AP Calculus 2002 AB (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The abscissa of the intersection point of the two curves shown in the diagram is the solution, a, of the equation

$$4 - a = \frac{a^3}{1 + a^2}, \text{ or, solving numerically,} \tag{1}$$

$$a \sim 1.48766. \tag{2}$$

The area of the region R is then given by

$$\int_0^a \left[(4 - 2x) - \frac{x^3}{1 + x^2} \right] dx \sim 3.21456, \tag{3}$$

where we have carried out the integration numerically.

Note: The integral is elementary:

$$\int \frac{x^3\,dx}{x^2 + 1} = \int \frac{x^3 + x - x}{x^2 + 1}\,dx \tag{4}$$

$$= \int \left[\frac{x\cancel{(x^2 + 1)}}{\cancel{x^2 + 1}} - \frac{x}{x^2 + 1} \right] dx \tag{5}$$

$$= \int x\,dx - \frac{1}{2} \int \frac{2x\,dx}{x^2 + 1} = \frac{1}{2} \left[x^2 - \ln(x^2 + 1) \right]. \tag{6}$$

But we know the upper limit of integration only approximately, so there is little point in spending the time necessary to find the "exact" integral.

1.2 Part b

By the method of washers, the required volume is

$$\pi \int_0^a \left[(4-2x)^2 - \frac{x^6}{(1+x^2)^2}\right] dx \sim 31.88487. \tag{7}$$

We have integrated numerically. The integration, although elementary, is tedious:

$$\int \frac{x^6\,dx}{(1+x^2)^2} = \int \left[x^2 - 2 + \frac{3x^2+2}{(1+x^2)^2}\right] dx \tag{8}$$

$$= \int \left[x^2 - 2 + \frac{3}{1+x^2} - \frac{1}{(1+x^2)^2}\right] dx. \tag{9}$$

All but the last of these integrals are fundamental (that is, ones that everyone who has completed a first calculus course should know how to carry out). We handle the last by making the substitution $x = \tan\theta$; $dx = \sec^2\theta\,d\theta$. This leads to

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} = \int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = \int \cos^2\theta\,d\theta \tag{10}$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{2}\theta + \sin\theta\cos\theta). \tag{11}$$

But if $\tan\theta = x$, then $\theta = \arctan x$, and

$$\sin\theta = \sin\arctan x = \frac{x}{\sqrt{1+x^2}}, \text{ and} \tag{12}$$

$$\cos\theta = \cos\arctan x = \frac{1}{\sqrt{1+x^2}}. \tag{13}$$

Thus,

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\arctan x + \frac{x}{1+x^2}. \tag{14}$$

1.3 Part c

The area of a cross section at $x = t$ is $\left[(4-2t) - \dfrac{t^2}{1+t^2}\right]^2$, so the required volume is

$$\int_0^a \left[(4-2t) - \frac{t^2}{1+t^2}\right]^2 dt \sim 8.99700. \tag{15}$$

Note: Again, we have integrated numerically.This integral, too, is elementary. But not many people would want to evaluate it symbolically by hand:

$$\left[(4-2t)-\frac{t^2}{1+t^2}\right]^2 = \left[3-2t-\frac{1}{1+t^2}\right]^2 \tag{16}$$

$$= 4t^2-12t+9-\frac{4t}{1+t^2}+\frac{6}{1+t^2}+\frac{1}{(1+t^2)^2}. \tag{17}$$

All but the last of these is a straightforward fundamental integration, and the last proceeds in the same way as the last of the integrals in the previous part of this problem.

2 Problem 2

2.1 Part a

If $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$, then P' is continuous on its domain, and $P'(9) \sim -0.64643 < 0$. This means that $P(t)$, the amount of pollutant in the lake at time t, is decreasing in the vicinity of $t = 0$.

Note: We phrase our answer this way because the phrase "increasing at a point" is almost never defined in elementary calculus texts. It is possible, in fact, for a function to have a derivative that is positive at a point but for the function to be increasing in no open interval centered at that point. This can't happen, however, if the derivative is continuous at the point in question.

