AP Calculus 2004 AB (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The curve intersects the x-axis at $x = 1$, so the desired area is

$$\int_1^{10} \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\Big|_1^{10} \tag{1}$$

$$= \frac{2}{3}(10-1)^{3/2} - \frac{2}{3}\cdot 0 = 18. \tag{2}$$

1.2 Part b

The volume generated when the region of Part a is revolved about the horizontal line $y = 3$ is

$$\pi \int_1^{10} \left[9 - \left(3 - \sqrt{x-1}\right)^2\right] dx = \pi \int_1^{10} \left[6\sqrt{1-x} + (1-x)\right] dx \tag{3}$$

$$= \pi \left[4(1-x)^{3/2} + x - \frac{x^2}{2}\right]\Bigg|_1^{10} = \frac{135}{2}\pi \sim 212.05750. \tag{4}$$

1.3 Part c

Solving the equation $y = \sqrt{x-1}$ for x in terms of y gives $x = y^2 + 1$. Hence, the volume generated by revolving the region about the vertical line $x = 10$ is

$$\pi \int_0^3 \left[10 - (y^2+1)\right]^2 \, dy = \pi \int_0^3 \left(y^4 - 18y^2 + 81\right) \, dy \tag{5}$$

$$= \pi \left(\frac{y^5}{5} - 6y^3 + 81y \right) \Big|_0^3 \tag{6}$$

$$= \pi \left(\frac{243}{5} - 6 \cdot 27 + 81 \cdot 3 \right) - 0 = \frac{648}{5}\pi \sim 407.15041. \tag{7}$$

2 Problem 2

2.1 Part a

Because $R(t) = 5\sqrt{t}\cos(t/5)$ is the rate of change of the number of mosquitos on the island and we have $R(6) \sim 4.43796 > 0$, it follows from the continuity of R that the number of mosquitos is increasing throughout some interval centered at $t = 6$.

Note: The statement that the number is increasing *at* $t = 6$ is problematic: The standard definition of the term *increasing* applies only on intervals, and not at an individual point.

2.2 Part b

$$R'(t) = \frac{5}{2\sqrt{t}} \cos\frac{t}{5} - \sqrt{t}\sin\frac{t}{5}, \text{ so} \tag{8}$$

$$R'(6) \sim -1.91319 < 0. \tag{9}$$

$R'(6) < 0$, and R' is continuous at $t = 6$. It follows that $R(t)$ is decreasing near $t = 6$. Thus, the number of mosquitos is increasing at a decreasing rate near $t = 6$. (But see the note to Part a, above.)

2.3 Part c

By the Fundamental Theorem of Calculus, the number $M(t)$ of mosquitos at time t is given by

$$M(t) = 1000 + \int_0^t R(\tau)\, d\tau. \tag{10}$$

Hence (carrying out the integration numerically)

$$M(31) = 1000 + \sqrt{5} \int_0^{31} \sqrt{\tau} \cos \frac{\tau}{5}\, d\tau \sim 964.33519. \tag{11}$$

To the nearest whole number, this is 964.

2.4 Part d

The maximum number of mosquitos for the period $0 \le t \le 31$ will occur when $t = 0$, or when $t = 31$, or when $R(t) = 0$. The latter condition obtains when $t = 5\pi/2$ and when $t = 15\pi/2$. Integrating numerically when necessary in (10), we find that

$$M(0) = 1000; \tag{12}$$

$$M\left(\frac{5\pi}{2}\right) \sim 1039.35691; \tag{13}$$

$$M\left(\frac{15\pi}{2}\right) \sim 842.40475; \tag{14}$$

$$M(31) \sim 964.33519. \tag{15}$$

Thus, the mosquito population peaks at about 1039 when $t = 5\pi/2$.

