AP Calculus 2004 AB FRQ Solutions

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1 Problem 1

1.1 Part a

The function $F(t) = 82 + 4\sin(t/2)$ gives the rate, in cars per minute, at which cars pass through the intersection. Thus, the total number of cars that pass through the intersection in the period $0 \le t \le 30$ is

$$\int_0^{30} F(t)\,dt = \int_0^{30} \left[82 + 4\sin\frac{t}{2}\right]\,dt \tag{1}$$

$$= \left[82t - 8\cos\frac{t}{2}\right]\Bigg|_0^{30} \tag{2}$$

$$= [2460 - 8\cos 15] - [0 - 8] \sim 2474.07750, \tag{3}$$

or 2474 to the nearest whole number.

1.2 Part b

$$F'(t) = 2\cos\frac{t}{2},\ \text{so} \tag{4}$$

$$F'(7) = 2\cos\frac{7}{2} \sim -1.87291 < 0, \tag{5}$$

and, F' being a continuous function, we conclude that traffic flow is decreasing near $t = 7$ because $F'(7) < 0$ and F' is continuous near $t = 7$. (We have phrased our answer this way, because the terms "increasing" and "decreasing" are almost always defined only for intervals, and not at individual points.)

1.3 Part c

The average value, in cars per minute, of traffic flow over the interval $10 \le t \le 15$ is

$$\frac{1}{15-10}\int_{10}^{15} F(t)\,dt = \frac{1}{5}\left[82t - 8\cos\frac{t}{2}\right]\Big|_{10}^{15} \tag{6}$$

$$= \frac{1}{5}\left(410 + 8\cos 5 - 8\cos\frac{15}{2}\right) \tag{7}$$

$$\sim 81.89924 \text{ cars per minute.} \tag{8}$$

1.4 Part d

The average rate of change of the traffic flow over the interval $10 \le t \le 15$ is

$$\frac{F(15) - F(10)}{15 - 10} = \frac{4\sin(15/2) - 4\sin 5}{5} \text{ cars per minute per minute} \tag{9}$$

$$\sim 1.51754 \text{ cars per minute per minute.} \tag{10}$$

2 Problem 2

Throughout this problem we understand that

$$f(x) = 2x(1-x) \text{ and} \tag{11}$$

$$g(x) = 3(x-1)\sqrt{x} \tag{12}$$

for $0 \le x \le 1$.

2.1 Part a

The graphs of the curves $y = f(x)$ and $y - g(x)$ intersect on the x-axis at $x = 0$ and at $x = 1$. Thus, the area between the two curves is

$$\int_0^1 [f(x) - g(x)]\,dx = \int_0^1 \left[2x(1-x) - 3(x-1)\sqrt{x}\right] dx \tag{13}$$

$$= \int_0^1 \left[3x^{1/2} + 2x - 3x^{3/2} - 2x^2\right] dx \tag{14}$$

$$= \left[2x^{3/2} + x^2 - \frac{6}{5}x^{5/2} - \frac{2}{3}x^3\right]\Big|_0^1 \tag{15}$$

$$= \left[2 + 1 - \frac{6}{5} - \frac{2}{3}\right] - 0 = \frac{17}{15}. \tag{16}$$

2.2 Part b

The volume of the solid generated by rotating the shaded region about the horizontal line $y = 2$ is

$$\int_0^1 \left[\pi[2 - g(x)]^2 - \pi[2 - f(x)]^2\right] dx \tag{17}$$

$$= \pi \int_0^1 \left(4x^4 - 17x^3 + 30x^2 + 12x^{3/2} - 17x - 12x^{1/2}\right) dx \tag{18}$$

$$= \pi \left(8x^{3/2} + \frac{17}{2}x^2 - \frac{24}{5}x^{5/2} - 10x^3 + \frac{17}{4}x^4 - \frac{4}{5}x^5\right)\Big|_0^1 \tag{19}$$

$$= \frac{103}{20}\pi \sim 16.17920. \tag{20}$$

2.3 Part c

The volume of the solid given is

$$\int_0^1 [h(x) - g(x)]^2\,dx = \int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx \tag{21}$$