2.2 Part b

The exponential $e^{-0.2\sqrt{t}}$ is decreasing on the interval $[0,\infty)$, so P' is an increasing function on that interval. Moreover, $P'(0) = -2$, while $\lim_{t\to\infty} P'(t) = 1$. It follows that P' has exactly one zero, and that P has exactly one critical point in the interval—where, moreover, P' changes sign from negative to positive. Consequently, P has an absolute minimum at this critical point. We find the critical point by solving $1 - 3e^{-0.2\sqrt{t_0}} = 0$ to obtain $t_0 = 25(\ln 3)^2 \sim 30.17372$.

The amount of pollutant in the lake is at a minimum when $t = t_0 \sim 30.17372$ days.

2.3 Part c

By the Fundamental Theorem of Calculus, the amount $P(t)$ of pollutant in the lake at time t is given by

$$P(t) = P(0) + \int_0^t P'(\tau)\, d\tau \tag{18}$$
$$= 50 + \int_0^t \left[1 - 3e^{-0.2\sqrt{\tau}}\right] d\tau. \tag{19}$$

Integrating numerically, we find that

$$P(t_0) = 50 + \int_0^{25(\ln 3)^2} \left[1 - 3e^{-0.2\sqrt{\tau}}\right] d\tau \tag{20}$$
$$\sim 35.10434 \text{ gallons.} \tag{21}$$

If the lake is considered safe when $P(t) \leq 40$, the lake is safe when the amount of pollutant reaches its minimum value of about 35.10424.

2.4 Part d

The linearization L at $t = 0$ has equation

$$L(t) = 50 + P'(0)t \tag{22}$$
$$= 50 - 2t. \tag{23}$$

Thus, the linearization predicts the arrival of safety when $50 - 2t = 40$, or when $t = 5$.

Note: Newton's Method, together with repeated numerical integration, gives the arrival of safety at $t \sim 10.16000$. The lake then remains safe until $t \sim 56.47974$, after which an unsafe condition will prevail.

3 Problem 3

3.1 Part a

See Figure 1.

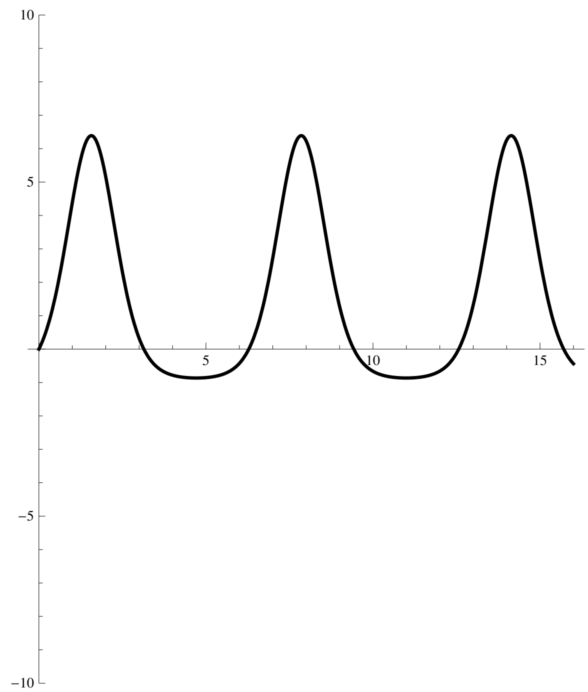

Figure 1: Problem 3, Part a

3.2 Part b

The particle moves to the left during those time-intervals when $v(t) < 0$—that is, where $e^{2\sin t} < 1$, which is to say where $\sin t < 0$. Thus, motion is leftward when $\pi < t < 2\pi$, when $3\pi < t < 4\pi$, and when $5\pi < t \leq 16$.

3.3 Part c

We compute distance traveled by integrating speed, which is the absolute value of velocity, over the time interval in question. The total distance traveled as t ranges from 0 to 4 is therefore

$$\int_0^4 |v(t)|\, dt = \int_0^\pi \left(e^{2\sin t} - 1\right)\, dt - \int_\pi^4 \left(e^{2\sin t} - 1\right)\, dt \sim 10.54247, \tag{24}$$

where we have carried out the integrations numerically—which we shall also do in Part d, below.