3 Problem 3

3.1 Part a

The Midpoint Rule with four subintervals of equal length gives

$$\int_0^{40} v(t)\, dt \sim v(5) \cdot (10 - 0) + v(15) \cdot (20 - 10) + v(25) \cdot (30 - 20) + v(35) \cdot (40 - 30) \tag{16}$$

$$\sim (9.2 + 7.0 + 2.4 + 4.3) \cdot 10 = 229 \text{ miles}. \tag{17}$$

The integral gives the distance, in miles, that the plane traveled during the time interval $0 \leq t \leq 40$.

3.2 Part b

By Rolle's Theorem, acceleration—which is $v'(t)$—must be zero at least once in the interval $0 \leq t \leq 15$ because $v(0) = v(15)$. Similarly, $v'(t)$ must be zero at least once in the interval $25 \leq t \leq 30$, because $v(25) = v(30)$. Thus, acceleration must vanish at least twice in the interval $0 \leq t \leq 40$.

3.3 Part c

If the plane's velocity is given by

$$f(t) = 6 + \cos\frac{t}{10} + 3\sin\frac{7t}{40}, \tag{18}$$

then

$$f'(t) = \frac{21}{40}\cos\frac{7t}{40} - \frac{1}{10}\sin\frac{t}{10} \tag{19}$$

gives acceleration. At $t = 23$, this gives acceleration as

$$f'(23) = \frac{21}{40}\cos\frac{161}{40} - \frac{1}{10}\sin\frac{23}{10} \text{ miles/min}^2 \tag{20}$$

$$\sim -0.40769 \text{ miles/min}^2. \tag{21}$$

3.4 Part d

Average velocity over $0 \leq t \leq 40$ is

$$\frac{1}{40}\int_0^{40}\left(6 + \cos\frac{t}{10} + 3\sin\frac{7t}{40}\right)dt = \frac{1}{40}\left[6t + 10\sin\frac{t}{10} - \frac{120}{7}\cos\frac{7t}{40}\right]\Bigg|_0^{40} \tag{22}$$

$$= \frac{1}{40}\left[240 + 10\sin 4 - \frac{120}{7}\cos 7\right] - \frac{1}{40}\left[\frac{120}{7}\right] \tag{23}$$

$$\sim 5.91627\text{miles/min}. \tag{24}$$

4 Problem 4

4.1 Part a

Inflection points are to be found where f'' changes sign—that is, where the slope of f' changes from positive to negative or vice versa. Consequently, the function f whose derivative is pictured has inflection points at $x = 1$ and at $x = 3$.

4.2 Part b

the function f is decreasing on the interval $[-1, 4]$ and increasing on the interval $[4, 5]$ because f' is non-positive, with only isolated zeros, on the first of these intervals and non-negative, with only an isolated zero on the second.

The absolute maximum vale of f must fall at one of the points $x = -1$ or $x = 5$. (There can be no absolute maximum for f at any point interior to $(-1, 5)$ because f' does not change signs from positive to negative anywhere in that interval.) The (unsigned) area bounded by f and the x-axis on the interval $[-1, 4]$ is clearly larger than the area between f and the x-axis on the interval $[4, 5]$, so

$$-\int_{-1}^{4} f'(t)\,dt = f(-1) - f(4) > f(5) - f(4) = \int_{4}^{5} f'(t)\,dt, \tag{25}$$

whence

$$f(-1) > f(5), \tag{26}$$

so the absolute maximum value taken on in the interval $[-1, 5]$ is $f(-1)$.

4.3 Part c

We are given that $g(x) = xf(x)$, so

$$g'(2) = f(2) + 2f'(2) = 6 + 2 \cdot (-1) = 4. \tag{27}$$

Also

$$g(2) = 2f(2) = 12. \tag{28}$$

An equation for the line tangent to the graph at $x = 2$ is therefore

$$y = 12 + 4(x - 2), \text{ or} \tag{29}$$

$$y = 4x + 4. \tag{30}$$

5 Problem 5

5.1 Part a

See Figure 1.