Thus, the desired equation is

$$\int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx = 15. \tag{22}$$

Note: Solving equation (22) is not required, so evaluation of the integral is also not necessary. However,

$$\int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx = \frac{1}{30}k^2 + \frac{32}{105}k + \frac{3}{4}, \tag{23}$$

and solution of the resulting quadratic equation for $k > 0$ gives

$$k = \frac{\sqrt{87886} - 64}{14} \sim 16.60398. \tag{24}$$

3 Problem 3

3.1 Part a

Acceleration, $a(t)$, at time t is

$$a(t) = v'(t) = \frac{d}{dt}\left[1 - \arctan e^t\right] \tag{25}$$

$$= -\frac{e^t}{1 + e^{2t}}. \tag{26}$$

Thus,

$$a(2) = -\frac{e^2}{1 + e^4} \sim -0.13290. \tag{27}$$

3.2 Part b

Speed, $\sigma(t)$, at time t is given by

$$\sigma(t) = |v(t)| = \sqrt{[v(t)]^2}, \tag{28}$$

so

$$[\sigma(t)]^2 = [v(t)]^2. \tag{29}$$

Thus,

$$\cancel{2}\sigma(t)\sigma'(t) = \cancel{2}v(t)v'(t), \tag{30}$$

and

$$\sigma'(t) = \frac{v(t)}{\sigma(t)} v'(t) = \frac{v(t)}{|v(t)|} v'(t). \tag{31}$$

We now find that

$$\sigma'(2) = \frac{v(2)}{|v(2)|} v'(2) \tag{32}$$

$$= \left(\frac{1 - \arctan e^2}{|1 - \arctan e^2|} \right) \cdot \left(-\frac{e^2}{1 + e^4} \right) \sim 0.13290 > 0. \tag{33}$$

This is a positive number at a point where σ' is continuous, so we conclude that speed is increasing near $t = 2$.

3.3 Part c

First we observe that

$$v(0) = 1 - \arctan e^0 = 1 - \frac{\pi}{4} > 0. \tag{34}$$

The function $t \mapsto \arctan e^t$ is a composition of increasing functions on the positive half of the x-axis, and so is increasing there. Moreover, $\lim_{t\to\infty} \arctan e^t = \pi/2$. Hence, v is a decreasing function on $[0, \infty)$ and $\lim_{t\to\infty} v(t) < 0$. It follows that $v(T) = 0$ for just one value of $T > 0$, and because $v(t)$ passes from a region where it is positive to a region where it is negative as t increases through T, that point must give a maximum value for position $y(t)$, where

$$y(t) = -1 + \int_0^t v(\tau)\, d\tau. \tag{35}$$

Solving for T in the equation $v(T) = 0$ we find that $T = \ln \tan 1 \sim 0.44302$. This is the time when $y(t)$ reaches its maximal value.

3.4 Part d

Taking position as given by equation (35), we integrate numerically to find that

$$y(2) = -1 + \int_0^2 v(\tau)\, d\tau \sim -1.36069. \tag{36}$$

Because, as we have seen in Part c, above, $y(t)$ is decreasing for all $t > T \sim 0.44302$, we conclude that the particle is moving away from the origin when $t = 2$.

4 Problem 4

4.1 Part a

From

$$x^2 + 4y^2 = 7 + 3xy \tag{37}$$

we obtain, by implicit differentiation with respect to x, treating y as (locally) a function of x,

$$2x + 8yy' = 3y + 3xy', \tag{38}$$

so that

$$8yy' - 3xy' = 3y - 2x, \tag{39}$$

or

$$\frac{dy}{dx} = y' = \frac{3y - 2x}{8y - 3x}. \tag{40}$$

4.2 Part b

If we are to have $y' = 0$ in Part a, above, then we must have, from (40),

$$0 = y' = \frac{3y - 2x}{8y - 3x}, \tag{41}$$

and from this we conclude that $3y - 2x = 0$. But we are given that $x = 3$, and so $y = 2$. These values for x and y give

$$x^2 + 4y^2 = 3^2 + 4 \cdot 2^2 = 9 + 16 = 25 = 7 + 18 = 7 + 3 \cdot 3 \cdot 2 = 7 + 3xy, \tag{42}$$

showing that the point $(3, 2)$ lies on the curve. The point $P = (3, 2)$ thus meets our requirements.

4.3 Part c

From Part a, above, we have

$$(8y - 3x)y' = 3y - 2x. \tag{43}$$

Another implicit differentiation with respect to x then gives

$$(8y' - 3)y' + (8y - 3x)y'' = 3y' - 2. \tag{44}$$

At $(3, 2)$, as we have seen above, we have $y' = 0$. Substituting these values for x, y, and y' in equation (44) gives

$$(8 \cdot 0 - 3) \cdot 0 + (8 \cdot 2 - 3 \cdot 3)y'' = 3 \cdot 0 - 2, \tag{45}$$

whence

$$y''\Big|_{(3,2)} = -\frac{2}{7} < 0. \tag{46}$$

We conclude, from the Second Derivative Test, that the curve has a local maximum at $(3, 2)$.

