$$

2 Problem 2

2.1 Part a

From the graph of runner A's velocity, we see that her velocity at time $t = 2$ is $\frac{20}{3}$ meters per second. Runner B's velocity at time t is given as $\frac{24t}{2t+3}$, so runner B's velocity at time $t = 2$ is $\frac{24 \cdot 2}{2 \cdot 2 + 3} = \frac{48}{7}$ meters per second.

2.2 Part b

Acceleration is the derivative, taken with respect to time, of velocity. In the case of runner A, the slope of the velocity curve at time $t = 2$ is $\frac{10}{3}$, so her acceleration at time $t = 2$ is $\frac{10}{3}$ meters per second per second. For runner B, we have

$$\frac{d}{dt}\left(\frac{24t}{2t+3}\right)\bigg|_{t=2} = \frac{24(2t+3) - 24t \cdot 2}{(2t+3)^2}\bigg|_{t=2} = \frac{72}{(2t+3)^2}\bigg|_{t=2} = \frac{72}{49} \text{ meters/sec/sec.} \tag{4}$$

2.3 Part c

When velocity is non-negative, as in the circumstances of this problem, distance traveled is the integral of velocity. Hence, reasoning from the graph of runner A's velocity, we find that the distance, in meters, runner A covered during $0 \leq t \leq 10$ is the sum of the area of a triangle of base 3, altitude 10 and the area of a rectangle of base 7, altitude 10, or

$$\frac{1}{2} \cdot 3 \cdot 10 + 7 \cdot 10 = 85 \text{ meters} \tag{5}$$

During the same time interval, runner B covered

$$\begin{aligned}
\int_0^{10} \frac{24t}{2t+3}\, dt &= \int_0^{10} \left[\frac{24t+36}{2t+3} - \frac{36}{2t+3}\right] dt && (6)\\
&= \int_0^{10} \left[12 - \frac{36}{2t+3}\right] dt && (7)\\
&= \left[12t - 18\ln(2t+3)\right]\Big|_0^{10} = 120 + 18\ln\frac{3}{23} \sim 83.33612 \text{ meters.} && (8)
\end{aligned}$$

3 Problem 3

3.1 Part a

According to the First Derivative Test, a differentiable function attains a relative minimum at a point where its derivative changes sign from negative to positive as the independent variable increases. There is just one such point for the derivative whose graph is shown: $x = -1$. Consequently, f has a relative minimum at $x = -1$.

3.2 Part b

According to the First Derivative Test, a differentiable function attains a relative maximum at a point where its derivative changes sign from positive to negative as the independent variable increases. There is just one such point for the derivative whose graph is shown: $x = -5$. Consequently, f has a relative maximum at $x = -5$.

3.3 Part c

The second derivative is negative throughout intervals where f' has a tangent line that slopes downward to the right. There are three such intervals: $(-7, -3)$, $(2, 3)$, and $(3, 5)$. (Note that the point where $x = 3$, where the tangent to the f' curve is vertical, corresponds to a point where f'' is not defined.)

3.4 Part d

By the Fundamental Theorem of Calculus,

$$f(t) = f(-7) + \int_{-7}^{t} f'(\tau)\, d\tau. \tag{9}$$

Thus, $f(t)$ gives the algebraic sum of the signed areas bounded by the x-axis, the curve $y = f'(x)$, and the vertical lines $x = -7$ and $x = t$. (We observe the usual convention that area above the horizontal axis is positive area, while area below the horizontal axis is negative area.) It is evident from the picture that the maximal such area, taking signs into account, is that for which $t = 7$, so the absolute maximum occurs at $t = 7$.

4 Problem 4

4.1 Part a

Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, so the tank loses

$$\int_0^3 \sqrt{t+1}\,dt = \frac{2}{3}\left[t+1\right]^{3/2}\Big|_0^3 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \text{ gallons} \tag{10}$$

during the interval $0 \leq t \leq 3$.

4.2 Part b

During the interval $0 \leq t \leq 3$, a total of $\frac{14}{3}$ gallons have leaked from the tank, while $3 \cdot 8 = 24$ gallons have entered it. Because there were 30 gallons of water in the tank at time $t = 0$, the tank contains

$$30 + 24 - \frac{14}{3} = \frac{148}{3} \text{ gallons} \tag{11}$$

of water when $t = 3$.