3.4 Part d

The particle moves to the right throughout the time interval $0 < t < \pi$. During that period it travels

$$\int_0^\pi v(t)\, dt \sim 10.10656, \tag{25}$$

and because its morion during this period is solely rightward, it doesn't return to the origin during this time period.

It then moves leftward throughout the period $\pi < t < 2\pi$. At time $t = 2\pi$, we find the particle

$$\int_0^{2\pi} v(t)\, dt \sim 8.03987 \tag{26}$$

units to the right of its initial position, and this is the leftmost point of its travel during that interval.

The particle, we conclude, begins at the origin but doesn't return to it again during the period $0 < t < 2\pi$.

During the succeeding interval of length 2π, that is, $[2\pi, 4\pi]$, the particle behaves as it did during the first interval of length 2π, but begins at a point just over 8 units to the right of the origin. It therefore doesn't return to the point where we found it at time $t = 2\pi$ during this second interval—let alone return even farther to the left to reach the origin. Similar reasoning shows that it doesn't return to the origin during any of the subsequent periods of the form $[2k\pi, 2(k+1)\pi]$, and therefore never returns to the origin.

4 Problem 4

4.1 Part a

If

$$g(x) = 5 + \int_6^x f(t)\,dt, \tag{27}$$

then $g(6) = 5$, because $\int_6^6 g(t)\,dt = 0$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so, according to the graph given in the statement of the problem, $g'(6) = 3$.

Also, $g''(x) = f'(x)$, and because it is given that the line tangent to the curve $y = f(x)$ at $x = 6$ is horizontal, we know that $f'(6) = 0$. Hence, $g''(6) = 0$.

4.2 Part b

We know that $g'(x) = f(x)$, and we also know that g is decreasing on any interval where g' is negative. Moreover, if a continuous function is decreasing on an open interval it is also decreasing on the closure of that interval. From the graph given, we see that $f(x) < 0$ on $[-3, 0)$ and on $(12, 15]$. We conclude that g is decreasing on $[-3, 0]$ and that g is decreasing on $[12, 15]$.

Notes:

- In recent history, the readers have not cared about the endpoints of intervals of monotonicity.
- We must not conclude that g is decreasing on $[-3, 0] \cup [12, 15]$. In fact, values of $g(u)$ will be negative when $u \in [-3, 0]$, and consequently smaller than any of the positive values that $g(v)$ assumes for each $v \in [12, 15]$.

4.3 Part c

A function is concave downward on any open interval where its derivative is decreasing. But $g'(x) = f(x)$ is decreasing on $[6, 15]$, and nowhere else. So g is concave downward on $(6, 15)$. Whether to include the endpoints of this interval is highly dependent upon the definition one chooses for the term "concave downward". Several distinct definitions appear in different textbooks, so the choice should not affect scoring.

4.4 Part d

The required trapezoidal approximation is

$$\int_{3}^{15} f(t)\,dt \sim \frac{3}{2}\big[f(-3) + 2f(0) + 2f(3) + 2f(6) + 2f(9) + 2f(12) + f(15)\big] \tag{28}$$

$$\sim \frac{3}{2}\big[(-1) + 0 + 2 + 6 + 2 + 0 + (-1)\big] = 12. \tag{29}$$

5 Problem 5

5.1 Part a

If the line $y = -2$ is tangent to the solution curve $y = f(x)$ to the differential equation

$$y' = \frac{3 - x}{y}, \tag{30}$$

then at the point of tangency we must have $y' = 0$, whence $3 - x = 0$, which means that $x = 3$. Because the y-coordinate of the point of tangency is $y = -2$ and solutions of differential equations are continuous on their domains, there is then an open interval I centered at $x = 3$ and throughout which $f(x) < 0$. We may assume that I does not contain zero. If $x \in I$ and $x < 3$, then $y' = (3 - x)/y < 0$ because $3 - x > 0$ and $y < 0$. If, on the other hand, $x \in I$ and $x > 3$, then $y' = (3 - x)/y > 0$ because $3 - x < 0$ and $y < 0$. Consequently, $x = 3$ gives a critical point for f, and $f'(x) < 0$ for x immediately to the left of $x = 3$ but $f'(x) > 0$ for x immediately to the right of $x = 3$. It follows from the First Derivative Test that f has a local minimum at $x = 3$.