5 Problem 5

5.1 Part a

$$g(0) = \frac{1}{2}(2 + 1) \cdot 3 = \frac{9}{2} \tag{47}$$

$$g'(0) = f(0) = 1. \tag{48}$$

5.2 Part b

The function g, being differentiable throughout its domain, attains a relative maximum only at points x_0 where $g'(x_0) = f(x_0) = 0$ and there is an $\epsilon > 0$ such that $g' = f$ is positive on the interval $(x_0 - \epsilon, x_0)$ but negative on the interval $(x_0, x_0 + \epsilon)$. The only such point is $x_0 = 3$.

5.3 Part c

Because g is differentiable throughout $(-5, 4)$, its absolute minimum value occurs either at a point where $g' = f$ has a zero or at an endpoint of the interval. We have $g'(x) = 0$ at $x = 3$, at $x = 1$, and at $x = -4$. Of these, only $x = -4$ is a possibility, because, by the First Derivative Test, neither of the other two gives a local minimum—and an absolute minimum interior to the interval must be a local minimum. It is geometrically evident that $g(-5) = 0$, that $g(-4) = -1$, and that $g(4)$ is substantially larger than 0. Consequently, the required abolute minimum value of g in the interval $[-5, 4]$ is $g(-4) = -1$.

5.4 Part d

The graph of g has an inflection point where the graph of $g' = f$ has a relative extremum. Consequently, g has inflection points at $x = -3$, at $x = 1$, and at $x = 2$.

6 Problem 6

6.1 Part a

See Figure 1.

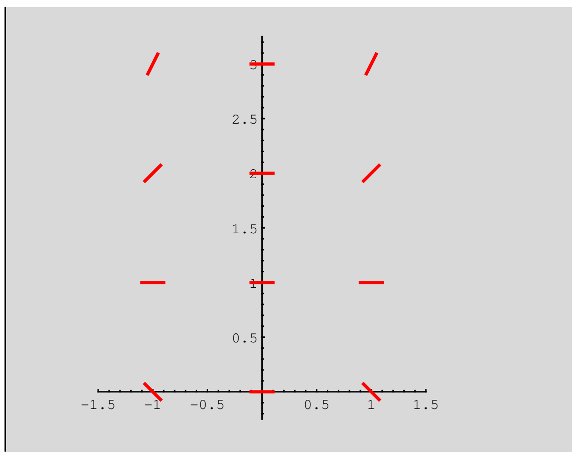

Figure 1: Problem 6, Part a

6.2 Part b

Because $y' = x^2(y-1)$ slope is positive only when $x \neq 0$ and $(y-1)$ is positive. Thus, slope is positive precisely where both $x \neq 0$ and $y > 1$.

6.3 Part c

If $y = f(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = x^2(y-1); \quad (49)$$
$$y(0) = 3, \quad (50)$$

then

$$\frac{f'(x)}{f(x)-1} = x^2, \tag{51}$$

so

$$\int_0^x \frac{f'(\xi)}{f(\xi)-1}\,d\xi = \int_0^x \xi^2\,d\xi, \tag{52}$$

as long as x is chosen so that $f(\xi) \neq 1$ anywhere in the closed interval whose endpoints are 0 and x. That such values of x exist follows from the the fact that f is the solution of the initial value problem for which $f(0) = 3$ and so is continuous on some interval centered at $x = 0$.

For such x,

$$\ln|f(\xi)-1|\Big|_0^x = \frac{\xi^3}{3}\Big|_0^x; \tag{53}$$

$$\ln|f(x)-1| - \ln|3-1| = \frac{x^3}{3}, \tag{54}$$

or

$$\ln|f(x)-1| = \frac{x^3}{3} + \ln 2, \tag{55}$$

and

$$|f(x)-1| = 2e^{x^3/3}. \tag{56}$$

Now $f(0)-1 = 3-1 = 2 > 0$, and f, and we have chosen x so that $f(x)-1 \neq 0$ anywhere in the interval determined by 0 and x. Thus, by continuity, $f(x)-1$ has the same sign as 2. Thus, $|f(x)-1| = f(x)-1$ and

$$f(x) = 1 + 2e^{x^3/3} \tag{57}$$

gives the solution we seek.