AP Calculus 2003 AB FRQ Solutions

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1 Problem 1

1.1 Part a

The two curves intersect when $x = a$, where $\sqrt{a} = e^{-3a}$. Solving numerically, we find that $a \sim 0.23873$. Thus, we find (after a numerical integration) that the area of the region R is

$$\int_a^1 \left(\sqrt{x} - e^{-3x}\right)\, dx \sim 0.44263. \tag{1}$$

Note: The exact integral is

$$\int_a^1 \left(\sqrt{x} - r^{-3x}\right)\, dx = \frac{1}{3}\left[2x^{3/2} + e^{-3x}\right]\Big|_a^1 \tag{2}$$

$$= \frac{1}{3}\left(2 + e^{-3}\right) - \frac{1}{3}\left(2a^{3/2} + e^{-3a}\right). \tag{3}$$

However, we know a only approximately, so "exact" integration is misleading.

1.2 Part b

This problem is most easily solved using the method of washers. The required volume, V, is

$$V = \pi \int_a^1 \left[\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right] dx \tag{4}$$

$$\sim 1.42356. \tag{5}$$

It is also possible—but probably not wise—to use the method of shells:

$$V = 2\pi \int_{e^{-3}}^{\sqrt{a}} (1-y)\left(1+\frac{1}{3}\ln y\right)\,dy + 2\pi \int_{\sqrt{a}}^{1} (1-y)(1-y^2)\,dy. \tag{6}$$

Note: For the sake of completeness (See the Note to Part a, above), we record the "exact" value:

$$V = \frac{1}{6}\pi\left(-8a^{3/2} + 3a^2 + e^{-6a} - 4e^{-3a} + 4e^{-3} - e^{-6} + 5\right). \tag{7}$$

1.3 Part c

The area $A(h)$ of the cross section meeting the x-axis at $x = h$ is

$$A(h) = 5\left(\sqrt{h} - e^{-3h}\right)^2 \tag{8}$$

The required volume is therefore

$$\int_a^1 A(x)\,dx \sim 1.55435. \tag{9}$$

The integral is not elementary, and we have carried out the integration numerically.

2 Problem 2

2.1 Part a

We are given

$$v(t) = -(t+1)\sin\frac{t^2}{2}, \tag{10}$$

so acceleration $a(t)$ is

$$a(t) = v'(t) = -t(1+t)\cos\frac{t^2}{2} - \sin\frac{t^2}{2}. \tag{11}$$

Setting $t = 2$ then gives

$$v(2) = -3\sin 2; \tag{12}$$
$$a(2) = -6\cos 2 - \sin 2. \tag{13}$$

Speed $\sigma(t)$ is given by $\sigma(t) = |v(t)|$, so $[\sigma(t)]^2 = [v(t)]^2$, or $\sigma^2 = v^2$. Thus, at least when $v \neq 0$, $2\sigma\sigma' = 2vv'$, or

$$\sigma'(t) = \frac{v(t)}{\sigma(t)} v'(t) = \frac{v(t)}{|v(t)|} v'(t). \tag{14}$$

Thus,

$$\sigma'(2) = \frac{v(2)}{|v(2)|} v'(2) \sim -1.58758 < 0, \tag{15}$$

and it follows from the continuity of σ' that speed is decreasing near $t = 2$.

2.2 Part b

Changes in the direction of motion correspond to local extrema for position—which can occur only where the derivative of position, *i.e.* velocity, changes sign. If $v(t) = 0$, then either $1 + t = 0$ or $\sin(t^2/2) = 0$. In the interval $(0, 3)$, $1 + t$ is positive, while $\sin(t^2/2)$ changes sign only at $t^2 = 2\pi$, or $t = \sqrt{2\pi}$. Thus, $t = \sqrt{2\pi}$ gives the only time at which a direction change occurs.

