AP Calculus 2003 AB (Form B) FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

July 30, 2017

1 Problem 1

1.1 Part a

If $f(x) = 4x^2 - x^3$ and $g(x) = 18 - 3x$, then the curves have an intersection in the first quadrant where $x = 3$. Setting $f(x) = g(x)$ we find that $x^3 - 4x^2 - 3x + 18 = 0$. We note that $x = 3$ is a solution of this equation, and that $f(3) = g(3) = 9$. Moreover, $f'(x) = 8x - 3x^2$, and thus, $f'(3) = -3$, which is precisely the slope of the line $y = 18 - 3x = g(x)$. It follows that the line $y = 18 - 3x$ is the tangent line to the graph of $y = f(x)$ at the point $x = 3$.

1.2 Part b

The solutions of the equation $f(x) = 0$ are $x = 0$ and $x = 4$. The solution of the equation $18 - 3x = 0$ is $x = 6$

The region R extends horizontally from $x = 3$ on the left to $x = 6$ on the right, so the area, A_R of R is given by

$$A_R = \int_3^4 \left[(18 - 3x) - \left(4x^2 - x^3\right)\right] dx + \int_4^6 (18 - 3x)\, dx \qquad (1)$$

Now

$$\int_3^4 \left[(18-3x)-\left(4x^2-x^3\right)\right] dx = \left[18x-\frac{3}{2}x^2-\frac{4}{3}x^3+\frac{1}{4}x^4\right]\Big|_3^4 \tag{2}$$

$$= \left(72-24-\frac{256}{3}+64\right)-\left(54-\frac{27}{2}-36+\frac{81}{4}\right) \tag{3}$$

$$= \frac{80}{3}-\frac{99}{4}=\frac{23}{12}\sim 1.91667, \tag{4}$$

while

$$\int_4^6 (18-3x)\, dx = \left(18x-\frac{3}{2}x^2\right)\Big|_4^6 \tag{5}$$

$$= (108-54)-(72-24)=54-48=6. \tag{6}$$

Thus

$$A_R = \frac{23}{12}+6=\frac{95}{12}\sim 7.91667. \tag{7}$$

1.3 Part c

The curve $y = 4x^2 - x^3$ intersects the x-axis, as we have seen in Part a, above, at $x = 0$ and at $x = 4$. thus, the volume generated when the region R is revolved about the x-axis is

$$\pi \int_0^4 \left(4x^2-x^3\right)^2 dx = \pi \int_0^4 \left(16x^4-8x^5+x^6\right) dx \tag{8}$$

$$= \pi\left(\frac{16}{5}x^5-\frac{4}{3}x^6+\frac{1}{7}x^7\right)\Big|_0^4 = \frac{16384}{105}\pi \sim 490.20813. \tag{9}$$

Remark: This is a calculator-active problem, and we can save time by doing the integrations of Parts b and c numerically.

2 Problem 2

Oil is pumped into the tank at $H(t)$ gallons per hour and is removed at the rate $R(t)$ gallons per hour for $0 \le t \le 12$, where H and R are given by

$$H(t) = 2+\frac{10}{1+\ln(1+t)}; \tag{10}$$

$$R(t) = 12\sin\frac{t^2}{47}. \tag{11}$$

2.1 Part a

The amount pumped into the tank during the time interval $0 \leq t \leq 12$ is given by

$$\int_0^{12} H(t)\,dt = \int_0^{12} \left[2 + \frac{10}{1+\ln(t+1)}\right]\,dt \sim 70.57086 \text{ gallons}, \tag{12}$$

where we have carried out the integration numerically.

2.2 Part b

At time t, the rate of change of volume of oil in the tank is $H(t) - R(t)$. At time $t = 6$, this is

$$H(6) - R(6) = 2 + \frac{10}{1+\ln 7} - 12\sin\frac{36}{47} \tag{13}$$

$$\sim -2.9419 \text{ gallons per hour.} \tag{14}$$

The level of oil in the tank is falling when $t = 6$ because the rate of change of volume is negative at that time.

2.3 Part c

We are given that there were 125 gallons of oil in the tank when $t = 0$. Thus, the volume $V(t)$ of oil in the tank at time t is, by the Fundamental Theorem of Calculus,

$$V(t) = 125 + \int_0^t [H(\tau) - R(\tau)]\,d\tau \tag{15}$$

$$= 125 + \int_0^t \left[2 + \frac{10}{1+\ln(1+\tau)} - 12\sin\frac{\tau^2}{47}\right]\,d\tau. \tag{16}$$

Integrating numerically, we find that the volume of oil in the amount tank at $t = 12$ is

$$V(12) = 25 + \int_0^{12} [H(\tau) - R(\tau)]\,d\tau \tag{17}$$

$$= 125 + \int_0^t \left[2 + \frac{10}{1+\ln(1+\tau)} - 12\sin\frac{\tau^2}{47}\right]\,d\tau \sim 122.02571 \text{ gallons.} \tag{18}$$

