AP Calculus 2002 AB FRQ Solutions

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1 Problem 1

1.1 Part a

See Figure 1 for a plot of the region given in this problem.

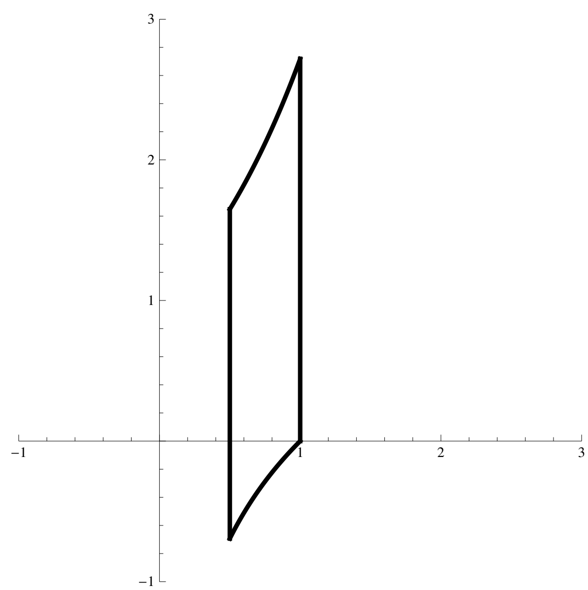

Figure 1: The region of Problem 1, Part a

The area of this region is

$$\int_{1/2}^{1} (e^x - \ln x)\ dx = (e^x + x - x\ln x)\Big|_{1/2}^{1} \tag{1}$$

$$= (e + 1 - 1\cdot 0) - \left(e^{1/2} + \frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \tag{2}$$

$$= e - e^{1/2} + \frac{1}{2} - \ln\sqrt{2} \sim 1.22299. \tag{3}$$

1.2 Part b

We find, using the method of washers, that the required volume is

$$\pi \int_{1/2}^{1} \left[(4 - \ln x)^2 - (4 - e^x)^2\right] dx \sim 23.60949. \tag{4}$$

The integral is elementary, but tedious—and it requires repeated integrations by parts. We have saved time by carrying out the integration numerically.

1.3 Part c

First, we seek the critical points of $h(x) = e^x - \ln x$ in $(1/2, 2)$ together with the singularities of h' in the same interval. Now $h'(x) = e^x - x^{-1}$, which is defined for all $x \in (0, 1)$. Numeric solution of the equation $e^x - x^{-1} = 0$ gives one critical point in $(1/2, 1)$ at $x = x_1 \sim 0.56714$.

We know that the absolute extrema of $h(x)$ in $[1/2, 1]$ occur at one of the points $x = 1/2$, $x = x_1$, or $x = 1$. We have

$$h(1/2) \sim 2.34187; \tag{5}$$

$$h(x_1) \sim 2.33037; \tag{6}$$

$$h(1) = e \sim 2.71828. \tag{7}$$

$$\tag{8}$$

The absolute minimum value of $h(x)$ on $[1/2, 1]$ is $h(x_1) \sim 2.33037$, and the absolute maximum value of $h(x)$ on $[1/2, 1]$ is $h(1) = e$.

2 Problem 2

2.1 Part a

The number of people who have entered the park by the time $t = 17$ is

$$\int_9^{17} \frac{15600\,dt}{t^2 - 24t + 160} \sim 6004.27032. \tag{9}$$

To the nearest whole number, this is 6004.

Note: The integral is elementary, but requires a technique that students of AB calculus may not have seen before:

$$\int \frac{dt}{t^2 - 24t + 160} = \int \frac{dt}{(t-12)^2 + 16} \tag{10}$$

$$= \frac{1}{4} \arctan \frac{t-12}{4} \Big|_9^{17} \tag{11}$$

$$\tag{12}$$

2.2 Part b

Revenue is given by

$$15 \int_9^{17} \frac{15600\,dt}{t^2 - 24t + 160} + 11 \int_{17}^{23} \frac{15600\,dt}{t^2 - 24t + 160} \sim 104048.16523. \tag{13}$$

To the nearest dollar, this is $\$104,048$.