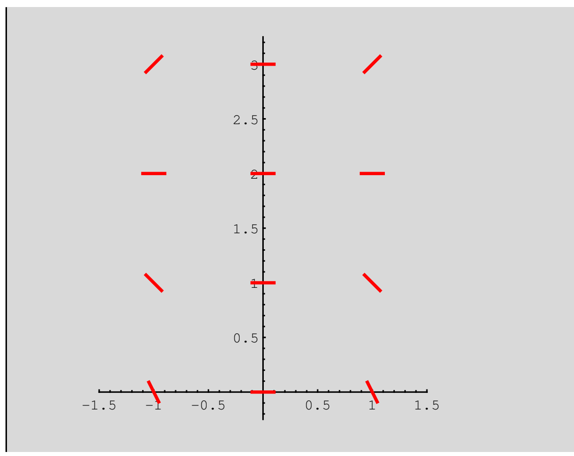

Figure 1: Problem 5, Part a

5.2 Part b

If $y' = x^4(y-2)$, then slope can be negative only when the product on the right side of the equation is negative. This is so just when $(y-2) < 0$, the points in the plane where slope is negative are the points (x, y) for which $y < 2$.

5.3 Part c

If $y = f(x)$, with $f(0) = 0$, is a solution to the differential equation $y' = x^4(y-2)$, then

$$f'(x) = x^4\left[f(x) - 2\right], \text{ or} \tag{31}$$

$$\frac{f'(x)}{f(x)-2} = x^4. \tag{32}$$

As a solution to a differential equation near $x = 0$, f must be a continuous function, at least on some interval centered at $x = 0$, so we can be sure that $(f(x) - 2) < 0$ when x is near 0. Choosing such an x, we integrate both sides of equation (32) from 0 to x:

$$\int_0^x \frac{f'(\xi)}{f(\xi)-2}\,d\xi = \int_0^x \xi^4\,d\xi. \tag{33}$$

Making use of the negativity of the denominator on the left side, as well as the fact that $f(0) = 0$, we obtain

$$\ln\left[2 - f(\xi)\right]\Big|_0^x = \frac{\xi^5}{5}\Big|_0^x, \text{ or} \tag{34}$$

$$\ln\left[2 - f(x)\right] = \ln 2 + \frac{x^5}{5}. \tag{35}$$

From this, it follows that

$$2 - f(x) = 2e^{x^5/5}, \text{ whence} \tag{36}$$

$$f(x) = 2\left(1 - e^{x^5/5}\right) \tag{37}$$

6 Problem 6

6.1 Part a

If $n > 1$, then

$$\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}. \tag{38}$$

6.2 Part b

If $n > 1$ and $y = x^n$, then

$$y' = nx^{n-1}, \text{ so} \tag{39}$$

$$y'\Big|_{x=1} = n. \tag{40}$$

It follows that the equation of the line tangent to $y = x^n$ at $(1, 1)$ is

$$y = 1 + n(x - 1). \tag{41}$$

This line crosses the x-axis at $x = 1 - \dfrac{1}{n}$, so that the base of the triangle T has length $\dfrac{1}{n}$. The altitude of T is one, so the area of T is $\dfrac{1}{2n}$.

6.3 Part c

From what we have seen in Parts a and b, above, the area, $A(n)$ of the region S, as a function of n, is

$$A(n) = \frac{1}{n+1} - \frac{1}{2n} = \frac{n-1}{2n^2+2n}. \tag{42}$$

Thus,

$$A'(n) = \frac{(2n^2+2n) - (n-1)(4n+2)}{4n^2(n+1)^2} \tag{43}$$

$$= -\frac{n^2-2n-1}{2n^2(n+1)^2}. \tag{44}$$

When $n > 0$, we see that $A'(n) = 0$ only for $n = 1+\sqrt{2}$, by the Quadratic Formula. Noting that $A'(n) > 0$ for $1 \leq n < 1+\sqrt{2}$ but that $A'(n) < 0$ for $1+\sqrt{2} < n$, we conclude, by the First Derivative Test, that the maximal area occurs when $n = 1+\sqrt{2}$.