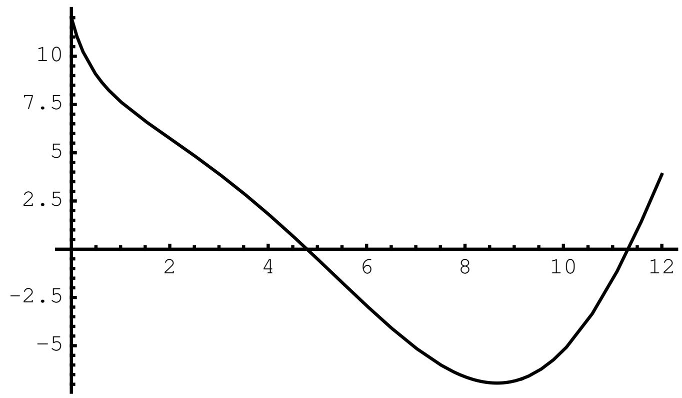

Figure 1: Plot of $H(t) - R(t)$ (Problem 2, Part d)

2.4 Part d

See Figure 1 for a plot of the rate at which the volume of the oil in the tank changes.

This rate is positive, and the volume $V(t)$ increases, from $t = 0$ until about $t = 5$. The rate is negative, and the volume decreases from about $t = 5$ until a little after $t = 11$. Thereafter, the volume increases. This means that the minimum amount of oil in the tank occurs when $t = 0$, or when $t = 12$, or when t is the value near 11 for which the plot crosses the t-axis. (We rule out the value of t near 5 because it gives a local maximum—which can't be a global minimum—for V.)

Solving numerically, we find that $t_0 \sim 11.31847$ gives the zero of $H(t) - V(t)$ near 11.

Using (16), and carrying out the required integrations numerically, we find that

$$V(0) = 125; \tag{19}$$

$$V(t_0) \sim 120.73818; \tag{20}$$

$$V(12) \sim 122.02571. \tag{21}$$

We conclude that the volume of oil in the tank is minimal when $t \sim 11.31847$.

3 Problem 3

3.1 Part a

Average radius is $\dfrac{1}{720}\displaystyle\int_0^{360} B(x)\,dx$.

3.2 Part b

The required midpoint Riemann sum is

$$\frac{1}{720}\big[20\cdot(120-0)+30\cdot(240-120)+24\cdot(360-240)\big]=14 \tag{22}$$

3.3 Part c

The integral $\pi\int_{125}^{275}\left[\frac{B(x)}{2}\right]^2 dx$ gives the volume, in cubic centimeters, of the segment of the blood vessel that extends from $x = 125$ mm to $x = 275$ mm.

3.4 Part d

The function B is given twice differentiable, so if $0 \le a < b \le 360$, then B is continuous on $[a, b]$ and differentiable on (a, b). Also, B' is continuous on $[a, b]$ and differentiable on (a, b). (We interpret continuity and differentiability at an endpoint of an interval in the appropriate one-sided sense.) Consequently, Rolle's Theorem can be applied to B on any interval, $0 \le a < b \le 360$ for which $B(a) = B(b)$. Similarly for B'.

We have $B(60) = B(180)$. By Rolle's Theorem, then, there is a number $\xi_1 \in (60, 180)$ for which $B'(\xi_1) = 0$. We also have $B(240) = B(360)$, so—again by Rolle's Theorem—there is a number $\xi_2 \in (240, 360)$ such that $B'(\xi_2) = 0$. It is clear that $\xi_1 < \xi_2$ because we know that $\xi_1 < 180 < 240 < \xi_2$.

Now $B'(\xi_1) = 0 = B'(\xi_2)$, and a third application of Rolle's Theorem, this time to B' on the interval $[\xi_1, \xi_2]$, yields a number $\eta \in (\xi_1, \xi_2)$ such that $B''(\eta) = 0$. Noting that $0 < \xi_1 < \eta < \xi_2 < 360$, we conclude that we have found $\eta \in (0, 360)$ such that $B''(\eta) = 0$.

4 Problem 4

4.1 Part a

If $v(t) = e^{1-t} - 1$ then acceleration is $a(t) = v'(t) = -e^{1-t}$, and $a(3) = -e^{-2}$.

4.2 Part b

Speed, $\sigma(t)$ at time t is $\sigma(t) = |v(t)|$, so $\sigma^2 = v^2$. Thus, $2\sigma\sigma' = 2vv'$ and

$$\sigma'(t) = \frac{v(t)}{\sigma(t)} v'(t) = \frac{v(t)}{|v(t)|} v'(t). \tag{23}$$

Thus

$$\sigma'(3) = \frac{v(3)}{|v(3)|} v'(3) = \frac{e^{-2} - 1}{|e^{-2} - 1|} \left(-e^{-2}\right) > 0. \tag{24}$$

The function σ' is continuous near $t = 3$, and so is therefore positive in some open interval centered at $t = 3$. Therefore $\sigma(t)$ is increasing near $t = 3$.