2.3 Part c

By the Fundamental Theorem of Calculus,

$$H'(t) = \frac{15600}{t^2 - 24t + 160} - \frac{9890}{t^2 - 38t + 370}, \text{ so that} \tag{14}$$

$$H'(17) = -\frac{202690}{533} \sim 380.28143. \tag{15}$$

$H(t)$ gives the number of people in the park at time t, where $9 \leq t \leq 23$. Thus, $H(17) \sim 3725$ gives the number of people in the park at 5:00 pm. $H'(t)$ gives the rate at which the number of people in the park is increasing at time t, again for $9 \leq t \leq 23$. $H'(17) \sim -380$ means that at 5:00 pm the number of people in the park is decreasing at the rate of 380 people per hour.

2.4 Part d

As we have seen in Part c, above,

$$H'(t) = \frac{15600}{t^2 - 24t + 160} - \frac{9890}{t^2 - 38t + 370}. \tag{16}$$

See Figure 2 for a plot of $H'(t)$.

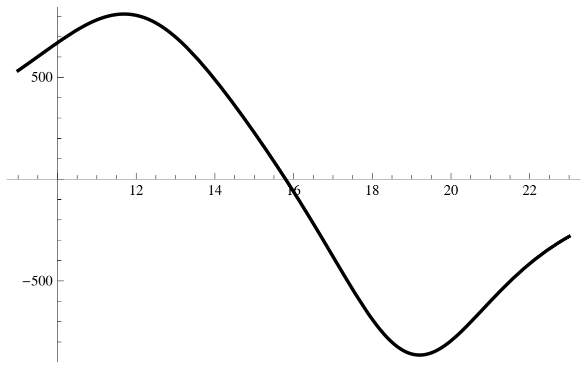

Figure 2: Problem 2, Part d: Plot of $H'(t)$

We see from the plot that there is a value t_0 near $t = 16$ for which $H'(t_0) = 0$, that $H'(t) > 0$ for $t < t_0$, and that $H'(t) < 0$ for $t > t_0$. So $H(t_0)$ must be the absolute maximum for $H(t)$, because $H(t)$ is increasing on $[9, t_0]$ and decreasing on $[t_0, 23]$. Setting $H(t) = 0$ and solving numerically gives $t_0 \sim 15.79481$.

We conclude that the model predicts the maximal number of people in the park just before 4:00 pm—when $t \sim 15.79481$, which is about 3:48 pm.

3 Problem 3

3.1 Part a

If velocity, $v(t)$ is given by

$$v(t) = \sin \frac{\pi t}{3}, \tag{17}$$

then acceleration $a(t)$ is given by

$$a(t) = v'(t) = \frac{d}{dt} \sin \frac{\pi t}{3} = \frac{\pi}{3} \cos \frac{\pi t}{3}. \tag{18}$$

Thus

$$a(4) = \frac{\pi}{3} \cos \frac{4\pi}{3} = -\frac{\pi}{6}. \tag{19}$$

The problem gives no units, so neither do we.

3.2 Part b

If $3 < t < 4.5$, then $\pi < \dfrac{\pi t}{3} < \dfrac{3\pi}{2}$, so that $\cos \dfrac{\pi t}{3}$ and $\sin \dfrac{\pi t}{3}$ are both negative.. Thus, for such t, $v(t) < 0$ and $v'(t) < 0$. It follows from the inequality $v'(t) < 0$ that $v(t)$ is decreasing on the interval $(3, 4.5)$.

Speed, $\sigma(t) = |v(t)|$ satisfies

$$[\sigma(t)]^2 = [v(t)]^2 \text{, so} \tag{20}$$

$$\not 2\, \sigma(t) \cdot \sigma'(t) = \not 2\, v(t) \cdot v'(t). \text{ Thus, when } \sigma(t) \neq 0, \tag{21}$$

$$\sigma'(t) = \frac{v(t)}{\sigma(t)} v'(t) = \frac{v(t)}{|v(t)|} v'(t). \tag{22}$$

We have seen above that for $t \in (3, 4.5)$, both $v(t)$ and $v'(t)$ are negative. It now follows from (22) that $\sigma'(t)$ is positive on $(3, 4.5)$, and from this that $\sigma(t)$ is increasing on that interval.

Statement I and Statement II are both correct.