2.3 Part c

Total distance traveled during $0 \leq t \leq 3$ is (integrating numerically)

$$\int_0^3 |v(t)|\, dt \sim 4.33382. \tag{16}$$

2.4 Part d

By the Fundamental Theorem of Calculus, position $x(t)$ at time t is given by

$$x(t) = 1 + \int_0^t v(\tau)\, d\tau. \tag{17}$$

Maximal distance from the origin for $0 \leq t \leq 3$ must then correspond to one of $t = 0$, $t = \sqrt{2\pi}$ or $t = 3$—the endpoints of the interval and the only critical point. Doing the necessary integrations numerically, we find that

$$x(0) = 1; \tag{18}$$

$$x\left(\sqrt{2\pi}\right) = 1 + \int_0^{\sqrt{2\pi}} v(\tau)\, d\tau \sim -2.26548 \tag{19}$$

$$x(3) = 1 + \int_0^3 v(\tau)\, d\tau \sim -1.19715. \tag{20}$$

Maximal distance from the origin therefore occurs when $t = \sqrt{2\pi}$, and is approximately 2.26548.

3 Problem 3

3.1 Part a

An approximate value for $R'(45)$ is given by

$$R'(45) \sim \frac{R(50) - R(40)}{50 - 40} = \frac{55 - 40}{10} = \frac{3}{2} \text{ gallons per minute.} \tag{21}$$

3.2 Part b

If $R(t)$ is increasing fastest at $t = 45$, then R' is maximal when $t = 45$. R' is differentiable, so this means that R' has a critical point at $t = 45$, or that $R''(45) = 0$.

3.3 Part c

The required left Riemann sum is

$$20 \cdot (30 - 0) + 30 \cdot (40 - 30) + 40 \cdot (50 - 40) + 55 \cdot (70 - 50) + 65 \cdot (90 - 70) = 3700. \tag{22}$$

Note that R is increasing on $[0, 90]$, and this means that $R(t)$ is minimal for each of the sub-intervals we consider at that sub-interval's left endpoint. We conclude that

$$3700 < \int_0^{90} R(t)\, dt. \tag{23}$$

3.4 Part d

If $0 \leq b \leq 90$, then $\int_0^b R(t)\,dt$ is the amount (in gallons) of fuel consumed between $t = 0$ and $t = b$. Thus $\frac{1}{b}\int_0^b R(t)\,dt$ is the average rate (in gallons per minute) at which fuel is consumed during the interval $0 \leq t \leq b$.

4 Problem 4

4.1 Part a

The graph of $y = f'(x)$, as given, lies above the x-axis only on the interval $[-3, -2)$, so f is increasing precisely on the interval $[-3, -2]$.

Note: Positivity of the derivative on $[-3, -2)$ guarantees that f is increasing on $[-3, -2)$. It is easily shown that a continuous function that is increasing on $[a, b)$, or, in fact, on (a, b), must be increasing on $[a, b]$. However, the readers have ignored this subtlety in the past.

4.2 Part b

Inflection points can be found at places where the derivative changes from increasing to decreasing, or vice versa. For the function f, we see from the graph of f' that one of these things happens at $x = 0$ and at $x = 2$.

4.3 Part c

We have $f'(0) = -2$, so the tangent line to $y = f(x)$ at the point with coordinates $(0, 3)$ is

$$y = 3 - 2x \tag{24}$$

4.4 Part d

The Fundamental Theorem of Calculus assures us that

$$f(x) = 3 + \int_0^x f'(\xi)\,d\xi, \tag{25}$$

so

$$f(-3) = 3 + \int_0^{-3} f'(\xi)\, d\xi \tag{26}$$

Now $\int_{-3}^{0} f(\xi)\, d\xi = -\int_0^{-3} f(\xi)\, d\xi$ is the area of a triangle of base 1 and height 1 minus the area of a triangle of base 2 and height 2, or $\frac{1}{2} - 2 = -\frac{3}{2}$. So