4.3 Part c

The amount $A(t)$, in gallons, of water in the tank at time t is

$$A(t) = 30 + 8t - \int_0^t \sqrt{\tau+1}\,d\tau. \tag{12}$$

This can be (although the question by no means makes it clear if it is expected that whether will be) rewritten as

$$A(t) = \frac{92}{3} + 8t - \frac{2}{3}(t+1)^{3/2}. \tag{13}$$

5 Problem 5

5.1 Part a

If (x, y) is a point on the curve given by the equation $xy^2 - x^3y = 6$ and the equation defines y implicitly as a function of x near that point, then

$$\frac{d}{dx}\left(xy^2 - x^3y\right) = \frac{d}{dx}6; \tag{14}$$

$$y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0; \tag{15}$$

$$(2xy - x^3)\frac{dy}{dx} = 3x^2y - y^2; \tag{16}$$

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}, \tag{17}$$

as required.

5.2 Part b

If the point $(1, y)$ lies on the curve, then

$$1 \cdot y^2 - 1^3 \cdot y = 6, \text{ or} \tag{18}$$

$$y^2 - y - 6 = 0; \tag{19}$$

$$(y - 3)(y + 2) = 0. \tag{20}$$

There are, consequently, two such points: $(1, 3)$ and $(1, -2)$. We have

$$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{3 \cdot 1^2 \cdot 3 - 3^2}{2 \cdot 1 \cdot 3 - 1^3} = 0, \tag{21}$$

so that the line tangent to the curve at $(1, 3)$ has equation $y = 3$.

At $(1, -2)$, we have

$$\left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{3 \cdot 1^2 \cdot (-2) - (-2)^2}{2 \cdot 1 \cdot (-2) - (1)^3} = \frac{-10}{-5} = 2. \tag{22}$$

An equation for the line tangent to the curve at $(1, -2)$ is therefore $y = -2 + 2(x - 1)$, or $y = 2x - 4$.

5.3 Part c

At a point where the tangent to the curve is vertical, we can't expect that the equation defines y implicitly as a function of x, so differentiation with respect to x is meaningless. Therefore, we assume that the equation gives x as a function of y, and we carry out an implicit differentiation with respect to y:

$$\frac{d}{dy}\left(xy^2 - x^3y\right) = \frac{d}{dy}6; \tag{23}$$

$$2xy + y^2\frac{dx}{dy} - x^3 - 3x^2y\frac{dx}{dy} = 0; \tag{24}$$

$$\frac{dx}{dy} = \frac{x^3 - 2xy}{y^2 - 3x^2y}. \tag{25}$$

At a point with a vertical tangent, $\frac{dx}{dy}$ must vanish, so $x^3 - 2xy$ must be zero—that is $x = 0$ or $y = x^2/2$. But $xy^2 - x^3y = 6$, so $x = 0$ is not possible. if $y = x^2/2$, on the other hand, then

$$xy^2 - x^3y = 6 \text{ becomes} \tag{26}$$

$$x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6. \tag{27}$$

The only real solution for this equation is easily seen to be $x = -\sqrt[5]{24}$, and the corresponding point on the curve is $\left(-\sqrt[5]{24}, \sqrt[5]{18}\right)$.

We would like to conclude that the tangent line to the curve at the point $\left(-\sqrt[5]{24}, \sqrt[5]{18}\right)$ is vertical. Strictly speaking, we must check to be sure that the denominator on the right side of equation (25) doesn't vanish at this point before we may draw this conclusion, but the readers probably didn't care. (To see why this last step is necessary, consider the curve $y^2 = x^2$ at the origin.)

6 Problem 6

6.1 Part a

Let f be a solution of the initial value problem

$$\frac{dy}{dx} = \frac{3x^2}{e^{2y}}; \tag{28}$$

$$y(0) = \frac{1}{2}. \tag{29}$$

Then

$$f'(x) = \frac{3x^2}{e^{2f(x)}}, \text{so} \tag{30}$$

$$e^{2f(x)} f'(x) = 3x^2. \tag{31}$$

Hence,

$$\int_0^x e^{2f(\xi)} f'(\xi)\, d\xi = 3 \int_0^x \xi^2\, d\xi; \tag{32}$$

$$\frac{1}{2} e^{2f(\xi)} \Big|_0^x = \xi^3 \Big|_0^x; \tag{33}$$

$$\frac{1}{2} e^{2f(x)} - \frac{1}{2} e^{2f(0)} = x^3 \tag{34}$$

$$e^{2f(x)} = 2x^3 + e; \tag{35}$$

$$f(x) = \frac{1}{2} \ln(2x^3 + e). \tag{36}$$

6.2 Part b

The domain of the function f of Part a, above, is $\{x \in \mathbb{R} \colon x > -\sqrt[3]{e/2}\}$. The equation $\nu = 2x^3 + e$ has a solution for every $\nu > 0$, the range of f is $\mathbb{R}$.