3.3 Part c

Total distance is the integral of speed, or

$$\int_0^4 \sigma(t)\, dt = \int_0^4 |v(t)|\, dt \tag{23}$$

$$= \int_0^4 \left| \sin \frac{\pi t}{3} \right| dt \tag{24}$$

$$= \int_0^3 \sin \frac{\pi t}{3}\, dt - \int_3^4 \sin \frac{\pi t}{3}\, dt \tag{25}$$

$$= -\frac{3}{\pi} \cos \frac{\pi t}{3} \Big|_0^3 + \frac{3}{\pi} \cos \frac{\pi t}{3} \Big|_3^4 \tag{26}$$

$$= -\frac{3}{\pi}(-1 - 1) + \frac{3}{\pi} \left[-\frac{1}{2} - (-1) \right] = \frac{15}{2\pi} \sim 2.38743. \tag{27}$$

3.4 Part d

If $x(t)$ denotes position at time t, and $x(0) = x_0 = 2$, the Fundamental Theorem of Calculus tells us that

$$
\begin{aligned}
x(t) &= x(0) + \int_0^t v(\tau)\, d\tau && (28)\\
&= 2 + \int_0^t v(\tau)\, d\tau && (29)\\
&= 2 + \int_0^t \sin\frac{\pi\tau}{3}\, d\tau && (30)\\
&= 2 - \frac{3}{\pi}\cos\frac{\pi\tau}{3}\Big|_0^t && (31)\\
&= 2 + \frac{3}{\pi} - \frac{3}{\pi}\cos\frac{\pi t}{3}. && (32)
\end{aligned}
$$

It now follows that

$$x(4) = 2 + \frac{3}{\pi} - \frac{3}{\pi}\cos\frac{4\pi}{3} = 2 + \frac{9}{2\pi} \sim 3.43239. \tag{33}$$

4 Problem 4

4.1 Part a

The function g is given by

$$g(x) = \int_0^x F(t)\, dt \tag{34}$$

Thus, $g(-1)$ is the negative of the area of a triangle of base 1 and height 3, or $\frac{3}{2}$.

The value $g'(-1)$ is, by the Fundamental Theorem of Calculus, $f(-1) = 0$.

The value $g''(-1)$ is, by the Fundamental Theorem of Calculus again, $f'(-1)$. But in the vicinity of $x = -1$, the graph of the function f is a straight line of slope 3, so $g''(-1) = f'(-1) = 3$.

4.2 Part b

The function g is increasing on the closures of intervals where $g'(x) = f(x) > 0$. Thus g is increasing on $[-1, 1]$.

Note: It can be shown that a function which is continuous on an interval $[a, b]$ and increasing on (a, b) must also be increasing on $[a, b]$, so even though $g'(-1) = 0 = g'(1)$, the function g is nevertheless increasing on $[-1, 1]$. In the past, the readers have nevertheless accepted $(-1, 1)$ for the answer to a question such as this one.

4.3 Part c

As we have indicated above, $g''(x) = f'(x)$. We know that

$$f'(x) = \begin{cases} 3 & \text{for } -2 < x < 0 \\ \\ -3 & \text{for } 0 < x < 2 \end{cases} \tag{35}$$

Thus g is concave downward on the interval $(0, 2)$, where $g''(x) = f'(x) < 0$.

Note: Calculus textbooks vary in which of several inequivalent definitions of concavity they give. The question of whether to include endpoints of these intervals depends on which of these definitions we choose—and upon a careful reading of the definition we pick.

4.4 Part d

A pair of integrations shows that

$$g(x) = \begin{cases} 3x + \dfrac{3}{2}x^2 & \text{for } -2 \le x \le 0 \\ \\ 3x - \dfrac{3}{2}x^2 & \text{for } 0 < x \le 2. \end{cases} \tag{36}$$

See Figure 3 for the required graph. (We could have solved Parts a–c using this alternate description of g.)

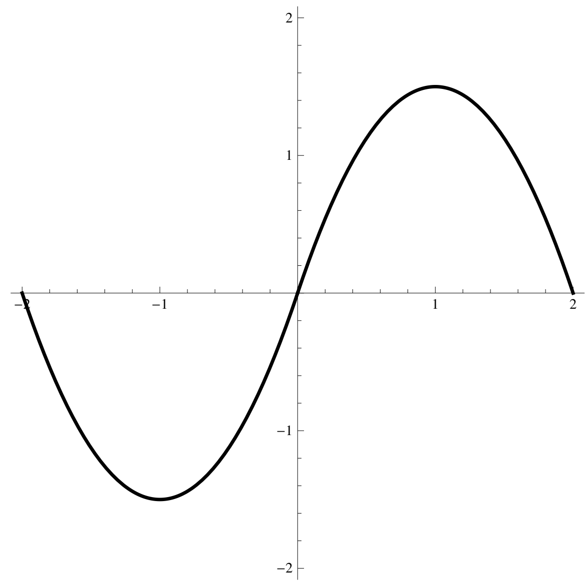

Figure 3: Graph for Problem 4, part d

5 Problem 5

5.1 Part a

By similar triangles, $r = h/2$. Thus, when $h = 5$,

$$v = \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi \left(\frac{5}{2}\right)^2 \cdot 5 = \frac{125}{12}\pi \text{ cm}^3. \tag{37}$$