5.2 Part b

If $y = g(x)$ is a solution to $y' = (3 - x)/y$ for which $g(6) = -4$, then

$$g'(x) = \frac{3 - x}{g(x)}, \text{ so} \tag{31}$$

$$g(x)g'(x) = 3 - x. \tag{32}$$

Integrating both sides of this latter equation from 6 to x, we have

$$\int_6^x g(\xi)g'(\xi)\,d\xi = \int_6^x (3-\xi)\,d\xi; \tag{33}$$

$$\frac{\cancel{1}}{\cancel{2}}\big[g(\xi)\big]^2\Big|_6^x = -\frac{\cancel{1}}{\cancel{2}}(3-\xi)^2\Big|_6^x; \tag{34}$$

$$\big[g(x)\big]^2 - \big[-4\big]^2 = -(3-x)^2 + 9; \tag{35}$$

$$\big[g(x)\big]^2 = 16 + 6x - x^2. \tag{36}$$

Now $g(6) = -4$, and, again by continuity, $g(x)$ must have constant sign throughout some neighborhood of $x = 6$. Consequently, we choose the negative square root, and we write the solution:

$$g(x) = -\sqrt{16 + 6x - x^2}. \tag{37}$$

Note: We can solve this differential equation, but with the initial condition $f(3) = -2$, in the same way, leading to $f(x) = -\sqrt{-(x^2 - 6x + 5)}$. This gives

$$f'(x) = \frac{x-3}{\sqrt{-x^2 + 6x - 5}}. \tag{38}$$

Then we can solve Part a of this problem by applying either the First Derivative Test or the Second Derivative Test at the critical point $x = 3$. The First Derivative Test is more efficient.

6 Problem 6

6.1 Part a

If $D(t)$ denotes the distance between the two ships at time t, then, by the Pythagorean Theorem,

$$\big[D(t)\big]^2 = \big[x(t)\big]^2 + \big[y(t)\big]^2. \tag{39}$$

If $x(t_0) = 4$ km and $y(t_0) = 3$ km, this gives $D(t_0) = 5$ km.

6.2 Part b

Differentiating equation (39) with respect to t gives

$$\cancel{2}D(t)D'(t) = \cancel{2}x(t)x'(t) + \cancel{2}y(t)y'(t), \text{ whence} \tag{40}$$

$$D'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{D(t)}. \tag{41}$$

But we know that $x'(t_0) = -15$ km/hr and $y'(t_0) = 10$ km/hr. Thus,

$$D'(t_0) = \frac{4 \cdot (-15) + 3 \cdot (10)}{5} = -6 \text{ km/hr}. \tag{42}$$

6.3 Part c

Let $\theta(t)$ denote the angle indicated. Then

$$\theta(t) = \arctan \frac{y(t)}{x(t)}, \text{ so} \tag{43}$$

$$\theta'(t) = \frac{1}{1 + [y(t)/x(t)]^2} \cdot \frac{x(t)y'(t) - x'(t)y(t)}{[x(t)]^2} \tag{44}$$

$$= \frac{x(t)y'(t) - x'(t)y(t)}{[x(t)]^2 + [y(t)]^2} = \frac{x(t)y'(t) - x'(t)y(t)}{[D(t)]^2} \tag{45}$$

Substituting our data for the time $t = t_0$, we find that

$$\theta'(t_0) = \frac{4 \cdot 10 - (-15) \cdot 3}{25} = \frac{85}{25} = \frac{17}{5} \text{ radians/hr}. \tag{46}$$