AP Calculus 2001 AB FRQ Solutions

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1 Problem 1

1.1 Part a

We solve numerically for the smallest positive number b such that $2-b^3 = \tan b$, obtaining $b \sim 0.90216$. The area of the region R is then

$$\int_0^b \tan x \, dx + \int_b^{2^{1/3}} \left(2 - x^3\right) dx \sim 0.72934 \tag{1}$$

where we have integrated numerically.

1.2 Part b

The area of the region S is, integrating numerically again,

$$\int_0^b \left[2 - x^3 - \tan x\right] dx \sim 1.16054. \tag{2}$$

Note: The integral is elementary, but we know the upper limit only approximately, so there is little point in carrying out an exact integration. Of course,

$$\int \left[2 - x^3 - \tan x\right] dx = 2x - \frac{1}{4}x^4 + \log|\cos x|, \tag{3}$$

1.3 Part c

Using the method of washers, we find that the area of the solid generated by revolving the region S about the x-axis is

$$\pi \int_0^b \left[(2 - x^3)^2 - \tan^2 x\right] dx \sim 8.33182. \tag{4}$$

Once again, we have carried out the integration numerically.

Note: Again, the integral is elementary:

$$\int \left[(2 - x^3)^2 - \tan^2 x\right] dx = \int \left[5 - 4x^3 + x^6 - \sec^2 x\right] dx \tag{5}$$

$$= 5x - x^4 + \frac{1}{7}x^7 - \tan x. \tag{6}$$

But there is little point in doing the integral symbolically.

2 Problem 2

2.1 Part a

We have

$$W'(12) \sim \frac{W(15) - W(9)}{15 - 9} = \frac{21 - 24}{6} = -\frac{1}{2} \text{ degrees C/day}. \tag{7}$$

2.2 Part b

The required trapezoidal approximation to the average value is

$$\frac{1}{15 - 0} \cdot \frac{20 + 2 \cdot 31 + 2 \cdot 28 + 2 \cdot 24 + 2 \cdot 22 + 21}{2} \cdot 3 = \frac{251}{10}. \tag{8}$$

2.3 Part c

If P is given by

$$P(t) = 20 + 10te^{-t/3}, \tag{9}$$

then

$$P'(t) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}, \tag{10}$$

and

$$P'(12) = -30e^{-4} \sim -0.54947. \tag{11}$$

This means that, at the beginning of the twelfth day, the water temperature is decreasing at a rate of about 0.54947 degrees Celsius per day.

2.4 Part d

The required average value is

$$\frac{1}{15}\int_0^{15} P(t)\,dt \sim 25.75743 \text{ degrees Celsius.} \tag{12}$$

3 Problem 3

3.1 Part a

When t is near 2, the graph shows that acceleration is near 15 ft/sec^2. This is a positive number, so velocity is increasing in the vicinity of $t = 2$.

Note: We have phrased our answer this way because the phrase "increasing at $t = 2$" is not defined in most calculus textbooks. In this context, the term "increasing" applies only to functions on intervals.

3.2 Part b

The portion of the acceleration curve on the interval $6 \leq t \leq 12$ is symmetric, about the point $(6, 0)$, with the portion of the acceleration curve on the interval $(0, 6)$. Consequently, the integral of acceleration from 0 to 12 (which is total change in velocity over that interval) is zero. Thus, velocity at $t = 12$ is 55 feet per second.

3.3 Part c

The car's absolute maximum velocity for $0 \leq t \leq 18$ is 115 ft/sec, which is the velocity it attains when $t = 6$. Thereafter velocity decreases as long as acceleration is negative—that is, while $6 \leq t \leq 14$. Finally, it increases again while $14 \leq t \leq 18$. However, the area under the acceleration curve on the latter interval is smaller than the area between the acceleration curve and the t-axis on the interval $6 \leq t \leq 14$, so the total increase in velocity that accrues while $14 \leq t \leq 18$ does not balance out the total decrease that accrued while $6 \leq t \leq 14$.

This means that velocity attains its absolute maximum for $0 \leq t \leq 18$ when $t = 6$. We calculate this maximum value by finding the area of the trapezoid over the interval $0 \leq t \leq 6$, which is

$$\frac{2+6}{2} \cdot 15 = 60, \tag{13}$$

and adding the initial velocity, 55, to obtain a maximal velocity of 115 ft/sec.

3.4 Part d

The car never reaches a velocity of 0 ft/sec. In fact, the absolute minimum velocity attained by the car occurs when $t = 16$, and this velocity is the sum of 55 ft/sec, the area of the region above the t-axis in the interval $[0, 6]$, and the negative of the area of the region below the t-axis in the interval $[6, 16]$, or $55 + 60 - 105 = 10$ ft/sec.