5.2 Part b

For all t,

$$V(t) = \frac{1}{3}\pi \left[r(t)\right]^2 h(t) = \frac{1}{12}\pi \left[h(t)\right]^3. \tag{38}$$

Therefore, it being given that $h'(t) \equiv -\frac{3}{10}$,

$$V'(t) = \frac{1}{4}\pi \left[h(t)\right]^2 h'(t) = -\frac{3}{40}\pi \left[h(t)\right]^2, \tag{39}$$

and

$$V'(5) = -\frac{3}{40}\pi \cdot 25 = -\frac{15}{8}\pi \text{ cm}^3/\text{hr}. \tag{40}$$

5.3 Part c

As we noted above, $h(t) = 2r(t)$. thus

$$V'(t) = -\frac{3}{40}\pi \left[h(t)\right]^2 = -\frac{3}{10}\pi \left[r(t)\right]^2. \tag{41}$$

But if $A(t)$ is the exposed area of the water's surface, then $A(t) = \pi \left[r(t)\right]^2$, and it follows that

$$V'(t) = -\frac{3}{10}\pi \left[r(t)\right]^2 = \frac{3}{10}A(t), \tag{42}$$

so that $V'(t)$, the rate of change of the volume of water in the container due to evaporation, is directly proportional to the exposed surface area of the water, with constant of proportionality $-\frac{3}{10}$.

6 Problem 6

6.1 Part a

By the Fundamental Theorem of Calculus,

$$\int_0^{1.5} \left[3f'(x) + 4\right] dx = \left[3f(x) + 4x\right] \Big|_0^{1.5} \tag{43}$$

$$= \left[3f(1.5) + 4 \cdot (1.5)\right] - \left[3f(0) + 4 \cdot 0\right] \tag{44}$$

$$= (-3 + 6) - (-21 + 0) = 24. \tag{45}$$

6.2 Part b

An equation of the tangent line at (x_0, y_0) is

$$y = y_0 + f'(x_0)(x - x_0), \tag{46}$$

so the required line has equation $y = -4 + 5(x - 2)$.

We obtain an approximate value for $f(1.2)$ by substituting $x = 1.2$ into the equation for the tangent line:

$$f(1.2) \sim 5 \cdot (1.2) - 9 = -3. \tag{47}$$

We know that $f'(x) > 0$ for all values of x in the region of interest, so the curve $y = f(x)$ is concave upward in that region, and (at least locally) the tangent line lies below the curve. Consequently, our estimate of -3 for $f(1.2)$ is an underestimate: $f(1.2) > -3$.

6.3 Part c

We know that $f''(x)$ exists for $x \in [-1.5, 1.5]$. Consequently, f' is continuous on $[-1.5, 1.5]$ and satisfies the hypotheses of the Mean Value Theorem. So there must be a number c strictly between $x = 0$ and $x = 0.5$ such that

$$f''(c)(0.5 - 0) = f'(0.5) - f'(0) = -3 - 0 = -3, \tag{48}$$

or $f''(c) = 6$. The required number is $r = 6$.

6.4 Part d

It is not possible that f and g are the same function. As we have seen in Part c, above, f' must be continuous throughout $[-1.5, 1.5]$ because we know that $f''(x)$ exists at every point of that interval. However, $g'(x) = 4x - 1$ when $x < 0$, and $g'(x) = 4x + 1$ when $x < 0$. Thus

$$\lim_{x \to 0^+} g'(x) = 1, \text{ but} \tag{49}$$

$$\lim_{x \to 0^-} g'(x) = -1. \tag{50}$$

This means that g' has a discontinuity at $x = 0$. But we know that f' is continuous throughout, $[-1.5, 1.5]$, and the contradiction shows that f and g are not identical.