Note: The term *increasing* is almost always defined for functions over intervals, and almost never at isolated points. The issue is that a function like

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) + \dfrac{x}{2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases} \tag{25}$$

has a positive (but discontinuous) derivative at the origin but fails to be increasing on any open interval that contains the origin.

Technically, the question, as phrased, is unanswerable. But the readers have, historically, paid no attention to this subtlety.

4.3 Part c

The particle changes direction where the derivative of position with respect to time, which is velocity, changes sign. This happens only when $e^{1-t} - 1 = 0$, or when $t = 1$.

4.4 Part d

The total distance traveled (as opposed to the total *displacement*) over the time interval $[0, 3]$ is the integral of speed over $0 \le t \le 3$. This is

$$\int_0^3 \sigma(t)\, dt = \int_0^1 \left(e^{1-t} - 1\right)\, dt + \int_1^3 \left(1 - e^{1-t}\right)\, dt \tag{26}$$

$$= \left(-e^{1-t} - t\right)\Big|_0^1 + \left(t + e^{1-t}\right)\Big|_1^3 \tag{27}$$

$$= [(-1-1) - (-e)] + \left[(3 + e^{-2}) - (1+1)\right] = e - 1 + e^{-2}. \tag{28}$$

5 Problem 5

5.1 Part a

$g(3)$ is the sum of the area of a rectangle of height 2, base 1, with the area of a triangle of height 2, base 1, which is $2+1 = 3$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(3) = f(3) = 2$. In the interval $[2, 4]$, the Fundamental Theorem of Calculus tells us that

$$g'(x) = f(x) = f(4) + \frac{f(4) - f(2)}{4 - 2}(x - 4) \tag{29}$$

$$= 0 + \frac{0 - 4}{4 - 2}(x - 4) = -2(x - 4) \tag{30}$$

$$= 8 - 2x, \tag{31}$$

so

$$g''(x) = -2 \tag{32}$$

on $[2, 4]$. Hence, $g''(3) = -2$.

5.2 Part b

The average rate of change of g on $[0, 3]$ is

$$\frac{g(3) - g(0)}{3 - 0} = \frac{3 - (-4)}{3} = \frac{7}{3}. \tag{33}$$

See Part a, above, for the calculation of $g(3)$. To find $g(0)$ we simply observe that

$$g(0) = \int_2^0 f(t)\, dt, \tag{34}$$

which is the negative of the area of a triangle of base 2, height 4, which is -4.

5.3 Part c

By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Thus, on the interval $(0, 3)$, the function $g'(x)$ takes on its average value $\frac{7}{3}$ just twice—where the horizontal line $y = \frac{7}{3}$ intersects the graph of f.

One intersection lies in the interval $[0, 2]$ where $f(x) = 2x$. Thus, this intersection is at $x = \frac{7}{6}$. The other intersection lies in the interval $[2, 4]$, where $f(x) = 8 - 2x$, as we have seen in Part a, above. This intersection must therefore be at $x = \frac{17}{6}$.

We conclude that the only such points lie at $x = \frac{7}{6}$ and at $x = \frac{17}{6}$.

5.4 Part d

Inflection points occur where the montonicity of the derivative changes from increasing to decreasing, or vice versa. There are two such points for $g'(x) = f(x)$ (which equality we know from the Fundamental Theorem of Calculus). They are at $x = 2$ and at $x = 5$.

6 Problem 6

6.1 Part a

If f satisfies $f'(x) = x\sqrt{f(x)}$, with $f(3) = 25$, then $f'(3) = 3 \cdot \sqrt{25} = 15$. Thus,

$$f''(3) = \sqrt{f(3)} + 3 \cdot \frac{f'(3)}{2\sqrt{f(3)}} \tag{35}$$

$$= \sqrt{25} + \frac{3 \cdot 15}{2 \cdot \sqrt{25}} \tag{36}$$

$$= 5 + \frac{9}{2} = \frac{19}{2}. \tag{37}$$

6.2 Part b

We have, from $f'(x) = x\sqrt{f(x)}$,

$$\frac{f'(x)}{\sqrt{f(x)}} = x. \tag{38}$$

Now $f(3) = 25$, and as the solution of a differential equation, f must be a continuous function near $\xi = 3$. It follows that $f(\xi) > 0$ on some open interval centered at $\xi = 3$. Choosing any value of x lying in such an interval, we may write

$$\int_3^x \frac{f'(\xi)}{\sqrt{f(\xi)}}\, d\xi = \int_3^x \xi\, d\xi. \tag{39}$$

Carrying out the anti-differentiations, we find that

$$2\sqrt{f(\xi)}\Big|_3^x = \frac{1}{2}\xi^2\Big|_3^x. \tag{40}$$

But $f(3) = 25$, so this becomes

$$2\sqrt{f(x)} - 2\sqrt{25} = \frac{x^2}{2} - \frac{9}{2}, \tag{41}$$

or

$$\sqrt{f(x)} = \frac{x^2}{4} + \frac{11}{4}. \tag{42}$$

We conclude that

$$f(x) = \frac{1}{16}(x^2 + 11)^2. \tag{43}$$