$$f(-3) = 3 + \frac{3}{2} = \frac{9}{2}. \tag{27}$$

On the other hand,

$$f(4) = 3 + \int_0^4 f(t)\, dt, \tag{28}$$

and this integral is the negative of the area that remains when a semicircle of radius 2 is removed from a rectangle of base 4 and height 2, or $8 - 2\pi$. Thus,

$$f(4) = 3 - (8 - 2\pi) = 2\pi - 5. \tag{29}$$

5 Problem 5

5.1 Part a

We have

$$V = \pi r^2 h = 25\pi h, \tag{30}$$

so

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}. \tag{31}$$

But it is given that

$$\frac{dV}{dt} = -5\pi\sqrt{h}. \tag{32}$$

Therefore

$$25\pi \frac{dh}{dt} = -5\pi\sqrt{h}, \tag{33}$$

and, dividing by 25π, we obtain

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5}. \tag{34}$$

5.2 Part b

Let $h = f(t)$ be the solution of the differential equation $5h' = -\sqrt{h}$ for which $h = 17$ when $t = 0$. Then f, being the solution of a differential equation with a positive initial value at $t = 0$, is a continuous function, remains positive over some interval centered at $t = 0$. We can therefore choose t so that $f(\tau)$ doesn't vanish for any value of τ that lies in the closed interval whose endpoints are 0 and t. For such values of τ we see that from

$$f'(\tau) = -\frac{\sqrt{f(\tau)}}{5}, \tag{35}$$

it follows that

$$\int_0^t \frac{f'(\tau)}{\sqrt{f(\tau)}}\, d\tau = -\frac{1}{5}\int_0^t d\tau. \tag{36}$$

Integrating, we obtain

$$2\sqrt{f(t)}\Big|_0^t = -\frac{1}{5}\tau\Big|_0^t, \tag{37}$$

or

$$2\sqrt{f(t)} - 2\sqrt{f(0)} = -\frac{t}{5}. \tag{38}$$

But $f(0) = 17$, so

$$\sqrt{f(t)} = \sqrt{17} - \frac{t}{10}, \tag{39}$$

and we conclude that

$$f(t) = \frac{1}{100}t^2 - \frac{\sqrt{17}}{5}t + 17. \tag{40}$$

The solution we seek is thus $h = f(t) = \frac{1}{100}t^2 - \frac{\sqrt{17}}{5}t + 17$.

5.3 Part c

The coffee pot is empty when $(\sqrt{17} - t/10)^2 = 0$, or when $t = 10\sqrt{17}$ seconds.

6 Problem 6

6.1 Part a

The function f is continuous at $x = 3$ iff $\lim_{x\to 3} f(x) = 3$. It is given that $f(3) = 2$, and it is clear from what is given that $\lim_{x\to 3^-} f(x) = 2$. When $x > 3$, we have $f(x) = 5 - x$ so we have $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5 - x) = 2$. Both one-sided limits exist and are equal to $2 = f(3)$, so we conclude that $\lim_{x\to 3} f(x) = 2 = f(3)$, meaning that f is continuous at $x = 3$.

6.2 Part b

The average value of f over the interval $[0, 5]$ is

$$\frac{1}{5-0}\int_0^5 f(x)\,dx = \frac{1}{5}\left(\int_0^3 \sqrt{x+1}\,dx + \int_3^5 (5-x)\,dx\right) \tag{41}$$

$$= \frac{1}{5}\left[\frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{x^2}{2}\right)\Big|_3^5\right] \tag{42}$$

$$= \frac{1}{5}\left[\frac{2}{3}\left(4^{3/2} - 1\right) + \left(\frac{25}{2} - \frac{21}{2}\right)\right] = \frac{4}{3}. \tag{43}$$

6.3 Part c

We are given

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases} \tag{44}$$

where m and k are unspecified constants.

If $g'(3)$ is to exist, then g must be continuous at $x = 3$, and, reasoning as in Part a, above, we find that this imposes the restriction that $2k = 3m + 2$, or that $m = (2k - 2)/3$.