4 Problem 4

4.1 Part a

If

$$h'(x) = \frac{x^2 - 2}{x} = \frac{\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)}{x}, \tag{14}$$

then $h'(x) = 0$ when $x = \pm\sqrt{2}$, so the graph of h has a horizontal tangent when $x = \pm\sqrt{2}$. We note that

- $h'(x) < 0$ for $x < -\sqrt{2}$;
- $h'(x) > 0$ for $-\sqrt{2} < x < 0$;

- $h'(x) < 0$ for $0 < x < \sqrt{2}$;
- $h'(x) > 0$ for $\sqrt{2} < x$.

Thus, by the First Derivative Test, h has a local minimum at $x = -\sqrt{2}$, and h has a local minimum at $x = \sqrt{2}$.

Note: The quantity $h'(0)$ is undefined, but $x = 0$ fails to be a critical point for h. This is because h itself need not be defined at $x = 0$.

4.2 Part b

We have

$$h''(x) = \frac{d}{dx}\left[x - 2x^{-1}\right] = 1 + \frac{2}{x^2}, \tag{15}$$

which is always positive—except, of course, when $x = 0$. Hence h is concave upward on $(-\infty, 0)$ and on $(0, \infty)$.

4.3 Part c

The equation of the line tangent to the graph of h at $x = 4$ is

$$6 = h(4) + h'(4)(x - 4), \text{ or} \tag{16}$$

$$y = (-3) + \frac{4^2 - 2}{4}(x - 4). \tag{17}$$

This can be rewritten as

$$y = \frac{7}{2}x - 17. \tag{18}$$

4.4 Part d

We have $h''(x) = 1 + 2x^{-2}$, so that $h''(x) > 1$ for all $x \neq 0$. Thus, h' is increasing on $[4, \infty)$, and $h'(x) > h(4) = 7/2$ for all $x > 4$. Consequently,

$$h(x) - x(4) = \int_4^x h'(\xi)\, d\xi > \int_4^x \frac{7}{2}\, d\xi = \frac{7}{2}(x - 4), \tag{19}$$

again for all $x > 4$. Thus, when $x > 4$, we have

$$h(x) > \frac{7}{2}(x-4) + h(4) = \frac{7}{2}x - 17. \tag{20}$$

But the right-hand side of (20) is just the right-hand side of the equation of the tangent line to h at $(4, -3)$ as given in (18). Thus, the line tangent to the graph of $y = h(x)$ at $x = 4$ lies below the graph of h for $x > 4$.

5 Problem 5

5.1 Part a

We have $f(x) = 4x^3 + ax^2 + bx + k$, so $f'(x) = 12x^2 + 2ax + b$ and $f''(x) = 24x + 2a$. But there is an inflection point at $x = -2$, so $0 = f''(-2) = -48 + 2a$. Thus, $a = 24$.

So $f'(x) = 12x^2 + 48x + b$, and, because of the local minimum at $x = -1$, it follows that $0 = f'(-1) = -36 + b$. Thus, $b = 36$.

We obtain $a = 24$ and $b = 36$. It follows that $f(x) = 4x^3 + 24x^2 + 36x + k$.

5.2 Part b

From Part a, above, we have $f(x) = 4x^3 + 24x^2 + 36x + k$, so

$$\begin{aligned} 32 &= \int_0^1 f(x)\,dx && (21) \\ &= \int_0^1 \left(4x^4 + 24x^2 + 36x + k\right)\,dx && (22) \\ &= \left(x^4 + 8x^3 + 18x^2 + kx\right)\Big|_0^1 = 27 + k, && (23) \end{aligned}$$

and it follows that $k = 5$.

6 Problem 6

6.1 Part a

If $y = f(x)$ passes through the point $(3, 1/4)$ and satisfies the equation

$$y' = y^2(6 - 2x), \tag{24}$$

then

$$y'\Big|_{(3,1/4)} = \left(\frac{1}{4}\right)^2 (6 - 2 \cdot 3) = 0, \tag{25}$$

and

$$y'' = 2yy'(6 - 2x) - 2y^2, \tag{26}$$

so that

$$y''\Big|_{(3,1/4)} = -\frac{1}{8}. \tag{27}$$

6.2 Part b

We have $f'(x) = \big[f(x)\big]^2(6-2x)$ and $f(3) = 1/4$. Therefore f, as the solution of a differential equation, is continuous on its domain and, in particular, $f(x)$ is positive in some open interval centered at $x = 3$. For x in that interval, we may write

$$\int_3^x \frac{f'(\xi)}{[f(\xi)]^2}\, d\xi = \int_3^x (6 - 2\xi)\, d\xi. \tag{28}$$

Making use of the facts that $f(3) = 1/4$ and that f remains non-zero throughout the interval in question, we carry out the integrations to find that

$$-\frac{1}{f(\xi)}\bigg|_3^x = \left(6\xi - \xi^2\right)\bigg|_3^x; \tag{29}$$

$$-\frac{1}{f(x)} + 4 = (6x - x^2) - (6 \cdot 3 - 3^2); \tag{30}$$

$$-\frac{1}{f(x)} = (6x - x^2) - 13; \tag{31}$$

$$f(x) = \frac{1}{x^2 - 6x + 13}. \tag{32}$$