Now suppose that G is a function continuous on some interval centered at $x = x_0$ and that $\lim_{x\to x_0^+} G'(x)$ exists and has the value L. (Note that for this limit to exist, G must be differentiable near x_0, though not necessarily at $x = x_0$ itself.) If $h > 0$ is small, then, G being continuous on $[x_0, x_0 + h]$ and differentiable on $(x_0, x_0 + h)$, we can, by the Mean Value Theorem. find $\xi_h \in (0, h)$ such that

$$\frac{G(x_0 + h) - G(x_0)}{h} = G'(\xi_h) \tag{45}$$

so that

$$\lim_{h\to 0^+} \frac{G(x_0+h)-G(x_0)}{h} = \lim_{h\to 0^+} G'(\xi_h), \tag{46}$$

provided that the latter limit exists.

But we have supposed that $\lim_{x\to x_0^+} G'(x) = L$, and $x_0 < \xi_h < x_0 + h$. Therefore $\xi_h \to x_0^+$ as $h \to 0^+$, and it follows that $\lim_{h\to 0^+} G'(\xi_h) = L$. We conclude that the right-hand derivative $G'_+(x_0)$ exists and equals L. From this, we see that, if G is any function which is continuous a near $x = x_0$, and for which $\lim_{x\to x_0^+} G'(x) = L$, then $G'_+(x_0) = L$.

A similar argument shows that if G is continuous near $x = x_0$ and $\lim_{x\to x_0^-} G'(x) = M$, then $G'_-(x_0) = M$.

Now $g'(x) = m$ for $3 < x \le 5$, so that $\lim_{x\to 3^+} g'(x) = \lim_{x\to 3^+} m = m$. If we require that $m = (2k-2)/3$, then g is continuous at $x = 3$, and we can conclude that $g'_+(3) = m$.

We have $g'(x) = k/(2\sqrt{x+1})$ for $0 \le x \le 3$, so that

$$\lim_{x\to 3^-} g'(x) = \lim_{x\to 3^-} \frac{k}{2\sqrt{x+1}} = \frac{k}{4}. \tag{47}$$

If we require that $m = (2k-2)/3$, as above, then g is continuous at $x = 3$, and we can conclude that $g'_-(3) = k/4$.

Thus, if $G'(3)$ is to exist (that is, that the two one-sided derivatives are both to exist and be equal), we now see that we must meet both of the conditions

$$m = \frac{2k-2}{3}, \text{ and} \tag{48}$$

$$m = \frac{k}{4}. \tag{49}$$

Thus, $3k = 8k - 8$, and $k = 8/5$. Finally, $m = k/4 = 2/5$. The desired values of the constants m and k are therefore

$$m = \frac{2}{5} \tag{50}$$

$$k = \frac{8}{5}. \tag{51}$$

Note: Problems like this appear in many calculus textbooks, but they generally encourage bad reasoning. The trouble is that equality of the quantities

$$\lim_{h\to 0} f'(x_0+h) \text{ and } \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \tag{52}$$

isn't an immediate consequence of the definitions. In fact, (52) asserts that

$$\lim_{x\to x_0}\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]=\lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right], \tag{53}$$

and this elucidates the problem: Such cavalier treatment of limit processes is not, in general, correct. A theorem is required to support these manipulations, and we proved one above:

Theorem *Let $\delta > 0$, and suppose that $f\colon [x_0, x_0+\delta) \longrightarrow \mathbb{R}$ is differentiable at every point of $(x_0, x_0+\delta)$. Suppose also that f is continuous from the right at $x = x_0$, and that*

$$\lim_{x\to x_0^+} f'(x) = L. \tag{54}$$

Then $f'_+(x_0)$ exists and equals L. Similarly for the left-hand derivative of f at x_0, necessary changes being made.•

That establishing this theorem seems to require the Mean Value Theorem suggests that the fact is non-trivial.